THE NUMERICAL APPROXIMATION TO SOLUTIONS FOR THE DOUBLE-SLIP AND DOUBLE-SPIN MODEL FOR THE DEFORMATION AND FLOW OF GRANULAR MATERIALS

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Chapter 1

Introduction

The subject matter of this thesis is the flow of granular materials. It is readily ob served that granular materials exhibit mechanical behaviour which differs from tha of ordinary solids, liquids, gases, or rather, they may readily exhibit behaviour anal ogous to all three phases in a manner which depends upon their environment. Thus a densely packed granular material which is confined in all directions behaves in solid-like manner. However, if there exists at least one direction in which the materia is not confined then, under a sufficiently high state of stress, the material may flow i a manner analogous to a liquid. Finally, if the material is unconfined and sufficien kinetic energy is supplied to the grains then the granular material behaves in a gaseou manner. The mathematical modelling of such materials has perplexed scientists du to these differences. In this thesis we shall only consider granular materials which ar in a dense, solid-like state. For low levels of stress the material responds elasticall i.e. the deformation is reversible. As the stress level is increased, at some point th material ceases to behave in an elastic manner and is said to be in a state of yield, i.e the material responds in an irreversible manner. If the stress level continues to rise i.e. the material is continually loaded up, then at some point the material fails and, uncontained, flows freely.

The concept of failure of granular was recognised very early on in the disciplin that is now called civil engineering and in 1776 Coulomb [5] postulated conditions t be satisfied when failure occurs in a granular material. Little further progress wa made for granular materials for another century and a half, however, the concept c yield proved very fruitful in studying the mechanical behaviour of metals, where i became clear that the constitutive behaviour could be assumed to comprise two parts, firstly a yield condition (an algebraic inequality to be satisfied by the components of stress) and secondly a flow rule (a relationship between the stress and deformation-rate tensors). A reasonably complete theory for the inelastic behaviour of metals was developed under the impetus of technological requirements in the period between 1860 and culminated in the book by Hill [28] in 1950. A crucial difference between the behaviour of metals and granular materials is that the yielding of metals is independent of the pressure, whereas the yielding of granular materials is markedly dependent on the magnitude of the pressure. The first suggestion that the methods of metal plasticity could be applied to problems of soil mechanics by the incorporation of a pressure dependent yield condition was made by Drucker & Prager [6]. They proposed such a yield condition, called the Drucker-Prager yield condition and used it as the plastic potential to derive an associated flow rule. One of the difficulties of modelling granular materials is that they possess two new physical properties, namely the pressure dependence on yield and the fact that shear is often accompanied by dilatation. To a first approximation, neither of these properties is exhibited by metals, for which the concept of an associated flow rule (i.e. identical yield and plastic potential functions) is very successful. For granular materials, a simple associated flow rule is inadequate due to the fact that there is only a single parameter to govern the magnitude of the above two physical quantities i.e. the pressure dependence and dilatation. However for real soils, the magnitude of dilatation is considerably less than the magnitude of the pressure dependence. This inadequacy gave rise to the adoption of a plastic potential function which is distinct from the yield function and the flow rule obtained in this way is called a non-associated flow rule [40]. It should be noted that some metals do exhibit some pressure dependence on yield but that incompressibilty is maintained (and so a non-associated flow rule would seem to be necessary for such materials). Also, it is possible, at the expense of added complication to the model, to obtain realistic results for the dilatation and pressure dependence using an associated flow rule, the prime example here being the critical state model developed by Roscoe and co-workers, see Schofield & Wroth [3].

