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Stochastic Taylor Methods for Stochastic Delay Differential Equations

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Abstract This paper demonstrates a systematic derivation of high order numerical methods from stochastic Taylor expansion for solving stochastic delay differential equations (SDDEs) with a constant time lag, $r > 0$. The stochastic Taylor expansion of SDDEs is truncated at certain terms to achieve the order of convergence of numerical methods for SDDEs. Three different numerical schemes of Euler method, Milstein scheme and stochastic Taylor method of order 1.5 have been derived. The performance of Euler method, Milstein scheme and stochastic Taylor method of order 1.5 are investigated in a simulation study.

Keywords Numerical Solution; Stochastic Delay Differential Equations; Stochastic Taylor Expansion.

2010 Mathematics Subject Classification 62L20, 65C20

1 Introduction

The system that behave in the presence of stochasticity and time delay can be modeled via stochastic delay differential equations (SDDEs). The analytical solution of SDDEs is hard to be found and we usually require numerical methods to solve the problems at hand. However, the researches on numerical methods for SDDEs is far from complete. Among the recent works are of Baker [1], Baker and Buckwar [2], Buckwar [3], Küchler and Platen [4], Hu *et al*. [5], Hofmann and Muller [6] and Kloeden and Shardlow [7]. Euler scheme for SDDEs was introduced by Baker [1] and Baker and Buckwar [2]. The derivation of numerical solutions for SDDEs from stochastic Taylor expansions with time delay showed a strong order of convergence of 1.0 was studied by Küchler and Platen [4]. Hu *et al*. [5] introduced Itô formula for tame function in order to derive the same order of convergence but with a different scheme. They provide the convergence proof of Milstein scheme to the solution of SDDEs with the presence of anticipative integrals in the remainder term. Moreover, Hofmann and Muller [6] presented an approximation of double stochastic integral involving time delay and introduce the modification of Milstein scheme. The exploration of numerical approximation to the strong solution of SDDEs is just relied on the truncating of stochastic Taylor expansions, up to 1.0 order of accuracy. Accordingly, the Euler--Maruyama and Milstein schemes with 0.5 and 1.0 strong order of convergence respectively had been proposed to apply them in practice or to study their properties. To achieve a high strong order methods it is necessary to derive a stochastic Taylor expansion of high order. This paper is prepared to demonstrate a systematic derivation of high order numerical methods from stochastic Taylor expansion for solving SDDEs with a constant time lag, $r > 0$.

2 Preliminary Background

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathcal{F}_t$ satisfying the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t=0}$ is right continuous, and each $\mathcal{F}_t$ contains all the sets of measure zero (P -null sets) in $\mathcal{F}$. For the constant delay $r > 0$, let $C([-r,0],\Re)$ is the Banach space of all continuous path from $[-r,0] \to \Re$ equipped with the sup-norm $\|\phi\|_C = sup|\phi(s)|$ where $|\cdot|$ denotes the Euclidean norm on $\Re$. Let $\mathcal{F}(t)$ is an $\mathcal{F}_0$-measurable $C([-r,0],\Re)$-valued random variable such that $\mathrm{E}\|\phi\|^2 < \infty$. Then, a scalar autonomous SDDE with constant time lag is written as

$$dx(t) = f(x(t), x(t-r))dt + g(x(t), x(t-r))dW(t), \quad t \le [-r,T]$$
$$x(t) = \Phi(t), \quad t \le [-r,0] \tag{1}$$

where $f: \Re \times \Re \to \Re, g: \Re \times \Re \to \Re$ and $\Phi(t)$ is an initial function defined on the interval $[-r,0]$ which is independent of $W(t)$. The process $W(t)$ be a one-dimensional Wiener process given on filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. The function f and g are assumed to satisfy the following conditions:

