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A Systematic Derivation of Stochastic Taylor Methods for Stochastic Delay Differential Equations

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Abstract. This article demonstrates a systematic derivation of stochastic Taylor methods for solving stochastic delay differential equations (SDDEs) with a constant time lag, $r > 0$. The derivation of stochastic Taylor expansion for SDDEs is presented. We provide the convergence proof of one–step methods when the drift and diffusion functions are Taylor expansion. It is shown that the approximation solutions for SDDEs converge in the L^2-norm.

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1. Introduction

The systems that behave in the presence of randomness and time delay can often be modelled via stochastic delay differential equations, SDDEs. In general, there is no closed form for analytical solution of SDDEs and we usually require numerical methods to solve the problems at hand. The researches on numerical methods for SDDEs are far from complete. Among the recent works are of Baker and Buckwar [1], Hofmann and Müller [3], Hu *et. al* [4], Kloeden and Shardlow [6] and Küchler and Platen [7]. Euler scheme in the sense of Itô SDDEs was introduced in [1]. The derivation of numerical solution from Itô-Taylor expansions with time delay showed a strong order of convergence of 1.0 was studied in [7]. While, Hofmann and Müller [3] presented the modification of Milstein scheme having order of convergence 1.0. Hu *et. al* [4] introduced Itô formula with a tamed function in order to derive the same order of convergence approximation method to the solution of SDDEs. However, the convergence proof as expounded in [4] are technically complicated due to the presence of anticipative integrals in the remainder term. Latter work was done by Kloeden

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and Shardlow [6] had used an elementary method to derive the Milstein scheme for SDDEs that did not involve anticipative integrals and anticipative calculus. The convergence proof is much simpler than the convergence proof provided in [4].

In this article we are interested in investigating the mean-square convergence of Taylor approximations to strong solutions of SDDEs. We refer the reader to [1] and [8], and the references cited therein, among others. Note that, the main difficulty to the development of higher order numerical schemes for SDDEs is the derivation of stochastic Taylor expansions for SDDE with arbitrarily high orders. Taylor expansion is a fundamental and frequently used in numerical analysis for the derivation of almost all of deterministic and stochastic numerical methods. An obvious distinction between Taylor expansion of SDDEs and SDEs is that Taylor expansion in SDDEs contains the multiple stochastic integrals involving time delay that have to be approximated.

The present article develops the numerical schemes to the solution of an SDDE from stochastic Taylor expansion. The paper is organized as follows. Section 2 presented preliminary background of SDDEs. Then, we showed a systematic derivation of stochastic Taylor expansion for SDDEs, provided that SDDE is in autonomous form with no time delay in diffusion function in Section 3. Numerical schemes and the convergence proof in a general way are carried out in Section 4.

2. Preliminary

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $(\mathscr{F}_t)$ satisfying the usual conditions, i.e. the filtration $(\mathscr{F}_t)_{t\geq 0}$ is right continuous, and each $\mathscr{F}_t, t \geq 0$ contains all the sets of measure zero $(P - null\ sets)$ in $\mathscr{F}$. For the constant delay $r > 0$, let $C([-r,0], \Re)$ is the Banach space of all continuous path from $[-r,0] \to \Re$ equipped with the sup-norm $\|\Phi\|_C = \sup_{s\in[-r,0]} |\Phi(s)|$ where $|\cdot|$ denotes the Euclidean norm on $\Re$. Let $\Phi(t)$ be an $\mathscr{F}_0$-measurable $C([-r,0], \Re)$-valued random variable such that $E\|\Phi\|^2 < \infty$. Then, a scalar autonomous SDDE with constant time lag is written as

$$
\begin{aligned}
dx(t) =& f(x(t), x(t-r))dt + g(x(t))dW(t), \quad t \in [-r, T] \\
x(t) =& \Phi(t), \quad t \in [-r, 0],
\end{aligned} \tag{2.1}
$$

where $f : \Re \times \Re \to \Re,\ g : \Re \to \Re$ and $\Phi(t)$ is an initial function defined on the interval $[-r,0]$ which is independent of $W(t)$. $W(t)$ be a one-dimensional Wiener process given on filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$. The function f, g, and Φ are assumed to satisfy the following conditions:

A1: The partial derivatives of f and g exist and are uniformly bounded at least up to $m_1 + 1$ and $m_2 + 1$ order in the interval of interest i.e. there exist positive constants L_i for $i = 1, \ldots, 5$ obeying

$$
\sup_{\Re\times\Re} \left| f_{x_1}^{(m_1+1)}(x_1, x_2) \right| \leq L_1, \tag{2.2}
$$

$$
\sup_{\Re\times\Re} \left| f_{x_2}^{(m_1+1)}(x_1, x_2) \right| \leq L_2, \tag{2.3}
$$

$$
\sup_{\Re\times\Re} \left| f_{x_1^{m_1} x_2}^{(m_1+1)}(x_1, x_2) \right| \leq L_3, \tag{2.4}
$$

$$\sup_{\Re\times\Re}\left|f^{(m_1+1)}_{x_1 x_2^{m_1}}(x_1,x_2)\right| \leq L_4, \tag{2.5}$$

$$\sup_{\Re\times\Re}\left|g^{(m_2+1)}_{x_1}(x_1)\right| \leq L_5. \tag{2.6}$$

A2: The initial function $\Phi(t)$ is Hölder-continuous with exponent $\gamma \in (0,1]$; that is there exist a constant $C_1 > 0$ such that for all $-r \leq s < t \leq 0$ and $p \geq 1$

$$E(|\Phi(t)-\Phi(s)|^p) \leq C_1|t-s|^{p\gamma}, \tag{2.7}$$

A2 restricted our attention, for the sake of simplicity to work with an SDDE in the form of

$$\begin{aligned} dx(t) &= f(x(t),x(t-r))dt + g(x(t))dW(t), \quad t\in[-r,T] \\ x(t) &= \Phi(t), \quad t\in[-r,0]. \end{aligned} \tag{2.8}$$

A1 guarantees the existence and uniqueness of the solution (2.1). The following Theorem 2.1 and Theorem 2.2 taken from [8] are useful to study the convergence of numerical schemes to the solution of SDDEs.

Theorem 2.1. *Let* $(2.2)-(2.6)$ *in* A1 *hold. Then the solution of Equation* (2.1) *has the property*

$$E\left(\sup_{t\in[-r,T]}|x(t)|^2\right) \leq L_6$$

with

$$L_6 := \left(1/2 + 4E|\Phi|^2\right)e^{6LT(T+4)}, \quad L := \max(L_1,\ldots,L_5)$$

Moreover, for any $p \geq 2$, $E\|\Phi\|^p < \infty$ *and* $0 \leq s \leq t \leq T$ *with* $t-s<1$, *we have*

$$E|x(t)-x(s)|^p \leq L_7(t-s)^{\frac{p}{2}}$$

where

$$L_7 = \frac{3}{4}2^p L^{\frac{p}{2}}(1+E\|\Phi\|^p)e^{CT}\left[(2T)^{\frac{p}{2}} + (p(p-1))^{\frac{p}{2}}\right]$$

and

$$C = p\left[2\sqrt{L} + (33p-1)L\right].$$

Proof. Proof of Theorem 2.1 can be found in [8]. ∎

Theorem 2.2. *Let*

$$E\int_0^T |g(x(s))|^p ds < \infty,$$

for $p \geq 2$. *Then*

$$E\left|\int_0^T g(x(s))dW(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(x(s))|^p ds.$$

Proof. Proof of Theorem 2.2 can be found in [8]. ∎

2.1. Discrete time approximation

Let the step size $\Delta = r/M$ for some positive integer M and let $T = N\Delta$ in which T is increasing for some integer $N > M$ and $t_n = n\cdot\Delta$ for $n = 0,\dots,N$. The increments of the Wiener process $\Delta W_n = W_{n+1} - W_n$, has Gaussian distribution with $E(\Delta W_n) = 0$ and $Var(\Delta W_n) = t_{n+1} - t_n = \Delta$ with the assumption that $W(t) = 0, t < 0$.