The inadequacy of the associated flow rule coupled with misgivings concerning the correctness of using non-associated flow rules led to the proposal of models which are based directly on assumptions concerning the underlying kinematics of the way in which granular materials flow. These physically based kinematic models are based on the assumption that granular materials flow as a combination of shear, dilatation and rotation and the resulting kinematic equations are used in conjunction with the stress equilibrium equations and a yield condition (usually the Coulomb yield condition). A number of authors have proposed such models, among them are Mandel [29], Geniev [20], de Josselin de Jong [21], [22], Spencer [1], [2], Mehrabadi & Cowin [34], Anand [30], Harris [15] and Harris & Grekova [16]. The shearing modes of deformation are sometimes referred to as *sliding* or *slip* modes and the rotational mode is sometimes referred to as *rotating* or *spin* modes. Historically the first such model to be proposed was that due to Mandel [29] but it made little impact at the time and was rediscovered later after other similar models had been proposed independently. Thus, Geniev [20] proposed a model in which the flow comprised a single shear aligned along one of the two possible Coulomb yield directions. In the context of incompressible flows, the double-sliding free-rotating model was proposed by de Josselin de Jong [21] on the basis of simultaneous shears in two slip directions which have orientations bounded by the Coulomb yield directions together with a spin of indeterminate magnitude. Using de Josselin de Jong's work as a basis, the double-shearing model was originally proposed for incompressible flows by Spencer [1]. It was extended to a certain class of dilatant materials on the assumption that they obeyed the Butterfield-Harkness [36] kinematic hypothesis by Mehrabadi & Cowin [34]. At the same time de Josselin de Jong [22] generalised the double-sliding free-rotating model to the same class of dilatant materials. The model was further generalised by Anand [30] who considered two arbitrary dilatant shearing directions.

An alternative formulation of the Butterfield-Harkness flow rule together with an alternative derivation of the Mehrabadi-Cowin equations is given in Harris [7]. In the context of the double shearing model applications to boundary value problems in mining engineering and geophysics were formulated and solved in Harris [8] and [10]. The model consisted of the stress equilibrium equations, Coulomb yield criterion, Mehrabadi-Cowin equations and continuity equation together with boundary conditions on the stress, velocity and density fields. In Harris [9], [10] and [12] various issues are examined relating to the numerical integration of the equations governing the stress field and the double shearing kinematic equations. A unification of the plastic potential model and the double shearing model was achieved in Harris [11] and [13] enabling a general treatment of dilatancy to be incorporated into the model. The issue of the linear ill-posedness of the governing equations is a recurring theme for granular materials and in [14] it was shown that both the non-associated flow rule and the double shearing model are linearly ill-posed. Ill-posedness of the governing equations gives rise to very bad instabilities in the solutions of the model that make construction of numerical approximations impossible. Harris [15] developed a well-posed single shearing model, while Harris & Grekova [16] proposed a well-posed planar model which may be called the double-slip and double-spin model. This model was generalised to a fully three dimensional model in Harris [17] which may be called the augmented plastic potential model. This approach combines the physically based modelling with the more usual constitutive modelling based on tensor formulations of the flow rule. Properties of this model were further developed in Harris [18]. Analytic solutions of the model for incompressible materials are presented in Alexandrov and Harris [37] and [38].

The issue of ill-posedness of the non-associated plastic potential model and the double shearing model has prevented the proper formulation and solution of boundary value problems which involve the flow of the material for applications to engineering and geophysics. The construction of a class of well-posed models in Harris & Grekova [16] and Harris [17] now enables such problems to be formulated and solved.

The main purpose of the work presented in this thesis is to generalise the numerical methods commonly used in metal plasticity so that they may be used to construct numerical approximations to solutions of a special case of the double-slip and double-spin model for the deformation and flow of granular materials. In this model the granular material is modelled as a single-phase continuum with three material parameters namely the cohesion c, the angle of internal friction ϕ and the angle of dilatancy ν.

In the problems considered in this thesis the deformational response to loading is assumed to be planar and rigid-plastic, i.e. the elastic deformation is neglected and the material which is not in yield is assumed to be rigid. In the problems considered here, the flow is assumed to consist at each point in the deforming region, of two simultaneous (possibly dilatant) shears, one in the direction of each tangent to the stress characteristic curves. The field variables are the Cauchy stress tensor $\boldsymbol{\sigma}$, the velocity vector $\boldsymbol{v}$ and the bulk density ρ. The field equations comprise a set of first-order partial differential equations, namely the stress-equilibrium equations, the double-slip and double-spin kinematic equations and continuity equation together with an algebraic inequality, namely the Coulomb yield criterion.