A1 The functions f and g are assumed to satisfy uniform Lipschitz and linear growth conditions i.e. there exist positive constants L_i for $i = 1,\ldots,4$ and K_j for $j = 1,2$ such that for ϕ_1, ϕ_2, φ_1, $\varphi_2 \in \Re$

$$|f(\phi_1,\phi_2) - f(\varphi_1,\varphi_2)| \le L_1|\phi_1 - \phi_2| + L_2|\varphi_1 - \varphi_2| \tag{2}$$

$$|g(\phi_1,\phi_2) - g(\varphi_1,\varphi_2)| \le L_3|\phi_1 - \phi_2| + L_4|\varphi_1 - \varphi_2| \tag{3}$$

and

$$|f(\phi_1,\phi_2)|^2 \le K_1(1 + |\phi_1|^2 + |\phi_2|^2) \tag{4}$$

$$|g(\phi_1,\phi_2)|^2 \le K_2(1 + |\phi_1|^2 + |\phi_2|^2) \tag{5}$$

A2 There is no time delay in diffusion function g, i.e. the diffusion is in the form of $g(x(t))$.

A3 The initial function $\Phi(t)$ is Hölder-continuous with exponent $\eta \in (0,1]$, i.e. there exist a constant $C_1 > 0$ such that for all $-r \le s < t = 0$ and $p = 1$

$$\mathrm{E}(\|\Phi(t) - \Phi(s)\|^p) = C_1|t-s|^{p\eta} \tag{6}$$

Assumption **A1** guarantees the existence and uniqueness of the solution to equation (1). Moreover, Assumption **A2** allows us for the sake of simplicity to work with SDDE in the form of

$$dx(t) = f(x(t), x(t-r))dt + g(x(t))dW(t), \qquad t \le [-r,T]$$
$$x(t) = \Phi(t), \qquad t \le [-r,0]. \tag{7}$$

3 Stochastic Taylor Expansion of SDDEs

In this section, we show a systematic derivation of stochastic Taylor expansion for SDDEs, provided that SDDE is autonomous with no delay in diffusion function. Let consider SDDE in autonomous form of (7). For every $t \in [-r,T]$, equation (7) can be expressed in the integral form as

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(t), x(t-r))dt + \int_{t_n}^{t_{n+1}} g(x(t))dW(t). \quad (8)$$

The derivation of stochastic Taylor expansion for SDDE is done by replacing the integrals in equation (8) with their corresponding Taylor expansions about $(x(t_n), x(t_{n-r}))$ and insert recursively lower order Taylor method into the series. For simplicity the following notation is introduced

$$f = f(x(t_n), x(t_n - r)),$$

$$\tilde{f} = f(x(t_n - r), x(t_n - 2r)),$$

$$g = g(x(t_n)),$$

$$\tilde{g} = g(x(t_n - r)),$$

$$f_0' = \frac{\partial f}{\partial x_{t_n}}(x(t_n), x(t_n - r)),$$

$$f_1' = \frac{\partial f}{\partial x_{t_n - r}}(x(t_n), x(t_n - r)),$$

$$g_0' = \frac{\partial g}{\partial x_{t_n}}(x(t_n)),$$

$$\tilde{g}_1' = \frac{\partial g}{\partial x_{t_n - r}}(x(t_n - r)),$$

$$g_{00}'' = \frac{\partial^2 g}{\partial x_{t_n}^2}(x(t_n)).$$

By applying Taylor expansion for drift function, f and diffusion function, g we therefore obtain

$$\begin{aligned} f(x(t), x(t-r)) = f &+ (x(t) - x(t_n))f_0' + (x(t-r) - x(t_n - r))f_1' \\ &+ \frac{1}{2}(x(t) - x(t_n))^2 f_{00}'' \\ &+ (x(t) - x(t_n))(x(t-r) - x(t_n - r))f_{01}'' \\ &+ \frac{1}{2}(x(t-r) - x(t_n - r))^2 \tilde{f}_{11}'' \\ &+ O_f(|x(t) - x(t_n)|^3) \\ &+ O_f(|x(t-r) - x(t_n - r)|^3) \end{aligned} \quad (9)$$