We cite the following definitions and Theorem 2.3 from [1], which are useful to study the convergence of the numerical schemes to the solution of SDDEs.

Definition 2.1. *Let I_Ψ be a finite number of multiple stochastic integrals of the form*

$$I_{(i_1,\dots,i_j),\Delta} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \cdots \int_{t_n}^{s_1} dWi_1(s_1)\dots dWi_{(j-1)}\left(s_{j-1}\right) dWi_j(t) \tag{2.9}$$

where $i_k \in \{0,1\}$ and $dW_0(t) = dt$ for $k = 1,\dots,j$. Then, the increment function $\Psi : (0,1)\times\Re\times\Re\times\Re \to \Re$ incorporates (2.9) *and generates the approximations $\bar{x}(t_n)$ is written as*

$$\Psi = \Psi(\Delta, \bar{x}(t_n), \bar{x}(t_{n-M}), I_\Psi). \tag{2.10}$$

Definition 2.2. *The one-step iteration can be expressed in term of the increment function as*

$$\bar{x}(t_{n+1}) = \bar{x}(t_n) + \Psi(\Delta, \bar{x}(t_n), \bar{x}(t_{n-M}), I_\Psi), \tag{2.11}$$

such that the increment function, (2.10) *can be written as*

$$\Psi(\Delta, \bar{x}(t_n), \bar{x}(t_{n-M}), I_\Psi) = \bar{x}(t_{n+1}) - \bar{x}(t_n). \tag{2.12}$$

where the initial values are given by $\bar{x}(t_{n-M}) := \Phi(t_{n-M})$, for $t_{n-M} \le 0$. $x(t_{n+1})$ and $\bar{x}(t_{n+1})$ be the value of actual and approximate solutions respectively obtained after one-step iteration at the mesh point t_{n+1}.

Definition 2.3. *The local error of* (2.11) *is the sequence of random variables*

$$\delta_{n+1} = x(t_{n+1}) - \bar{x}(t_{n+1}), \tag{2.13}$$

for $n = 0,\dots,N-1$

We cite the following Definition 2.4 from [9].

Definition 2.4. *Equation* (2.11) *is consistent with order p_1 in the absolute mean and order p_2 in the mean square sense if the following estimates hold as $\Delta \to 0$ (C is constant and does not depend on Δ)*

$$\max_{0\le n\le N-1} |E(\delta_{n+1})| \le C\Delta^{p_1}, \tag{2.14}$$

and

$$\max_{0\le n\le N-1} \left(E|\delta_{n+1}|^2\right)^{\frac{1}{2}} \le C\Delta^{p_2}, \tag{2.15}$$

with

$$p_2 \ge 1/2, \tag{2.16}$$

and

$$p_1 \ge p_2 + 1/2. \tag{2.17}$$

Theorem 2.3. *Assume that the assumptions A1 to A3 are fulfilled and the increment function Ψ has the following properties*

$$|E(\Psi(\Delta,x,y,I_\Psi)-\Psi(\Delta,\bar{x},\bar{y},I_\Psi)| \leq C_2\Delta(|x-\bar{x}|+|y-\bar{y}|), \tag{2.18}$$

$$E(|(\Psi(\Delta,x,y,I_\Psi)-\Psi(\Delta,\bar{x},\bar{y},I_\Psi)|^2) \leq C_3\Delta(|x-\bar{x}|^2+|y-\bar{y}|^2), \tag{2.19}$$

and

$$E(|\Psi(\Delta,x,y,I_\Psi)|^2) \leq C_4\Delta\left(1+|x|^2+|y|^2\right). \tag{2.20}$$

where C_2, C_3 and C_4 are positive constants and $x,\bar{x},y,\bar{y} \in \Re$. Suppose the method defined by (2.12) *is consistent with p_1 in absolute mean and p_2 in mean square sense, with p_1 and p_2 satisfying* (2.16) *and* (2.17) *respectively, and the increment function Ψ in* (2.12)*satisfies the estimates* (2.18), (2.19) *and* (2.20)*. Then, the approximation* (2.12) *to SDDE* (2.1) *is convergent in L^2 as $\Delta \to 0$ with $r/\Delta \in N$ with order $p = p_2 - 1/2$.*

Proof. Proof of Theorem 2.3 please refer to [1]. ∎

3. Derivation of stochastic Taylor expansion for SDDEs

In this section, we show a systematic derivation of stochastic Taylor expansion for SDDEs with no delay argument in diffusion function. Strong Taylor approximations up to 1.5 order of convergence were constructed. The methods derived and analysed in this article has strong order of convergence and we are focusing on the pathwise convergence or convergence in the L^2-sense.

3.1. Stochastic Taylor expansion for autonomous SDDEs

Let consider SDDE (2.1). For every $t \in [-r,T]$, Equation (2.1) can be expressed in the integral form as

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(t),x(t-r))dt + \int_{t_n}^{t_{n+1}} g(x(t))dW(t). \tag{3.1}$$

For simplicity the following notation is introduced

$$\begin{aligned}
f =& f(x(t_n),x(t_n-r)) \\
\tilde{f} =& f(x(t_n-r),x(t_n-2r)) \\
\approx\!\!\!\!\!f =& f(x(t_n-2r),x(t_n-3r)) \\
g =& g(x(t_n)),\ \tilde{g} = g(x(t_n-r)) \\
\approx\!\!\!\!\!g =& g(x(t_n-2r)) \\
f_0' =& \frac{\partial f}{\partial x_{t_n}}(x(t_n),x(t_n-r)),\ g_0' = \frac{\partial g}{\partial x_{t_n}}(x(t_n)) \\
\tilde{f}_1' =& \frac{\partial f}{\partial x_{t_n-r}}(x(t_n-r),x(t_n-2r)) \\
\tilde{g}_1' =& \frac{\partial g}{\partial x_{t_n-r}}(x(t_n-r))
\end{aligned}$$

$$
\begin{aligned}
f_1^{'} =& \frac{\partial f}{\partial x_{t_n - r}} \left(x(t_n), x(t_n - r)\right) \\
\tilde{f}_2^{'} =& \frac{\partial f}{\partial x_{t_n - 2r}} \left(x(t_n - r), x(t_n - 2r)\right) \\
\approx{g}_2^{'} =& \frac{\partial g}{\partial x_{t_n - 2r}} \left(x(t_n - 2r)\right) \\
f_{0,0}^{''} =& \frac{\partial^2 f}{\partial x_{t_n}^2} \left(x(t_n), x(t_n - r)\right), \; g_{0,0}^{''} = \frac{\partial^2 g}{\partial x_{t_n}^2} \left(x(t_n)\right) \\
f_{0,1}^{''} =& \frac{\partial^2 f}{\partial x_{t_n} \partial x_{t_n - r}} \left(x(t_n), x(t_n - r)\right), \\
f_{1,1}^{''} =& \frac{\partial^2 f}{\partial x_{t_n - r}^2} \left(x(t_n), x(t_n - r)\right).
\end{aligned}
$$