However, due to the ill-posedness of the plastic potential and double-shearing models it has not previously been possible to obtain the velocity fields corresponding to the stress fields determined by the stress boundary conditions and hence it has not been possible to solve the complete boundary value problem. The purpose of the work presented in this thesis is to use the model developed in Harris & Grekova [16] and Harris [17] to solve for the stress, velocity and density fields, thereby enabling problems in the mechanics of granular materials to be solved as completely as problems in metal plasticity. We shall refer to this model as the double-slip and double-spin model. The relations along the characteristics for the double-slip and double-spin model have not been published.

We now outline the contents of each chapter of the thesis. In Chapter 2 we present the double-slip and double-spin model for the stress, velocity and density fields which has, as a special case the plastic potential model (both associated and non-associated) and which, for certain special flows, is identical with the double-shearing model. In Chapter 3 the numerical integration of the stress, velocity and density equations is considered. In particular, the construction of the stress and velocity field is considered in detail. It is also shown how to calculate the work-rate, which must be a non-negative quantity for the solution to have physical significance. In Chapter 4 the model considered in chapter 2 and the methods of numerical calculation discussed in chapter 3 are used to solve various boundary value problems from a number of applications. Finally, we discuss conclusions and future work in Chapter 5.

Chapter 2

Mathematical formulation of the equations governing the model

2.1 Introduction

In this chapter we shall present equations which govern the stress, velocity and density distribution of a granular material occupying a given region of space. Firstly, however, we motivate the mechanical model in general terms. It is usual to describe the constitutive equation for a solid-like material in terms of the relationship that exists between stress and strain. An idealised, but typical, stress-strain curve is shown in Figure 2.1. The stress is derived from measurement of the load applied to the specimen and the strain is derived from the subsequent deformation of the material. The curve is unique for each material. In Figure 2.1 the strain is plotted on the horizontal axis and stress on the vertical axis. The straight line from the origin O to point A indicates that the relationship between stress and strain in this initial region is linear and proportional. This is true for many materials and the material behaves linearly elastically on OA. Beyond the point A the response is no longer linear and hence the stress at A, σ_A, is called the proportional limit. The curve then continues to rise to the right of A (but with a reduced gradient compared with the linear elastic part), and the material behaviour is non-linearly elastic. On OB the deformation is reversible. Beyond the point B, the material behaviour is irreversible and the material is said to be in a state of yield on this portion of the curve. The point B is also called the yield point and σ_B is also called the yield stress. If the curve has a positive gradient, the material is said to exhibit plastic work- or strain-hardening. If the gradient becomes zero at some point C the material is said to be in a state of failure and will flow freely if unconstrained. If, to the right of the point C, the curve has a negative gradient then the material is said to exhibit work- or strain-softening.

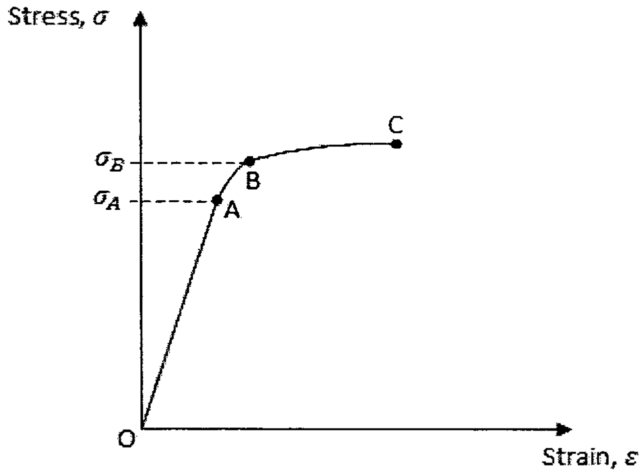

Figure 2.1: The stress-strain curve

A further simplification that can be made under circumstances in which the material is flowing is to neglect the elastic component of the deformation and consider the material to behave as a rigid body at stress levels below the yield point. In this case the point B lies on the σ-axis and OB is the portion of the σ-axis between O and B. The plastic portion of the curve BC is assumed to be a horizontal straight line. Such models are referred to as rigid-perfectly plastic models and, in the rest of the thesis, we shall restrict attention to such models. On the horizontal portion of the graph there is not a one-to-one correspondence between stress and strain, infinitely many values of the strain corresponding to a single value of the stress. See Figure 2.1.