$$\begin{aligned} g(x(t)) = g &+ (x(t) - x(t_n))g_0' + \frac{1}{2}(x(t) - x(t_n))^2 g_{00}'' \\ &+ O_g(|x(t) - x(t_n)|^3) \end{aligned} \quad (10)$$

where $O_f(|x(t)-x(t_n)|^3)$, $O_f(|x(t-r)-x(t_n-r)|^3)$ and $O_g(|x(t)-x(t_n)|^3)$ representing higher order term for drift and diffusion functions respectively. Substituting equation (9) and equation (10) into equation (8) we then obtain

$$\begin{aligned}
x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} &\{ f + (x(t)-x(t_n))f_0' + (x(t-r)-x(t_n-r))f_1' \\
&+\frac{1}{2}(x(t)-x(t_n))^2 f_{00}'' \\
&+(x(t)-x(t_n))(x(t-r)-x(t_n-r))f_{01}'' \\
&+\frac{1}{2}(x(t-r)-x(t_n-r))^2 \tilde{f}_{11}'' \\
&+O_f(|x(t)-x(t_n)|^3) + O_f(|x(t-r)-x(t_n-r)|^3)\} dt \\
&+\int_{t_n}^{t_{n+1}} \{ g + (x(t)-x(t_n))g_0' + \frac{1}{2}(x(t)-x(t_n))^2 g_{00}'' \\
&+O_g(|x(t)-x(t_n)|^3)\} dW(t)
\end{aligned} \tag{11}$$

Rearrange equation (11), we then have

$$\begin{aligned}
x(t_{n+1}) = x(t_n) &+ f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t_{n+1}} dW(t) \\
&+\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))f_0' dt \\
&+\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))g_0' dW(t) \\
&+\int_{t_n}^{t_{n+1}} (x(t-r)-x(t_n-r))f_1' dt \\
&+\frac{1}{2}\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^2 g_{00}'' dW(t) \\
&+\frac{1}{2}\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))^2 f_{00}'' dt \\
&+\int_{t_n}^{t_{n+1}} (x(t)-x(t_n))(x(t-r)-x(t_n-r))f_{01}'' dt \\
&+\frac{1}{2}\int_{t_n}^{t_{n+1}} (x(t-r)-x(t_n-r))^2 \tilde{f}_{11}'' dt \\
&+\int_{t_n}^{t_{n+1}} O_f(|x(t)-x(t_n)|^3) dt \\
&+\int_{t_n}^{t_{n+1}} O_f(|x(t-r)-x(t_n-r)|^3) dt \\
&+\int_{t_n}^{t_{n+1}} O_g(|x(t)-x(t_n)|^3) dW(t)
\end{aligned} \tag{12}$$

Based on equation (12) the following integrals are identified.

1. The second term in equation (12) is computed as

$$f\int_{t_n}^{t_{n+1}} dt = f \cdot \Delta \tag{13}$$

2. The third term of equation (12) is computed as

$$g\int_{t_n}^{t_{n+1}} dW(t) = g\cdot(W(t_{n+1}) - W(t_n)) \tag{14}$$

3. The fourth term on the right hand side of equation (12)

$$f_0'\int_{t_n}^{t_{n+1}} (x(t) - x(t_n))dt \tag{15}$$

can be expanded by Taylor series as follows

$$\begin{aligned} f_0'\int_{t_n}^{t} (x(t) - x(t_n))dt = {} & f_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt \\ & + f_0' f_0'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (x(s) - x(t_n))dsdt \\ & + f_0' f_1'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (x(s-r) - x(t_n - r))dsdt \\ & + f_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\ & + f_0' g_0'\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (x(s) - x(t_n))dW(s)dt \\ & + \textit{higher order term} \end{aligned} \tag{16}$$

The term $x(t) - x(t_n)$ in equation (16) is written as lower order stochastic Taylor method

$$\begin{aligned} x(t) - x(t_n) &= f(x(t_n), x(t_n - r))(s - t_n) + g(x(t_n))(W(s) - W(t_n)) \\ &= f\cdot(s - t_n) + g\cdot(W(s) - W(t_n)) \end{aligned} \tag{17}$$