The derivation of stochastic Taylor expansion for SDDE is done by replacing the integrals (3.1) with their corresponding Taylor expansions about $(x_{t_n}, x_{t_n - r})$, where $x_{t_n} = x(t_n)$ and $x_{t_n - r} = x(t_n - r)$. The methods considered here are based on [11]. By applying Taylor expansion for drift function f and diffusion function, g we therefore obtain

$$
\begin{aligned}
f(x(t), x(t-r)) =& f + (x(t) - x(t_n)) f_0^{'} + (x(t-r) - x(t_n - r)) f_1^{'} \\
& + 1/2 \left(x(t) - x(t_n)\right)^2 f_{0,0}^{''} \\
& + (x(t) - x(t_n))(x(t-r) - x(t_n - r)) f_{0,1}^{''} \\
& + 1/2 (x(t-r) - x(t_n - r)) f_{1,1}^{''} \\
& + O_f \left(|x(t) - x(t_n)|^3 \right) \\
& + O_f \left(|x(t-r) - x(t_n - r)|^3 \right)
\end{aligned} \tag{3.2}
$$

$$
g(x(t)) = g + (x(t) - x(t_n)) g_0^{'} + 1/2 (x(t) - x(t_n)) g_{0,0}^{''} + O_g \left(|x(t) - x(t_n)|^3 \right) \tag{3.3}
$$

where $O_f \left(|x(t) - x(t_n)|^3 \right), O_f \left(|x(t-r) - x(t_n - r)|^3 \right)$ and $O_g \left(|x(t) - x(t_n)|^3 \right)$ representing higher order term for drift and diffusion functions respectively. Substituting (3.2) and (3.3) into (3.1) we then obtain

$$
\begin{aligned}
x(t_{n+1}) =& x(t_n + \int_{t_n}^{t_{n+1}} \Big\{ f + (x(t) - x(t_n)) f_0^{'} + (x(t-r) - x(t_n - r)) f_1^{'} \\
& + 1/2 \left(x(t) - x(t_n)\right)^2 f_{0,0}^{''} + (x(t) - x(t_n))(x(t-r) - x(t_n - r)) f_{0,1}^{''} \\
& + 1/2 (x(t-r) - x(t_n - r)) f_{1,1}^{''} \\
& + O_f \left(|x(t) - x(t_n)|^3 \right) + O_f \left(|x(t-r) - x(t_n - r)|^3 \right) \Big\} dt \\
& + \int_{t_n}^{t_{n+1}} \Big\{ g + (x(t) - x(t_n)) g_0^{'} + \frac{1}{2} (x(t) - x(t_n)) g_{0,0}^{''} \\
& + O_g \left(|x(t) - x(t_n)|^3 \right) \Big\} dW(t),
\end{aligned} \tag{3.4}
$$

or in general Equation (3.4) can be written as

$$
\begin{aligned}
&x(t_{n+1})\\
&=x(t_n)+\int_{t_n}^{t_{n+1}}\sum_{j=0}^{m_1}\left\{\frac{1}{j!}\left[(x(t)-x(t_n))\frac{\partial}{\partial z_0}+(x(t-r)-x(t_n-r))\frac{\partial}{\partial z_1}\right]^j\right.\\
&\left.\times f(z_0,z_1)\right\}dt+\int_{t_n}^{t_{n+1}}\sum_{j=0}^{m_2}\left(\frac{g^{(j)}}{j!}(x(t)-x(t_n))^j\right)dW(t)
\end{aligned}
\tag{3.5}
$$

where $z_0 = x_{t_n}$ and $z_1 = x_{t_n-r}$. To obtain higher order numerical schemes to the solution of SDDEs we need to expand (3.4) in the following way. Rearrange (3.4) ,we then have

$$
\begin{aligned}
x(t_{n+1}) =& x(t_n)+f\int_{t_n}^{t_{n+1}}dt+g\int_{t_n}^{t_{n+1}}dW(t)\\
&+\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))f_0^{'}dt\\
&+\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))g_0^{'}dW(t)\\
&+\int_{t_n}^{t_{n+1}}(x(t-r)-x(t_n-r))f_1^{'}dt\\
&+1/2\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))g_{0,0}^{''}dW(t)\\
&+1/2\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2f_{0,0}^{''}dt\\
&+\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))(x(t-r)-x(t_n-r))f_{0,1}^{''}dt\\
&+1/2\int_{t_n}^{t_{n+1}}(x(t-r)-x(t_n-r))f_{1,1}^{''}dt\\
&+\int_{t_n}^{t_{n+1}}O_f\left(|x(t)-x(t_n)|^3\right)dt\\
&+\int_{t_n}^{t_{n+1}}O_f\left(|x(t-r)-x(t_n-r)|^3\right)dt\\
&+\int_{t_n}^{t_{n+1}}O_g\left(|x(t)-x(t_n)|^3\right)dW(t).
\end{aligned}
\tag{3.6}
$$

Based on (3.6) the following multiple integrals together with their elementary functions are identified.

(a) $f\int_{t_n}^{t_{n+1}}dt = f\cdot\Delta$
(b) $g\int_{t_n}^{t_{n+1}}dW(t) = g\cdot(W(t_{n+1})-W(t_n))$
(c) $f_0^{'}\int_{t_n}^{t}(x(t)-x(t_n))dt$

To solve (c), $x(t)-x(t_n)$ is expanded in the form of Taylor series which lead to the following representation;

$$
x(t)-x(t_n) = f\int_{t_n}^{t}dt+f_0^{'}\int_{t_n}^{t}(x(t)-x(t_n))dt
$$

$$
\begin{aligned}
&+ f_1' \int_{t_n}^{t} (x(t-r) - x(t_n - r))dt \\
&+ g \int_{t_n}^{t} dW(t) + g_0' \int_{t_n}^{t} (x(t) - x(t_n))dW(t) \\
&+ higher\ order\ terms.
\end{aligned} \tag{3.7}
$$

Then, we have

$$
\begin{aligned}
f_0' \int_{t_n}^{t_{n+1}} (x(t) - x(t_n))dt =& f_0' f \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dsdt \\
&+ f_0' f_0' \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (x(s) - x(t_n))dsdt \\
&+ f_0' f_1' \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (x(s-r) - x(t_n - r))dsdt \\
&+ f_0' g \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW(s)dt \\
&+ f_0' g_0' \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (x(s) - x(t_n))dW(s)dt \\
&+ higher\ order\ terms.
\end{aligned} \tag{3.8}
$$

Term $x(t) - x(t_n)$ in (3.8) is written as a lower order Taylor method;

$$
\begin{aligned}
t_n) =& f(x(t_n), x(t_n - r))(s - t_n) + g(x(t_n))(W(s) - W(t_n)) \\
=& f \cdot (s - t_n) + g \cdot (W(s) - W(t_n)),
\end{aligned} \tag{3.9}
$$

while $x(t-r) - x(t_n - r)$ in (3.8) is written as

$$
\begin{aligned}
x(s-r) - x(t_n - r) =& f(x(t_n - r), x(t_n - 2r))(s - t_n) \\
&+ g(x(t_n - r))(W(s-r) - W(t_n - r)) \\
=& \tilde{f} \cdot (s - t_n) + \tilde{g} \cdot (W(s-r) - W(t_n - r)).
\end{aligned} \tag{3.10}
$$

Substituting (3.9) and (3.10) into (3.8), the following is obtained;

$$
\begin{aligned}
& f_0' \int_{t_n}^{t_{n+1}} (x(t) - x(t_n))dt \\
=& f_0' f \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dsdt + f_0' g \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW(s)dt \\
&+ f_0' f_0' f \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (s - t_n)dsdt \\
&+ f_0' f_0' g \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (W(s) - W(t_n))dsdt \\
&+ f_0' f_1' \tilde{f} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (s - t_n)dsdt \\
&+ f_0' f_1' \tilde{g} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (W(s-r) - W(t_n - r))dsdt \\
&+ f_0' g_0' f \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (s - t_n)dW(s)dt
\end{aligned}
$$

$$+ f_0' g_0' g \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (W(s) - W(t_n)) dW(s) dt$$

$$+ higher\ order\ terms. \tag{3.11}$$

(d) $f_1' \int_{t_n}^{t} (x(t-r) - x(t_n - r)) dt$.