The physical meaning of this is that the material may flow freely if it is not completely constrained by its environment. The material is then said to be in a state of failure. For the perfectly plastic model the yield and failure points coincide. For hardening materials they will generally be distinct. For a general state of stress, the yield point is determined by a yield criterion which is an inequality and is based upon a scalar valued function (called a yield function) of the components of stress. Many yield criteria have been proposed and compared with experiment. One of the simplest and most successful is the Mohr-Coulomb yield criterion and this is considered in the next section.

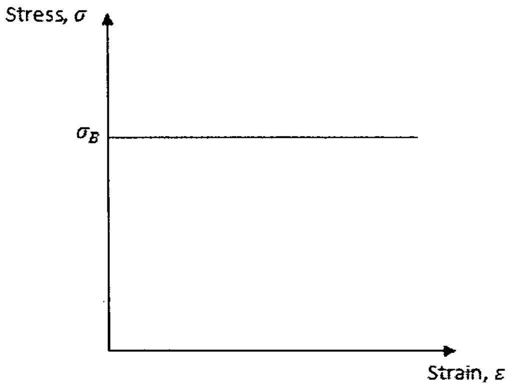

Figure 2.2: The stress-strain curve

2.1.1 Mohr-Coulomb yield criterion

As stated in the previous chapter, in the eighteenth century, Coulomb [5] introduced the classical model of failure of a granular material, which has been so successful that it persists to the present day. The material is assumed to be a continuous medium with a yield criterion which incorporates frictional effects via the pressure dependency. Real granular materials such as sand and clay are very complicated systems involving a large number of discrete grains. The voids between the grains may be contain fluid, e.g. air and water. We shall consider only simple, continuum models with the intention of only retain the essential properties of granular materials. There are several approaches to formulating the behaviour of granular materials.

The basic idea underlying the Mohr-Coulomb criterion is that the magnitude of the shearing stress, τ_n, on any section through a point in an isotropic granular material must not exceed a quantity which is linearly dependent upon the compressive normal stress, σ_n^c, acting on that section. The magnitude of the shear stress is assumed to satisfy the relationship

$$|\tau_n| \leq c + \sigma_n^c \tan\phi \qquad (2.1)$$

where the material parameters $c \geq 0$ and ϕ, where $0 \leq \phi < \pi/2$, are called the cohesion and the angle of internal friction, respectively, of the granular material. Tensile stres is taken to be positive. The inequality (2.1) is called the Mohr-Coulomb yield criterion The material is rigid if strict inequality holds and is said to be in a state of yield (o failure or in a plastic state) if the equality sign holds in relation (2.1). If the state o stress varies throughout the material then in some regions strict inequality may hol(while in the remainder, equality will hold. The set of points for which strict inequalit: holds is called the rigid region or regions, while the set of points for which equalit: holds is called the plastic region or regions.

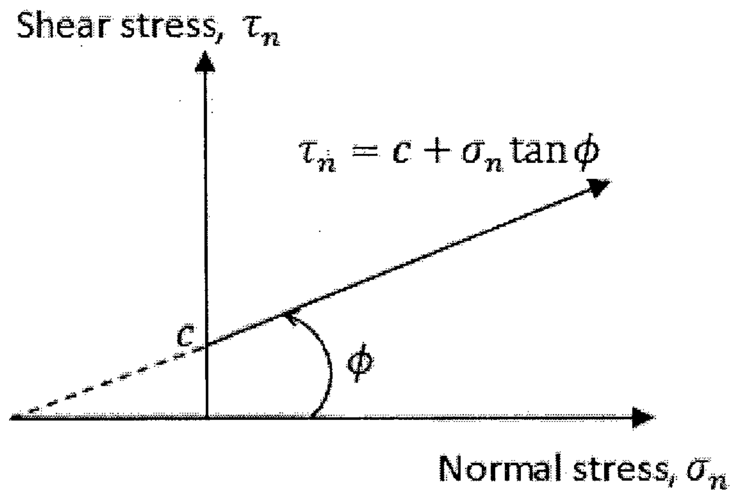

Figure 2.3: Mohr-Coulomb yield surface

The line in Figure 2.3 represents the condition of yielding for the material. Fc points below the line, the material response will be either rigid or elastic. If the shea stress is increased for a given normal stress such that the stress state of the materiε is exactly on the yield line, then plastic strain or yielding will result. If $\phi = 0$, the the material is said to be frictionless or purely cohesive. The inequality then reducε to those which determine the stress in the plane strain theory of metal plasticity. : $\phi \neq 0$ and $c = 0$, then the material is said to be cohesionless.