In a similar manner with equation (17), the term $x(t-r) - x(t_n - r)$ is written as

$$\begin{aligned} x(t-r) - x(t_n - r) &= f(x(t_n - r), x(t_n - 2r))(s - t_n) \\ &\quad + g(x(t_n - r))(W(s-r) - W(t_n - r)) \\ &= \tilde{f}\cdot(s - t_n) + \tilde{g}\cdot(W(s-r) - W(t_n - r)) \end{aligned} \tag{18}$$

Substituting equation (17) and equation (18) into equation (16), the following is obtained

$$\begin{aligned} & f_0'\int_{t_n}^{t} (x(t) - x(t_n))dt \\ & = f_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + f_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\ & + f_0' f_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (s - t_n)dsdt + f_0' f_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (W(s) - W(t_n))dsdt \\ & + f_0' f_1' \tilde{f}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (s - t_n)dsdt + f_0' g_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (s - t_n)dW(s)dt \\ & + f_0' g_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (W(s) - W(t)_n)dW(s)dt \\ & + \textit{higher order term} \end{aligned} \tag{19}$$

By expanding the other terms on the right hand side of equation (12) in a similar manner with the fourth term and adding together all the terms, the stochastic Taylor expansion for SDDE is obtained as

$$
\begin{aligned}
&x(t_{n+1})\\
&= x(t_n) + f\cdot\Delta + g\cdot(W(t_{n+1}) - W(t_n)) + g_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t)\\
&+(f_0' f + f_1'\tilde{f})\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + g_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW(t)\\
&+(f_0' g + f_1'\tilde{g})\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt + \frac{1}{2} g_{00}''(g,g)\int_{t_n}^{t_{n+1}} (W(t) - W(t_n))^2 dW(t)\\
&+g_0' g_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (W(s) - W(t)_n)dW(s)dW(t)\\
&+\cdots+\int_{t_n}^{t_{n+1}} O_f(|\, x(t) - x(t_n)\,|)^3 dt + \int_{t_n}^{t_{n+1}} O_f(|\, x(t-r) - x(t_n - r)\,|)^3 dt\\
&+\int_{t_n}^{t_{n+1}} O_g(|\, x(t) - x(t_n)\,|)^3 dW(t)
\end{aligned}
\tag{20}
$$

or in general equation (20) can be written as

$$
\begin{aligned}
x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} \sum_{j=0}^{m_1} \Big\{ &\frac{1}{j!}\Big[(y_0(t) - z_0(t_n))\frac{\partial}{\partial z_0} + (y_1(t) - z_1(t_n))\frac{\partial}{\partial z_1} \Big]^j \\
\times\; f(z_0, z_1)\Big\} dt &+ \int_{t_n}^{t_{n+1}} \sum_{j=0}^{m_2} \Big\{ \frac{g_0^{(j)}}{j!}(y_0(t) - z_0(t_n))^j \Big\} dW(t)
\end{aligned}
\tag{21}
$$

where $y_0(t) = x(t)$, $y_1(t) = x(t-r)$, $z_0 = x(t_n)$ and $z_1 = x(t_n - r)$.

4 Stochastic Taylor Methods

Taylor expansion is a fundamental and repeatedly used method of approximation in numerical analysis of most deterministic and stochastic numerical algorithms. The same procedure takes place in SDDEs. By truncating the stochastic Taylor expansion in equation (20), it enables us to construct a numerical method of high order. Let's start with the Euler--Maruyama scheme, that was initiated by Baker [2]. It represents the simplest strong Taylor approximation and had been proven in [2] and [3] that it attains a strong order of convergence of 0.5. The Euler--Maruyama for SDDEs is represented by

$$
x(t_{n+1}) = x(t_n) + f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t_{n+1}} dW(t) + \Re_1\,,
\tag{22}
$$