To solve (d), $(x(t-r) - x(t_n - r))$ is expanded using Taylor expansion as below

$$\begin{aligned} x(t-r) - x(t_n - r) = & \int_{t_n}^{t} \Big\{ \tilde{f} + (x(s-r) - x(t_n - r)) \tilde{f}_1' \\ & + (x(s-2r) - x(t_n - 2r)) \tilde{f}_2' \Big\} ds \\ & + \int_{t_n}^{t} \Big\{ \tilde{g} + (x(s-r) - x(t_n - r)) \tilde{g}_1' \Big\} dW(s). \\ & + higher\ order\ terms. \end{aligned}$$

Then $x(s-r) - x(t_n - r)$ is replaced by lower order method given by (3.10) which yielding to the following expression

$$\begin{aligned} x(t-r) - x(t_n - r) = & \tilde{f} \int_{t_n}^{t} ds + \tilde{f}_1' \tilde{f} \int_{t_n}^{t} (s - t_n) ds \\ & + \tilde{f}_1' \tilde{g} \int_{t_n}^{t} (W(s-r) - W(t_n - r)) ds \\ & + \tilde{f}_2' \approx{f} \int_{t_n}^{t} (s - t_n) ds \\ & + \tilde{f}_2' \approx{g} \int_{t_n}^{t} (W(s-2r) - W(t_n - 2r)) ds \\ & + \tilde{g} \int_{t_n}^{t} dW(s) + \tilde{g}_1' \tilde{f} \int_{t_n}^{t} (s - t_n) dW(s) \\ & + \tilde{g}_1' \tilde{g} \int_{t_n}^{t} (W(s-r) - W(t_n - r)) dW(s) \\ & + higher\ order\ terms. \end{aligned} \tag{3.12}$$

Therefore, we obtain

$$\begin{aligned} & f_1' \int_{t_n}^{t_{n+1}} (x(t-r) - x(t_n - r)) dt \\ & = f_1' \tilde{f} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} ds dt + f_1' \tilde{f}_1' \tilde{f} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (s - t_n) ds dt \\ & + f_1' \tilde{f}_1' \tilde{g} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (W(s-r) - W(t_n - r)) ds dt \\ & + f_1' \tilde{f}_2' \approx{f} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (s - t_n) ds dt \\ & + f_1' \tilde{f}_2' \approx{g} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (W(s-2r) - W(t_n - 2r)) ds dt \\ & + f_1' \tilde{g} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW(s) dt + f_1' \tilde{g}_1' \tilde{f} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} (s - t_n) dW(s) dt \end{aligned}$$

$$+f_1^{'}\tilde{g}_1^{'}\tilde{g}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s-r)-W(t_n-r))dW(s)dt$$
$$+\,higher\ order\ terms. \tag{3.13}$$

(e) $g_0^{'}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))dW(t)$

With the same technique as in (c), the term (e) can be expanded as follow:

$$\begin{aligned}
&g_0^{'}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))dW(t)\\
&=g_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dsdW(t)\\
&+g_0^{'}f_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-t_n)dsdW(t)\\
&+g_0^{'}f_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(t_n))dsdW(t)\\
&+g_0^{'}f_1^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-t_n)dsdW(t)\\
&+g_0^{'}f_1^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s-r)-W(t_n-r))dsdW(t)\\
&+g_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}dW(s)dW(t)\\
&+g_0^{'}g_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(s-t_n)dW(s)dW(t)\\
&+g_0^{'}g_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(t_n))dW(s)dW(t)
\end{aligned}$$
$$+\,higher\ order\ terms. \tag{3.14}$$

(f) $f_{0,0}^{''}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2dt.$

The term (f) is expanded as follow;

$$\begin{aligned}
&1/2f_{0,0}^{''}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2dt\\
&=1/2f_{0,0}^{''}\int_{t_n}^{t_{n+1}}\left(f\cdot(t-t_n)dt+g\cdot(W(t)-W(t_n))\right)^2dt\\
&=1/2f_{0,0}^{''}(f,f)\int_{t_n}^{t_{n+1}}(t-t_n)^2dt\\
&+f_{0,0}^{''}(f,g)\int_{t_n}^{t_{n+1}}(t-t_n)(W(t)-W(t_n))dt
\end{aligned}$$
$$+1/2f_{0,0}^{''}(g,g)\int_{t_n}^{t_{n+1}}(W(t)-W(t_n))^2dt. \tag{3.15}$$

(g) $1/2g_{0,0}^{''}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2dW(t).$

We employed the same procedure in (f). Then we obtain

$$1/2g_{0,0}^{''}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))^2dt$$

$$
\begin{aligned}
&= 1/2 g''_{0,0}(f,f)\int_{t_n}^{t_{n+1}} (t-t_n)^2 dt \\
&+ g''_{0,0}(f,g)\int_{t_n}^{t_{n+1}} (t-t_n)(W(t)-W(t_n))dt \\
&+ 1/2 g''_{0,0}(g,g)\int_{t_n}^{t_{n+1}} (W(t)-W(t_n))^2 dt.
\end{aligned}
\tag{3.16}
$$

(h) $f''_{0,1}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))(x(t-r)-x(t_n-r))dt.$

The term (h) is expanded in the following way.

$$
\begin{aligned}
&f''_{0,1}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))(x(t-r)-x(t_n-r))dt \\
&= f''_{0,1}\int_{t_n}^{t_{n+1}} \{(f\cdot(t-t_n)+g\cdot(W(t)-W(t_n))) \\
&\quad \left(\tilde{f}\cdot(t-t_n)+\tilde{g}\cdot(W(t-r)-W(t_n-r))\right)\} dt
\end{aligned}
\tag{3.17}
$$

Equation (3.17) can be written as

$$
\begin{aligned}
&f''_{0,1}\int_{t_n}^{t_{n+1}}(x(t)-x(t_n))(x(t-r)-x(t_n-r))dt \\
&= f''_{0,1}\left(f,\tilde{f}\right)\int_{t_n}^{t_{n+1}}(t-t_n)^2 dt \\
&+ f''_{0,1}\left(f,\tilde{g}\right)\int_{t_n}^{t_{n+1}}(t-t_n)\left(W(t-r)-W(t_n-r)\right)dt \\
&+ f''_{0,1}\left(g,\tilde{g}\right)\int_{t_n}^{t_{n+1}}\left(W(t)-W(t_n)\right)\left(W(t-r)-W(t_n-r)\right)dt \\
&+ f''_{0,1}\left(g,\tilde{f}\right)\int_{t_n}^{t_{n+1}}\left(W(t)-W(t_n)\right)(t-t_n)\,dt.
\end{aligned}
\tag{3.18}
$$

(i) $1/2\tilde{f}''_{1,1}\int_{t_n}^{t_{n+1}}(x(t-r)-x(t_n-r))^2 dt.$

The term (i) can be expanded as

$$
\begin{aligned}
&1/2\tilde{f}''_{1,1}\int_{t_n}^{t_{n+1}}(x(t-r)-x(t_n-r))^2 dt \\
&= 1/2\tilde{f}''_{1,1}\left(\tilde{f},\tilde{f}\right)\int_{t_n}^{t_{n+1}}(t-t_n)^2 dt \\
&+ \tilde{f}''_{1,1}\left(\tilde{f},\tilde{g}\right)\int_{t_n}^{t_{n+1}}(t-t_n)(W(t-r)-W(t_n-r)dt \\
&+ 1/2\tilde{f}''_{1,1}\left(\tilde{g},\tilde{g}\right)\int_{t_n}^{t_{n+1}}(W(t-r)-W(t_n-r)^2 dt.
\end{aligned}
\tag{3.19}
$$

(j) $\int_{t_n}^{t_{n+1}} O_f\left(|x(t)-x(t_n)|^2\right)dt + \int_{t_n}^{t_{n+1}} O_f\left(|x(t-r)-x(t_n-r)|^2\right)dt$
$+\int_{t_n}^{t_{n+1}} O_g\left(|x(t)-x(t_n)|^3\right)dW(t).$

Adding together (a)–(j), the stochastic Taylor expansion for SDDE is

$$\begin{aligned}
x(t_{n+1}) - x(t_n) = {} & f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t_{n+1}} dW(t) + g_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) \\
& + f_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt + f_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt \\
& + g_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW(t) + f_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\
& + f_1^{'}\tilde{f}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + f_1^{'}\tilde{g}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\
& + 1/2g_{0,0}^{''}(g,g)\int_{t_n}^{t} (W(t) - W(t_n))^2 dW(t) \\
& + g_0^{'}g_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (W(s) - W(t_n))dW(s)dW(t) \\
& + \ldots + \int_{t_n}^{t_{n+1}} O_f\left(|x(t) - x(t_n)|^2\right) dt \\
& + \int_{t_n}^{t_{n+1}} O_f\left(|x(t-r) - x(t_n - r)|^2\right) dt \\
& + \int_{t_n}^{t_{n+1}} O_g\left(|x(t) - x(t_n)|^3\right) dW(t).
\end{aligned} \tag{3.20}$$