2.2 Stress field

A rectangular Cartesian coordinate system consists of an orthonormal basis of un: vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and an origin, O. Right-handed Cartesian coordinate systems aı considered and the axes in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ directions are denoted by (x, y, z). The stat of stress at a point with respect to the rectangular Cartesian coordinate system is represented by the Cauchy stress tensor which comprises three normal stress components $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ and six shear stress components $(\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32})$ as in Figure 2.4 and may be written in matrix form as

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{2.2}$$

Each component is referenced to the coordinate system x, y and z and named by two subscripts. For example, the normal stress acting over the x, y-plane, parallel to z-axis is named σ_{33} and the two shear components σ_{31} and σ_{32}. The stress components are considered to be positive in tension. It is well known that in the absence of couple stress and body couples, moment equilibrium demands the following relationship on the Cauchy stress tensor

$$\sigma_{ij} = \sigma_{ji}. \tag{2.3}$$

i.e. the Cauchy stress tensor is symmetric, As a result there are only six independent stress components, the three normal stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ as before and three shearing stresses $\sigma_{12}, \sigma_{13}, \sigma_{23}$.

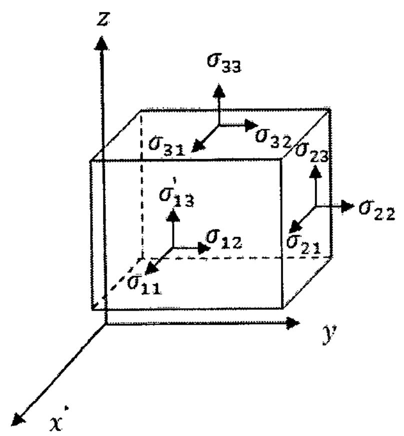

Figure 2.4: Components of stress in three dimensions

2.2.1 Principal stresses and stress invariants

For a symmetric tensor there exist three orthogonal directions **n** such that the shear components of stress are zero across a plane section with normal **n**. These three directions may be used to define a coordinate system called the principal stress coordinate system and relative to this system, only the normal stresses may be non-zero. The plane sections with normal vectors **n** are called the principal planes and the directions themselves, **n**, are called the principal directions. The three stresses normal to these principal planes and parallel to the principal directions are called the principal stresses. Principal stresses can be determined by using the concepts of eigenvalues and eigenvectors. The condition that **n** be a principal direction may be expressed as

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \lambda\, \mathbf{n}. \tag{2.4}$$

where λ is a scalar multiplier, the value of which is to be found, and is called an eigenvalue. For a given eigenvalue, the corresponding vector **n** is called an eigenvector. Once λ has been found, the eigenvector **n** can be obtained by solving the equivalent system of homogeneous linear equations

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \mathbf{n} = 0 \tag{2.5}$$

where **I** is the identity tensor. A nontrivial solution for **n** exists only for values of λ such that the matrix is singular, which occurs when its determinant is zero.

$$\det(\boldsymbol{\sigma} - \lambda \mathbf{I}) = 0. \tag{2.6}$$

Expanding the determinant gives the cubic equation for λ

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0. \tag{2.7}$$

where the coefficients are the invariants of the stress tensor given by

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \tag{2.8}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2, \tag{2.9}$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{13}. \tag{2.10}$$

The coefficients I_1, I_2 and I_3 are called the first, second and third stress invariants of the stress tensor, respectively. Equation (2.6) has three real roots, σ_I, σ_{II} and σ_{III} which are the principal stresses or eigenvalues. In the case where all three principal stresse: are distinct, the greatest and least principal stresses are called the major principa stress and minor principal stress, respectively. The remaining principal stress is callec the intermediate principal stress. In some applications it is convenient to agree t(number the principal stresses so that σ_I is the maximum principal stress and σ_{III} is th(minimum principal stress which leads to the conventional definition of $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$ In some circumstances it may be more convenient to have $\sigma_{III} \geq \sigma_{II} \geq \sigma_I$, or on(of the other four possible permutations of the three indices. Relative to the principa stress coordinate system (i.e. the coordinate directions are parallel to the principa directions) only one subscript is required to distinguish the normal components o stress and the stress tensor is written as