where $\Re_1$ is the remainder term. The integrals $\int_{t_n}^{t_{n+1}} dt = \Delta$ and $\int_{t_n}^{t_{n+1}} dW(t) = \Delta W(t)$. Then, Euler-Maruyama scheme is written as

$$
x(t_{n+1}) = x(t_n) + f\cdot\Delta + g\cdot\Delta W(t) + \Re_1\,.
\tag{23}
$$

By truncating equation (20) at the fifth-term, we shall obtain a Milstein scheme

$$
\begin{aligned}
x(t_{n+1}) &= x(t_n) + f\int_{t_n}^{t_{n+1}} dt + g\int_{t_n}^{t_{n+1}} dW(t)\\
&\quad + g_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) + \Re_2
\end{aligned}
\tag{24}
$$

It was shown in [9], the Stratonovich integral $\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t)$ is

$$\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) = \frac{1}{2}(\Delta W(t))^2 \tag{25}$$

The discretization of Milstein scheme then can be written as

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + f\cdot\Delta + g\cdot(\Delta W(t)) \\ &\quad + \frac{1}{2}g_0' g(\Delta W(t))^2 + \Re_2 \end{aligned} \tag{26}$$

where $\Re_2$ is the remainder term of Milstein scheme. As in SDEs, if the integrals up to $\int_{t_n}^{t_{n+1}}\int_{t_n}^{t}(W(s)-W(t_n))dW(s)dW(t)$ are retained, a strong Taylor method with the order of convergence of 1.5 is obtained as follows

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + f\cdot\Delta + g\cdot(W(t_{n+1})-W(t_n)) \\ &\quad + f_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + f_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\ &\quad + f_1'\tilde{f}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdt + f_1'\tilde{g}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dt \\ &\quad + g_0' f\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dsdW(t) + g_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} dW(s)dW(t) \\ &\quad + g_0' g_0' g\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} (W(s)-W(t)_n)dW(s)dW(t) \\ &\quad + \frac{1}{2}g_{00}''(g,g)\int_{t_n}^{t}(W(s)-W(t)_n)^2 dW(t) + \Re_3 \end{aligned} \tag{27}$$

where $\Re_3$ is the remainder term. With little amount of numerical methods for SDDEs, stochastic Taylor method of equation (27) improves the convergence rate of approximation methods appearing in the references therein. More accurate strong Taylor schemes can be obtained by the inclusion of further multiple stochastic integrals from stochastic Taylor expansion to the scheme, [8]. The stochastic integrals provide the information about the sample path of the Wiener process. This information is useful for determining the rate of convergence of the underlying methods.

4.1 Order of Convergence

This subsection is devoted to the order of convergence of the numerical schemes presented in Section 4.

Definition [2] **Multiple Stochastic Integrals.**
Let J_Ψ be a finite number of multiple stochastic integrals of the form

$$J_{(i_1,\cdots,i_j),\Delta} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t}\cdots\int_{t_n}^{s_1} dW_{i_1}(s_1)\ldots dW_{i_{j-1}}(s_{j-1})dW_{i_j}(t) \tag{28}$$

where $i_k \in (0,1)$ and $dW_0(t) = dt$ for $k = 1,\ldots,j$. Then, the increment function $\Psi : (0,1)\times\Re\times\Re\times\Re \to \Re$ incorporated equation (28) and generate the approximation $\overline{x}(tn) = \overline{x}(t_n)$ is written as

$$\Psi = \Psi(\Delta, \bar{x}(t_n), \bar{x}(t_{n-r}), J_\Psi) \tag{29}$$

The following Lemma 4.1 was cited from [9], guided us to compute the order of smallness of the integral in equation (28).