4. Strong Taylor methods for SDDEs

Taylor expansion is a fundamental and repeatedly used method of approximation in numerical analysis for the derivation of most deterministic and stochastic numerical algorithms. The truncating Taylor expansion provides a numerical scheme up to certain order of convergence. The same procedure takes place in SDDEs. The iterated stochastic Taylor expansion for SDDE (3.20) offers higher order numerical schemes to be attained. We shall begin with the Euler-Maruyama scheme, which already presented in [1]. It represents the simplest strong Taylor approximation and had been proved in [1] and [2] that it attains the order of strong convergence 0.5 which has the following form

$$x(t_{n+1}) = x(t_n) + f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t_{n+1}} dW(t) + R_1, \tag{4.1}$$

where $\int_{t_n}^{t} dt = \Delta$ and $\int_{t_n}^{t} dW(t) = \Delta W(t)$. Then, Euler-Maruyama scheme is given as

$$x(t_{n+1}) = x(t_n) + f\cdot\Delta + g\cdot(\Delta W(t)) + R_1, \tag{4.2}$$

where R_1 is remainder term. By truncating (3.20) at fifth term, we shall obtain a Milstein scheme

$$\begin{aligned}
x(t_{n+1}) = {} & x(t_n) + f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t} dW(t) \\
& + g_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) + R_2.
\end{aligned} \tag{4.3}$$

It was shown in [5], the integral

$$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) = 1/2\left((\Delta W(t))^2 - \Delta\right)$$

for Itô SDEs and for Stratonovich SDEs it is

$$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) = 1/2\,(\Delta W(t))^2.$$

The discretization of Milstein scheme is

$$x(t_{n+1}) = x(t_n) + \hat{f}\cdot\Delta t + g\cdot(\Delta W(t)) + 1/2g_0^{'}g\cdot\left((\Delta W(t))^2-\Delta\right) + R_2, \tag{4.4}$$

in Itô form, while for Stratonovich form

$$x(t_{n+1}) = x(t_n) + f\cdot\Delta + g\cdot(\Delta W(t)) + 1/2g_0^{'}g\cdot(\Delta W(t))^2 + R_2, \tag{4.5}$$

where $\hat{f} = f + 1/2g_0^{'}g.$ The derivation of (4.4) had been studied in [4], while Milstein scheme in Stratonovich form of (4.5) had been considered in [3]. Both methods have order of convergence 1.0. For the convergence proof of (4.4), please refer to [4] and [6]. The convergence proof described in [4] however is quite complicated due to the presence of anticipative calculus in the remainder term. Kloeden and Shardlow [6], on the other hand showed a simpler way to prove the convergence of Milstein scheme without relying on the use of anticipative integrals in the remainder term. As in SDEs if the integrals up to $\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(t_n))dW(s)dW(t)$ is retained, we shall have strong Taylor method with order of convergence of 1.5 as follow;

$$\begin{aligned} x(t_{n+1}) = x(t_n) &+ f\cdot\Delta + g\cdot(\Delta W(t)) + 1/2g_0^{'}g\cdot(\Delta W(t))^2 \\ &+ f_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + g_0^{'}f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW(t) \\ &+ f_1^{'}\tilde{f}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + f_0^{'}g\cdot\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\ &+ f_1^{'}\tilde{g}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\ &+ g_0^{'}g_0^{'}g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(t_n))dW(s)dW(t) \\ &+ 1/2g_{0,0}^{''}(g,g)\int_{t_n}^{t}(W(t)-W(t_n))^2dW(t) + R_3, \end{aligned} \tag{4.6}$$

where R_3 is the remainder term. Numerical scheme given by (4.6) improved the convergence rate of approximation methods appearing in the references therein. The convergence proof in more general way is presented in the following section.

4.1. Convergence proof

The convergence proof of the numerical schemes in general way is given here. Let $\bar{x}(t_{n+1}) = y_0(t), \bar{x}(t_{n+1}-r) = y_1(t), \bar{x}(t_n) = z_0(t_n)$ and $\bar{x}(t_n-r) = z_1(t_n)$. The following theorem stated our main result.

Theorem 4.1. *If the functions f, g and Φ satisfy the assumptions $A1$ to $A3$ and the increment function Ψ in* (2.12)*satisfies the estimates* (2.18), (2.19) *and* (2.20), *then the Taylor methods approximation is consistent with order $p_1 = \min(((m+3)/2),((m+1)\gamma)+1)$ in the absolute mean and with order $p_2 = \min(((m+2)/2),((m+1)\gamma)+1/2)$ in mean-square, where γ is the exponent of Hölder-inequality of Φ in assumption* A3 *and $m = \min(m_1, m_2)$,* *which then imply the convergence of Taylor approximations in L^2 (as $\Delta \to 0$ with $r/\Delta \in N$) with order $p = p_2 - 1/2$.*

Proof. For sufficiently large $N_r \le N$, we define the step by $\Delta = r/N_r$, where $\Delta \in (0,1)$. Then the approximation solution to (2.1) is computed by $x(t) = \Phi(t)$ on $t \in [-r, 0]$. Obviously for any $t \ge 0$, there exists integer $n \ge 0$ such that $t \in [t_n, t_{n+1}]$ and in reference to (3.5) we have

$$\begin{aligned}\bar{x}(t_{n+1}) =& x(t_n) + \int_{t_n}^{t_{n+1}} \sum_{j=0}^{m_1} \Big\{ \frac{1}{j!} \Big[(y_0(t) - z_0(t_n)) \frac{\partial}{\partial z_0} + (y_1(t) - z_1(t_n)) \frac{\partial}{\partial z_1} \Big]^j \\ &\times f(z_0, z_1) \Big\} dt + \int_{t_n}^{t_{n+1}} \sum_{j=0}^{m_2} \Big\{ \frac{g_0^{(j)}}{j!} (y_0(t) - z_0(t_n))^j \Big\} dW(t)\end{aligned} \tag{4.7}$$

By Definition 2.2, the increment function is

$$\begin{aligned}&\Psi(\Delta, x(t_n), x(t_n - r), \Delta W_n) \\ &= \int_{t_n}^{t_{n+1}} \sum_{j=0}^{m_1} \Big\{ \frac{1}{j!} \Big[(y_0(t) - z_0(t_n)) \frac{\partial}{\partial z_0} + (y_1(t) - z_1(t_n)) \frac{\partial}{\partial z_1} \Big]^j f(z_0, z_1) \Big\} dt \\ &+ \int_{t_n}^{t_{n+1}} \sum_{j=0}^{m_2} \Big\{ \frac{g_0^{(j)}}{j!} (y_0(t) - z_0(t_n))^j \Big\} dW(t)\end{aligned} \tag{4.8}$$

Now we aim to prove the consistency in absolute mean with order p_1. For simplicity, the following notation is introduced

$$\begin{aligned}A(y_0(t), y_1(t), z_0(t), z_1(t)) &= \sum_{j=0}^{m_1} \Big\{ \frac{1}{j!} \Big[(y_0(t) - z_0(t_n)) \frac{\partial}{\partial z_0} \\ &\qquad + (y_1(t) - z_1(t_n)) \frac{\partial}{\partial z_1} \Big]^j f(z_0, z_1) \Big\} \\ B(y_0(t), z_0(t)) &= \sum_{j=0}^{m_2} \Big\{ \frac{g_0^{(j)}}{j!} (y_0(t) - z_0(t_n))^j \Big\}\end{aligned} \tag{4.9}$$

By Definition 2.3, the local error, δ_{n+1} is given by

$$\begin{aligned}\delta_{n+1} =& x(t_{n+1}) - x(t_n) - \int_{t_n}^{t_{n+1}} A(y_0(t), y_1(t), z_0(t), z_1(t)) dt \\ &- \int_{t_n}^{t_{n+1}} B(y_0(t), z_0(t)) dW(t)\end{aligned} \tag{4.10}$$

Substituting (3.1) into (4.10) we obtain

$$\begin{aligned}\delta_{n+1} =& \Big[\int_{t_n}^{t_{n+1}} (f(x(t), x(t-r)) - A(y_0(t), y_1(t), z_0(t), z_1(t))) dt \Big] \\ &+ \Big[\int_{t_n}^{t_{n+1}} (g(x(t)) - B(y_0(t), z_0(t))) dW(t) \Big]\end{aligned} \tag{4.11}$$