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \tag{2.11}$$

2.2.2 The mean stress and deviatoric stress

The mean stress is defined as the average of the three normal stresses, which can b(expressed as follows

$$\sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}I_1. \tag{2.12}$$

The deviatoric stress tensor is defined to be the tensor with components defined by

$$s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} \tag{2.13}$$

where δ_{ij} is the Kronecker delta whose value is 1 when $i = j$ and is equal to 0 whe: $i \neq j$. The three invariants of the deviatoric stress are

$$J_1 = s_{kk} = 0 \tag{2.14}$$

$$J_2 = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{3}(I_1 + 2I_3) \tag{2.15}$$

$$= \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \tag{2.16}$$

$$= \frac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{13}^2 \tag{2.17}$$

$$J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki} = \frac{1}{27}(2I_1 + 9I_1I_2 + 27I_3). \tag{2.18}$$

Physically I_1 indicates the effect of mean stress, J_2 represents the magnitude of th shear stress and J_3 determines the direction of the shear stress.

2.2.3 Stress tensor transformation under a coordinate system rotation

In this thesis we are concerned with the plane deformation of granular materials and it will turn out that the flow will take place in the plane defined by the major and minor principal stress directions. This plane will be taken as the Oxy-plane and the intermediate principal stress will act in the Oz-direction. In this section we shall consider transformations of the coordinate system which keep Oz fixed. Under such transformations σ_{13}, σ_{23} remain zero and σ_{33} is invariant. Thus, the stress tensor takes the form

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \tag{2.19}$$

and which is symmetric. The stress components with respect to the x, y, z coordinate system can be transformed to a new coordinate system, $O\bar{x}\bar{y}z$, by a rotation about the Oz axis by an arbitrary angle θ measured from the positive x-axis towards the positive $\bar{x}$-axis, with a counterclockwise rotation taken as positive. See Figure 2.5.

Thus, after such a transformation, the state of stress is described by the set of stress components $(\bar{\sigma}_{11}, \bar{\sigma}_{12}, \bar{\sigma}_{21}, \bar{\sigma}_{22}, \sigma_{33})$ relative to the set of axes $O\bar{x}\bar{y}z$. The transformation of the stress matrix $\boldsymbol{\sigma}$ in (2.19) from the coordinates $Oxyz$ to the stress matrix $\bar{\boldsymbol{\sigma}}$ relative to the coordinates $O\bar{x}\bar{y}z$ is obtained according to the tensor transformation rule given by

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{R}^T \boldsymbol{\sigma} \boldsymbol{R}$$

where $\boldsymbol{R}$ is the rotation matrix

$$\boldsymbol{R} = (R_{ij}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.20}$$

In the case of the transformation for the state of stress given in equation (2.19) we obtain

$$\begin{pmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} & 0 \\ \bar{\sigma}_{21} & \bar{\sigma}_{22} & 0 \\ 0 & 0 & \bar{\sigma}_{33} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.21}$$

which gives the following four equations

$$\bar{\sigma}_{11} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22})\cos 2\theta + \sigma_{12}\sin 2\theta, \tag{2.22}$$

$$\bar{\sigma}_{12} = \bar{\sigma}_{21} = \frac{1}{2}(\sigma_{22} - \sigma_{11})\sin 2\theta + \sigma_{12}\cos 2\theta, \tag{2.23}$$

$$\bar{\sigma}_{22} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{22} - \sigma_{11})\cos 2\theta - \sigma_{12}\sin 2\theta, \tag{2.24}$$

$$\bar{\sigma}_{33} = \sigma_{33}. \tag{2.25}$$

Now suppose that the $Oxyz$ is an arbitrary coordinate system while the $O\bar{x}\bar{y}z$ coordinate system is taken to coincide with the principal axes of stress. Let ψ denote the corresponding value of θ, see Figure 2.5