Equation (4.11) can now be written as

$$\delta_{n+1} = \int_{t_n}^{t_{n+1}} \Big\{ \frac{1}{(m_1+1)!} \Big[(y_0(t) - z_0(t)) \frac{\partial}{\partial z_0} + (y_1(t) - z_1(t)) \frac{\partial}{\partial z_1} \Big]^{(m_1+1)}$$

$$\times f(z_0,z_1)\Big\}dt + \int_{t_n}^{t_{n+1}} \Big\{ \frac{g_{z_0}^{(m_2+1)!}}{(m_2+1)!}(y_0(t)-z_0(t))^{(m_2+1)}\Big\}dW(t) \tag{4.12}$$

Taking expectation and absolute on both sides of (4.12) and by the property $|a+b| \leq |a|+|b|$, the following is attained

$$\begin{aligned}
&|E(\delta_{n+1})| \\
&= \Big|E\Big(\int_{t_n}^{t_{n+1}} \Big\{\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)}dt \\
&\times f(z_0,z_1)\Big\} + \int_{t_n}^{t_{n+1}} \Big\{\frac{g_{z_0}^{(m_2+1)!}}{(m_2+1)!}(y_0(t)-z_0(t))^{(m_2+1)}\Big\}dW(t)\Big)\Big| \\
&\leq \Big|E\int_{t_n}^{t_{n+1}} \Big\{\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)} \\
&\times f(z_0,z_1)\Big\}dt\Big| + \Big|E\int_{t_n}^{t_{n+1}} \Big\{\frac{g_{z_0}^{(m_2+1)!}}{(m_2+1)!}(y_0(t)-z_0(t))^{(m_2+1)}\Big\}dW(t)\Big|
\end{aligned} \tag{4.13}$$

In employing the Binomial expansion, the first term at the right hand side of (4.13) can easily be simplified as

$$\begin{aligned}
&\Big|E\int_{t_n}^{t_{n+1}} \Big\{\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)} f(z_0,z_1)\Big\}dt\Big| \\
&\leq \Big|\frac{1}{(m_1+1)!}\Big|E\int_{t_n}^{t_{n+1}} \Big|\Big[(y_0(t)-z_0(t))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)} f(z_0,z_1)\Big|dt \\
&\leq L_{f,1}E\int_{t_n}^{t_{n+1}} (|y_0(t)-z_0(t_n)|^{(m_1+1)}|f_{z_0}^{(m_1+1)}| \\
&+(m_1+1)|y_0(t)-z_0(t_n)|^{m_1}|y_1(t)-z_1(t_n)||f_{z_0^{m_1}z_1}^{(m_1+1)}| \\
&+\frac{(m_1+1)m_1}{2!}|y_0(t)-z_0(t_n)|^{(m_1-1)}|y_1(t)-z_1(t_n)|^2|f^2_{z_0^{(m_1-1)}z_1}| \\
&+\cdot s+|y_1(t)-z_1(t_n)|^{(m_1+1)}|f_{z_1}^{(m_1+1)}|)dt
\end{aligned} \tag{4.14}$$

where $L_{f,1} = |1/(m_1+1)!|$. It is clear that $|y_0(t)-z_0(t_n)| \approx |y_1(t)-z_1(t_n)|$ and let

$$f^* = \max\Big\{|f_{z_0}^{(m_1+1)}|, (m_1+1)|f_{z_0^{m_1}z_1}^{(m_1+1)}|, \ldots, (((m_1+1)m_1\ldots 2)/m_1!)|f_{z_0z_1^{m_1}}^{(m_1+1)}|\Big\},$$

then the following is hold

$$\begin{aligned}
&\Big|E\int_{t_n}^{t_{n+1}} \Big\{\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)} f(z_0,z_1)\Big\}dt\Big| \\
&\leq L_{f,1}\int_{t_n}^{t_{n+1}} \Big(|f^*|E|y_0(t)-z_0(t_n)|^{(m_1+1)}+|f_{z_1}|^{(m_1+1)}E|y_1(t)-z_1(t_n)|^{(m_1+1)}\Big)dt \\
&\leq L_{f,1}\int_{t_n}^{t_{n+1}} |f^*|E|y_0(t)-z_0(t_n)|^{(m_1+1)}dt \\
&+L_{f,1}\int_{t_n}^{t_{n+1}} |f_{z_1}|^{(m_1+1)}E|y_1(t)-z_1(t_n)|^{(m_1+1)}dt
\end{aligned} \tag{4.15}$$

The inequality (4.15) can be divided into two following cases

(i) $t-r \leq 0$ for $t \in [t_n, t_{n+1}]$. So we have $x(t-r) = \Phi(t-r)$ for $t-r \leq 0$,
(ii) $t_n - r > 0$.

By Theorem 2.1, the first term at the right hand side of (4.15) can be written as

$$\begin{aligned}
L_{f,1}|f^*|\int_{t_n}^{t_{n+1}} E|y_0(t)-z_0(t_n)|^{(m_1+1)}dt &\leq L_{f,1}|f^*|\int_{t_n}^{t_{n+1}} L_7(t-t_n)^{\frac{(m_1+1)}{2}}dt \\
&\leq L_{f,2}\int_{t_n}^{t_{n+1}} L_7(t-t_n)^{\frac{(m_1+1)}{2}}dt \\
&\leq L_{f,2}L_7\left[\frac{(t-t_n)^{\frac{(m_1+1)}{2}+1}}{\frac{(m_1+1)}{2}+1}\right]_{t_n}^{t_{n+1}} \\
&\leq C_{f,1}(t_{n+1}-t_n)^{\frac{(m_1+3)}{2}} \\
&\leq C_{f,1}\Delta^{\frac{(m_1+3)}{2}} \qquad (4.16)
\end{aligned}$$

where $L_{f,2} = L_{f,1}|f^*|$ and $C_{f,1} = (2L_{f,2}L_7)/(m_1+3)$. By assumption **A3** and for case (i), the second term at the right hand side of (4.15) can be solved as

$$\begin{aligned}
&L_{f,1}\,|f_{z_1}|^{(m_1+1)}\int_{t_n}^{t_{n+1}} E|y_1(t)-z_1(t_n)|^{(m_1+1)}dt \\
&= L_{f,1}\,|f_{z_1}|^{(m_1+1)}\int_{t_n}^{t_{n+1}} E\Big|\Phi_1(t)-\Phi_1(t_n)\Big|^{(m_1+1)}dt \\
&\leq L_{f,1}\,|f_{z_1}|^{(m_1+1)}C_1(t_{n+1}-t_n)^{(m_1+1)\delta+1} \\
&\leq C_{f,3}\Delta^{(m_1+1)\delta+1} \qquad (4.17)
\end{aligned}$$

where $C_{f,3} = L_{f,1}\,|f_{z_1}|^{(m_1+1)}\,(C_1/((m_1+1)\delta+1))$ for a constant $C_1 > 0$. While for the second term at the right hand side of (4.15) and case (ii) we obtain

$$\begin{aligned}
&L_{f,1}\,|f_{z_1}|^{(m_1+1)}\int_{t_n}^{t_{n+1}} E\Big|y_1(t)-z_1(t_n)\Big|^{(m_1+1)}dt \\
&= L_{f,1}\,|f_{z_1}|^{(m_1+1)}\int_{t_n}^{t_{n+1}}(t-t_n)^{\frac{(m_1+1)}{2}}dt \\
&\leq L_{f,3}\left[\frac{(t-t_n)^{\frac{(m_1+1)}{2}+1}}{\frac{(m_1+1)}{2}+1}\right]_{t_n}^{t_{n+1}} \\
&\leq C_{f,4}\Delta^{\frac{(m_1+3)}{2}} \qquad (4.18)
\end{aligned}$$

where $L_{f,3} = L_{f,1}\,|f_{z_1}|^{(m_1+1)}$ and $C_{f,4} = (2L_{f,3})/(m_1+3)$. Then, for the second term at the right hand side of (4.13) and by using Theorem 2.2, the following is attained