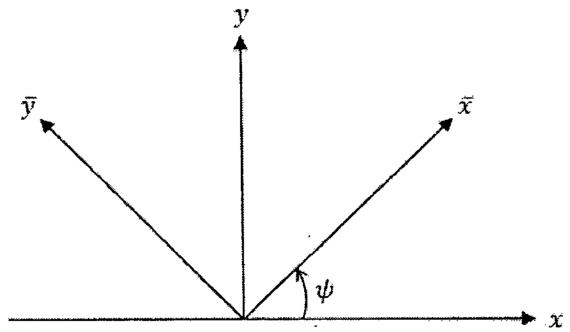

Figure 2.5: Angle ψ between stress axes (x, y) and principal stress axes $(\bar{x}, \bar{y})$

To find an expression for the angle ψ in terms of the stress components relative to the $Oxyz$ coordinate system, we note that $\bar{\sigma}_{12} = 0$ relative to the $O\bar{x}\bar{y}z$ coordinate system, and so equation (2.23) gives

$$\bar{\sigma}_{12} = \frac{1}{2}(\sigma_{22} - \sigma_{11})\sin 2\psi + \sigma_{12}\cos 2\psi = 0$$

and this yields the desired expression for the angle ψ, namely

$$\tan 2\psi = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}. \tag{2.26}$$

Since there are only three stress components which change under the transformation, namely the two normal stresses $(\sigma_{11}, \sigma_{22})$ and the single shear stress component σ_{12} it is convenient to consider omitting the third row and third column of the stress tensor and consider the planar symmetric Cauchy stress tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \tag{2.27}$$

It is instructive to re-calculate the eigenvalues and eigenvectors for this planar stress tensor. Let the eigenvectors be denoted by $\mathbf{n}_1 = \begin{pmatrix} n_1^1 \\ n_2^1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} n_1^2 \\ n_2^2 \end{pmatrix}$. Given the stress matrix $\boldsymbol{\sigma}$, the eigenvector $\mathbf{n}$ and eigenvalue λ must satisfy the equation 2.4. It is then possible to find a vector $\mathbf{n} \neq 0$ and a scalar λ using the equation analogous to equation (2.5).

The characteristic polynomial of the stress matrix equation (2.27) is

$$\lambda^2 - \lambda(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0. \tag{2.28}$$

Solving for λ gives the two eigenvalues λ_1 and λ_2,

$$\lambda = -p \pm q \tag{2.29}$$

where the invariant quantity

$$p = -\frac{1}{2}(\sigma_{11} + \sigma_{22}) \tag{2.30}$$

may be interpreted as the mean pressure in the plane and the invariant quantity

$$q = \frac{1}{2}\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} \tag{2.31}$$

may be interpreted as the maximum shearing stress. The eigenvector $\boldsymbol{n}_1$ associated with eigenvalue $\lambda_1 = -p + q$ satisfies (2.5) and leads to the simultaneous equations

$$(\sigma_{11} + p - q)n_1^1 + \sigma_{12}n_2^1 = 0$$
$$\sigma_{12}n_1^1 + (\sigma_{22} + p - q)n_2^1 = 0.$$

Without loss of generality, let $n_2^1 = 1$ then these equations yield the eigenvector

$$\mathbf{n}_1 = \begin{pmatrix} n_1^1 \\ n_2^1 \end{pmatrix} = \begin{pmatrix} \frac{-(\sigma_{22}+p-q)}{\sigma_{12}} \\ 1 \end{pmatrix} \tag{2.32}$$

The eigenvector $\boldsymbol{n}_2$ corresponding to the eigenvalue $\lambda_2 = -p - q$ leads to the simultaneous equations,

$$(\sigma_{11} + p + q)n_1^2 + \sigma_{12}n_2^2 = 0$$
$$\sigma_{12}n_1^2 + (\sigma_{22} + p + q)n_2^2 = 0.$$

Let $n_1^2 = 1$ then these equations yield the eigenvector

$$\mathbf{n}_2 = \begin{pmatrix} n_1^2 \\ n_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-(\sigma_{11}+p+q)}{\sigma_{12}} \end{pmatrix} \tag{2.33}$$

The eigenvectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ give the principal directions of $\boldsymbol{\sigma}$ and are orthogonal to one another,

$$\begin{aligned}\mathbf{n}_1 \cdot \mathbf{n}_2 &= \frac{-(\sigma_{22} + p - q)}{\sigma_{12}} + \frac{-(\sigma_{11} + p + q)}{\sigma_{12}} \\ &= \frac{-1}{\sigma_{12}}(\sigma_{22} + \sigma_{11} - \sigma_{11} - \sigma_{22}) \\ &= 0.\end{aligned}$$

See Figure 2.6

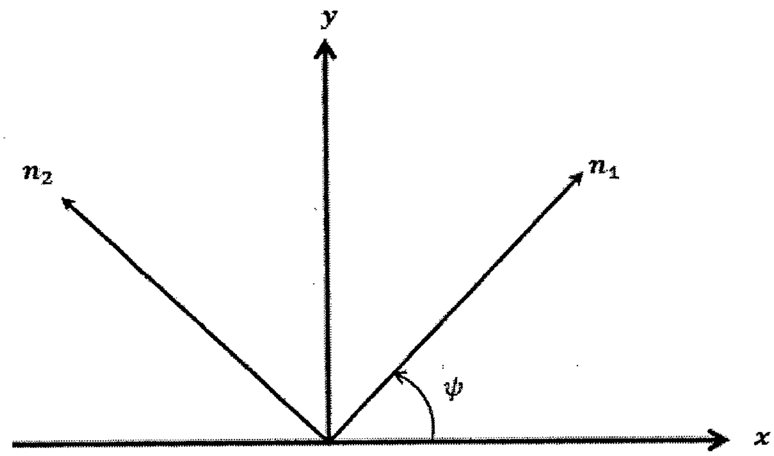

Figure 2.6: The eigenvectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are orthogonal to one another

Finally, noting that relative to the $O\bar{x}\bar{y}z$ axes, the quantities p and q become

$$p = -\frac{1}{2}(\bar{\sigma}_{11} + \bar{\sigma}_{22}), \qquad (2.34$$

and

$$q = \frac{1}{2}(\bar{\sigma}_{11} - \bar{\sigma}_{22}) \qquad (2.35$$

and inverting the relations (2.21), we obtain the following representation for σ_{ij} i: terms of p, q and ψ,

$$\sigma_{11} = -p + q\cos 2\psi, \qquad (2.36$$

$$\sigma_{22} = -p - q\cos 2\psi, \qquad (2.37$$

$$\sigma_{12} = q\sin 2\psi. \qquad (2.38$$

2.2.4 Coulomb yield criterion

We now specify in detail the yield condition. At a point P in the material consider an element of surface with normal vector $\mathbf{n}$ and let $\mathbf{T}$ be the traction acting across the surface. Let σ_n and τ_n be the normal and shear components of the traction across the surface. The traction vector is related to the stress tensor by

$$\begin{aligned}\mathbf{T} &= \boldsymbol{\sigma} \cdot \mathbf{n} &(2.39)\\ &= \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2\end{bmatrix} \\ &= \begin{bmatrix}\sigma_{11}n_1 + \sigma_{12}n_2 \\ \sigma_{12}n_1 + \sigma_{22}n_2\end{bmatrix} &(2.40)\end{aligned}$$

The normal stress across the surface is related to the traction vector by

$$\begin{aligned}\sigma_n &= \mathbf{n} \cdot \mathbf{T} \\ &= \begin{bmatrix} n_1 & n_2\end{bmatrix} \begin{bmatrix}\sigma_{11}n_1 + \sigma_{12}n_2 \\ \sigma_{12}n_1 + \sigma_{22}n_2\end{bmatrix} \\ &= \sigma_{11}n_1^2 + 2\sigma_{12}n_1n_2 + \sigma_{22}n_2^2 &(2.41)\end{aligned}$$

and the magnitude of the shear component is

$$|\tau_n| = \sqrt{\mathbf{T} \cdot \mathbf{T} - \sigma_n^2}. \qquad (2.42)$$

Let the direction of the normal make an angle θ with the positive x-axis then components of the unit normal to the surface, n_1 and n_2 become

$$n_1 = \cos\theta, \quad n_2 = \sin\theta, \qquad (2.43)$$