$$\begin{aligned}
&\left|E\int_{t_n}^{t_{n+1}}\left\{\frac{g_{z_0}^{(m_2+1)}}{(m_2+1)!}(y_0(t)-z_0(t_n))^{(m_2+1)}\right\}dW(t)\right| \\
&\leq L_{g,2}E\left|\int_{t_n}^{t_{n+1}}(y_0(t)-z_0(t_n))^{(m_2+1)}dW(t)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq L_{g,2}E\int_{t_n}^{t_{n+1}}|y_0(t)-z_0(t_n)|^{(m_2+1)}dW(t)\\
&\leq L_{g,2}E\int_{t_n}^{t_{n+1}}\Big[\Big|\int_{t_n}^{t}f(x(s),x(s-r))ds\Big|^{(m_2+1)}\\
&+\Big|\int_{t_n}^{t}g(x(s))dW(s)\Big|^{(m_2+1)}\Big]dW(t)\\
&\leq L_{g,2}E\int_{t_n}^{t_{n+1}}\Big|\int_{t_n}^{t}f(x(s),x(s-r))ds\Big|^{(m_2+1)}dW(t)\\
&+L_{g,2}\Big(\frac{m_2(m_2-1)}{2}\Big)^{\frac{m_2}{2}}\Delta^{\frac{m_2-2}{2}}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\Big|g(x(s))\Big|^{(m_2+1)}dsdW(t)\\
&\leq L_{g,2}E\int_{t_n}^{t_{n+1}}\Big|\int_{t_n}^{t}f(x(s),x(s-r))ds\Big|^{(m_2+1)}dW(t)\\
&+L_{g,3}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\Big|g(x(s))\Big|^{(m_2+1)}dsdW(t) \qquad (4.19)
\end{aligned}$$

where $L_{g,2}=(g_{z_0}^{(m_2+1)})/((m_2+1)!)$ and $L_{g,3}=L_{g,2}\,((m_2(m_2-1))/2)^{m_2/2}\,\Delta^{(m_2-2)/2}$. Based on [9], it is true

$$E\Big(\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\cdots\int_{t_n}^{s_1}dW_{i_j}(s_2)\ldots dW_{i_j}(s_{j-1})dW_{i_j}(s_t)\Big)=0$$

if at least one $i_k\neq 0$, for $k=1,\ldots,j$. Thus, it is obvious that (4.19)

$$\Big|E\int_{t_n}^{t_{n+1}}\Big\{\frac{g_{z_0}}{(m_2+1)!}(y_0(t)-z_0(t_n))^{(m_2+1)}\Big\}dW(t)\Big|=0$$

In summary we obtain

$$|E(\delta_{n+1})|\leq C_f\Delta^{\min((m_1+3)/2,(m_1+1)\delta+1)} \qquad (4.20)$$

where $C_f=C_{f,1}+C_{f,3}+C_{f,4}$. So the first condition of Theorem 4.1 follows. Now we will prove the consistency in mean–square, with order p_2.

$$\begin{aligned}
E|\delta_{n+1}|^2=E\Big|&\Big[\int_{t_n}^{t_{n+1}}(f(x(t),x(t-r))-A(y_0(t),y_1(t),z_0(t),z_1(t)))dt\Big]\\
&+\Big[\int_{t_n}^{t_{n+1}}(g(x(t))-B(y_0(t),z_0(t)))dW(t)\Big]\Big|^2,
\end{aligned}$$

By the elementary inequality $|a+b|^k\leq 2^{k-1}(|a|^k+|b|^k)$, which for $k=2$ implies

$$\begin{aligned}
&E|\delta_{n+1}|^2\\
&=E\Big|\int_{t_n}^{t_{n+1}}\Big(\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t_n))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t_n))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)}\\
&\quad\times f(z_0,z_1)\Big)dt+\int_{t_n}^{t_{n+1}}\Big(\frac{g_{z_0}}{(m_2+1)!}(y_0(t)-z_0(t_n))^{(m_2+1)}dW(t)\Big)\Big|^2\\
&\leq 2E\Big|\int_{t_n}^{t_{n+1}}\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t_n))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t_n))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)}\\
&\quad\times f(z_0,z_1)dt\Big|^2+2E\Big|\int_{t_n}^{t_{n+1}}\frac{g_{z_0}}{(m_2+1)!}(y_0(t)-z_0(t_n))^{(m_2+1)}dW(t)\Big|^2
\end{aligned}$$

$$\leq 2E\int_{t_n}^{t_{n+1}}\Big|\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t_n))\frac{\partial}{\partial z_0}+(y_1(t)-z_1(t_n))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)}$$

$$\times f(z_0,z_1)dt\Big|^2+2E\int_{t_n}^{t_{n+1}}\Big|\frac{g_{z_0}}{(m_2+1)!}(y_0(t)-z_0(t_n))^{(m_2+1)}dW(t)\Big|^2 \tag{4.21}$$

The first term at the right hand side of (4.21) is written as follows

$$\begin{aligned}
&2E\int_{t_n}^{t_{n+1}}\Big|\frac{1}{(m_1+1)!}\Big[(y_0(t)-z_0(t_n))\frac{\partial}{\partial z_0}\\
&\quad+(y_1(t)-z_1(t_n))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)}f(z_0,z_1)dt\Big|^2\\
&=2\Big(\frac{1}{(m_1+1)!}\Big)^2E\int_{t_n}^{t_{n+1}}\Big|\Big[(y_0(t)-z_0(t_n))\frac{\partial}{\partial z_0}\\
&\quad+(y_1(t)-z_1(t_n))\frac{\partial}{\partial z_1}\Big]^{(m_1+1)}f(z_0,z_1)\Big|^2dt\\
&\leq K_{f,1}E\int_{t_n}^{t_{n+1}}\Big(|y_0(t)-z_0(t_n)|^{2(m_1+1)}(f^*)^{2(m_1+1)}\\
&\quad+|y_1(t)-z_1(t_n)|^{2(m_1+1)}f_{z_1}^{2(m_1+1)}\Big)dt\\
&\leq K_{f,2}\int_{t_n}^{t_{n+1}}E\Big|y_0(t)-z_0(t_n)\Big|^{2(m_1+1)}dt\\
&\quad+K_{f,3}\int_{t_n}^{t_{n+1}}E\Big|y_1(t)-z_1(t_n)\Big|^{2(m_1+1)}dt
\end{aligned} \tag{4.22}$$

where $K_{f,1}=2\,(1/(m_1+1)!)^2$, $K_{f,2}=K_{f,1}(f^*)^{2(m_1+1)}$ and $K_{f,3}=K_{f,1}f_{z_1}^{2(m_1+1)}$. For case (i), i.e. $t_n-r<0$, by Theorem 2.1 and assumption **A3**, we have

$$\begin{aligned}
&K_{f,2}\int_{t_n}^{t_{n+1}}E\Big|y_0(t)-z_0(t_n)\Big|^{2(m_1+1)}dt+K_{f,3}\int_{t_n}^{t_{n+1}}E\Big|y_1(t)-z_1(t_n)\Big|^{2(m_1+1)}dt\\
&\leq K_{f,2}\int_{t_n}^{t_{n+1}}L_2\Big|t-t_n\Big|^{\frac{2(m_1+1)}{2}}dt+K_{f,3}\int_{t_n}^{t_{n+1}}E\Big|\Phi(t)-\Phi(t_n)\Big|^{2(m_1+1)}dt\\
&\leq K_{f,4}\int_{t_n}^{t_{n+1}}\Big|t-t_n\Big|^{\frac{2(m_1+1)}{2}}dt+K_{f,3}C_2\int_{t_n}^{t_{n+1}}\Big|t-t_n\Big|^{2(m_1+1)\delta}dt\\
&\leq K_{f,4}\frac{\Delta^{m_1+2}}{m_1+2}+K_{f,5}\frac{\Delta^{2(m_1+1)\delta+1}}{2(m_1+1)\delta+1}\\
&\leq C_{f,5}\Delta^{m_1+2}+C_{f,6}\Delta^{2(m_1+1)\delta+1}
\end{aligned} \tag{4.23}$$

where $K_{f,4}=K_{f,2}L_2$, $K_{f,5}=K_{f,3}C_2$ for a positive constant C_2, $C_{f,5}=K_{f,4}/(m_1+2)$ and $C_{f,6}=K_{f,5}/(2(m_1+1)\delta+1)$. For case (ii), i.e. $t_n-r>0$, by Theorem 2.1 we obtain

$$\begin{aligned}
K_{f,3}\int_{t_n}^{t_{n+1}}E\Big|y_1(t)-z_1(t_n)\Big|^{2(m_1+1)}dt&\leq K_{f,3}\int_{t_n}^{t_{n+1}}L_7(t-t_n)^{\frac{2(m_1+1)}{2}}dt\\
&\leq C_{f,7}\int_{t_n}^{t_{n+1}}(t-t_n)^{\frac{2(m_1+1)}{2}}dt\\
&\leq C_{f,7}\Delta^{m_1+2}
\end{aligned} \tag{4.24}$$

where $C_{f,7} = K_{f,3}L_7$ for a constant $L_7 > 0$. For second term at the right hand side of (4.21) and by Theorem 2.1 we have

$$\begin{aligned}
&2E\int_{t_n}^{t_{n+1}}\left|\frac{g_{z_0}^{m_2+1}}{(m_2+1)!}(y_0(t)-z_0(t_n))^{m_2+1}\right|^2 dt\\
&\le 2\left|\frac{g_{z_0}^{m_2+1}}{(m_2+1)!}\right|^2 E\int_{t_n}^{t_{n+1}}|y_0(t)-z_0(t_n)|^{2(m_2+1)}dt\\
&\le K_{g,1}\int_{t_n}^{t_{n+1}}L_7|t-t_n|^{\frac{2(m_2+1)}{2}}dt\\
&\le C_{f,7}\Delta^{m_2+2} \qquad (4.25)
\end{aligned}$$

where $K_{g,1} = 2\left|g_{z_0}^{m_2+1}/(m_2+1)!\right|^2$ and $C_{f,7} = K_{g,1}L_7$. This implies

$$E\left|\delta_{n+1}\right|^2 \le C_g\Delta^{\min((m+2),2((m+1)\delta)+1)},$$

where $C_g = C_{f,5}+C_{f,6}+C_{f,7}$ and $m = \min(m_1,m_2)$. Thus, we obtain

$$\left(E\left|\delta_{n+1}\right|^2\right)^{1/2} \le C_g\Delta^{\min(\frac{m}{2}+1,(m+1)\delta+1/2)}. \qquad (4.26)$$

which completes the proof. ∎

Specifically, the following corollary are obtained.

Corollary 4.1. *If $m = 0, \gamma = 1/2$ then the method is said to have a consistency in absolute mean with $p_1 = 3/2$ and in mean-square with $p_2 = 1$, which from Theorem 4.1 implies the convergence rate of $p = 1/2$.*

Corollary 4.2. *If $m = 1, \gamma = 1/2$ then the method is said to have a consistency in absolute mean with $p_1 = 2$ and in mean-square with $p_2 = 3/2$, which from Theorem 4.1 implies the convergence rate of $p = 1$.*

Corollary 4.3. *If $m = 2, \gamma = 1/2$ then the method is said to have a consistency in absolute mean with $p_1 = 5/2$ and in mean-square with $p_2 = 2$, which from Theorem 4.1 implies the convergence rate of $p = 3/2$.*

It can be seen by Corollary 4.1, 4.2 and 4.3 represent the specific case of Theorem 4.1 and presented the convergence rate for Euler-Maruyama (4.2), Milstein scheme (4.4) and Taylor method order of 1.5 (4.6) respectively.

5. Numerical example

We illustrate a numerical example that shall indicate the usefulness of the order 1.5 strong Taylor methods in comparison to the Euler-Maruyama and Milstein schemes. The following linear SDDE was taken from [7] is used as a test equation for the numerical methods developed here. Let us consider

$$\begin{aligned}
dX(t) &= \{aX(t)+bX(t-1)\}dt + cX(t)dW(t), \quad t\in[-1,T]\\
\Phi(t) &= 1+t, \quad t\in[-1,0]. \qquad (5.1)
\end{aligned}$$

The exact solution of (5.1) is

$$X(t) = \Phi_{t,k-1}\left(X(k-1) + \int_{k-1}^{t} bX(s-1)\,\Phi_{s,k-1}^{-1}\,ds\right)$$

where $\Phi_{t,k-1}^{-1}$ is an inverse function of $\Phi_{t,k-1}$, $X(s-1) = X(0)$ for $s \in [0,1]$ and

$$\Phi_{t,t_0} = \exp\left((a - c/2)(t - t_0) + c\left(W(t) - W(t_0)\right)\right)$$

To construct a numerical example we have used the set of coefficients as

$$a = -2,\ b = 0.1,\ c = 0.5,\ T = 3.0,\ X(0) = 1.0 \text{ and } \Delta = 0.01.$$

We numerically simulate 100 sample paths of the strong solution SDDE (5.1) via three different numerical schemes namely Euler-Maruyama, Milstein scheme and Taylor method order 1.5. The average of the sample paths of the exact and numerical solutions were computed. Numerical coding was performed in C language and the results for Euler-Maruyama, Milstein scheme and Taylor method order 1.5 are illustrated in Figure 1, Figure 2 and Figure 3 respectively.

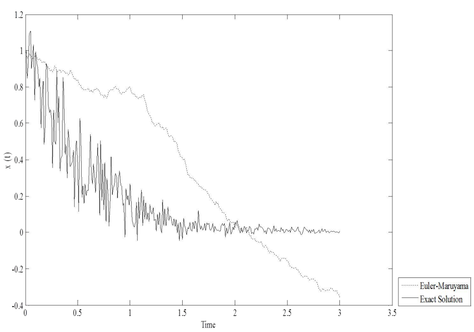

Figure 1. Strong approximation of SDDEs via Euler-Maruyama

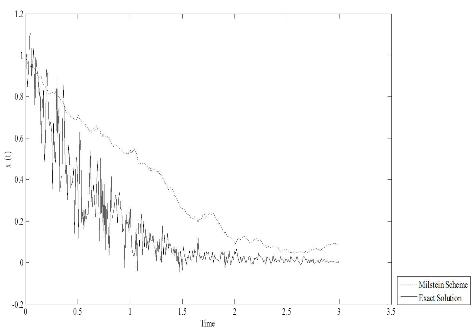

Figure 2. Strong approximation of SDDEs via Milstein scheme

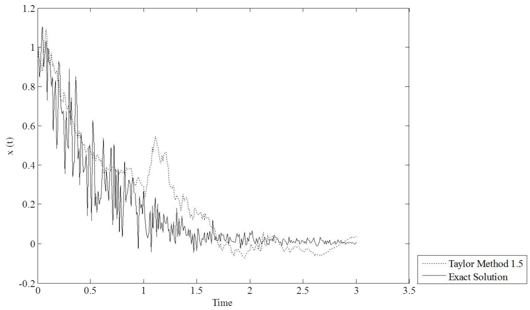

Figure 3. Strong approximation of SDDEs via Taylor method order 1.5

A glance at Figure 1, Figure 2 and Figure 3 reveals that the result illustrated by Figure 3 shows better performance than the results display in Figure 1 and Figure 2. Next, mean-square error between simulated solution and exact solution are calculated. The results were shown in Table 1.

Table 1. Mean-Square Error of Numerical Solution and Exact Solution.

Numerical Scheme	Euler-Maruyama	Milstein	1.5 Stochastic Taylor Method
MSE	0.1341	0.0581	0.0218

This example visually demonstrates that higher-order method can significantly improve the accuracy of the solution.

6. Conclusion

This article provides the derivation of numerical schemes of higher order to the solution of SDDEs from stochastic Taylor expansion. Based on stochastic Taylor expansion, we can see that the complexity arises as one need a method higher than 1.5, as multiple stochastic integral is nontrivial task to determine as well as more partial derivatives of high order are required. However, it is easier to derive the derivative-free method such as stochastic Runge-Kutta for SDDEs where our future work will be based on.

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