Lemma 4.1 *[9]*
We have

$$(E(J_{i_1,\cdots,i_j})^2)^{\frac{1}{2}} = O\left(\Delta \sum_{k=1}^{j} \frac{2-i_k}{2}\right), \tag{30}$$

where

$$ik = \begin{cases} 0, & if \quad ik = 0, \\ 1, & if \quad ik \neq 0. \end{cases} \tag{31}$$

From Lemma 4.1, it is clear that the following rules hold; dt contributes one to the order of smallness and $dW(t)$ contributes half. We shall now investigate the convergence rate of Euler-Maruyama, Milstein scheme and 1.5 strong Taylor method. The convergence rate of Euler-Maruyama, Milstein scheme and 1.5 strong Taylor method is identified by observing the presence of stochastic integrals of multiplicity one, two and three respectively.

Euler-Maruyama is the simplest method with 0.5 order of convergence. It is obtained by truncating the stochastic Taylor series in equation (20) at the third term. The convergence rate of Euler-Maruyama can be specified by the presence of $\int_{t_n}^{t_{n+1}} dW(t) = W(t_{n+1}) - W(t_n) = \Delta W$ in the scheme. The increment of the Wiener process, ΔW is normally distributed with mean, 0 and variance Δ. Hence, the mean--square of stochastic integral of multiplicity one is

$$(E(\Delta W)^2)^{\frac{1}{2}} = \Delta^{\frac{1}{2}} \tag{32}$$

Thus, the Euler-Maruyama converge to the true solution with the rate of $\frac{1}{2}$. We shall now examine the convergence rate of a Milstein scheme (26). The Milstein scheme is obtained by the inclusion of the term ninth into Euler-Maruyama scheme. Stochastic integral of multiplicity two, $\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dw(s) dW(t) = \frac{1}{2}(\Delta W)^2$ contribute to the convergence rate of this method. The mean-square of stochastic integral of multiplicity two is

$$\frac{1}{2}(E(\Delta W)^4)^{\frac{1}{2}} = \frac{1}{2}(\Delta^2)^{\frac{1}{2}} = \frac{1}{2}\Delta \tag{33}$$

which turns out to be the order 1.0. The convergence rate of 1.5 Taylor method is determined by the triple stochastic integral of

$$\iiint dW(u) dW(s) dW(t) = \frac{1}{6}(\Delta W)^3$$

The mean-square of stochastic integral of multiplicity three is

$$\frac{1}{6}(E(\Delta W)^6)^{\frac{1}{2}} = \frac{1}{6}(\Delta^3)^{\frac{1}{2}} = \frac{1}{6}\Delta^{\frac{3}{2}} \tag{34}$$

Subsequently, a numerical example that indicates the usefulness of the order 1.5 strong Taylor method in comparison to the Euler-Maruyama and Milstein scheme will be presented in the next section.

5 Numerical Example

The following linear SDDE taken from [3] is used as a test equation. Let us consider

$$\begin{aligned} dx(t) &= [ax(t) + bx(t-1)]dt + cx(t)dW(t), \quad t \in [-1, T] \\ \Phi(t) &= 1 + t, \quad t \in [-1, 0]. \end{aligned} \tag{35}$$

The exact solution of equation (35) is

$$x(t) = \Phi_{t,k-1}\left(x(k-1) + \int_{k-1}^{t} bx(s-1)\Phi_{s,k-1}^{-1} ds \right) \tag{36}$$

where $\Phi_{s,k-1}^{-1}$ is an inverse function of $\Phi_{t,k-1}$, $x(s-1) = x(0)$ for $s \in [0,1]$ and

$$\Phi_{t,t_0} = \exp\left(\left(a - \frac{1}{2}c \right)(t - t_0) + c(W(t) - W(t_0)) \right) \tag{37}$$

To construct a numerical example we have used the following set of coefficients

$$a = -2, \quad b = 0.1, \quad c = 0.5, \quad T = 2.0, \quad x(0) = 1.0, \quad \Delta = 0.01 \tag{38}$$

We numerically simulate 100 sample paths for each numerical method. Then, the average of the sample paths for each method is computed and the average values are compared with the exact solution. The results obtained via those three numerical schemes and the exact solution of equation (36) are illustrated in Figure 1, Figure 2 and Figure 3 respectively.

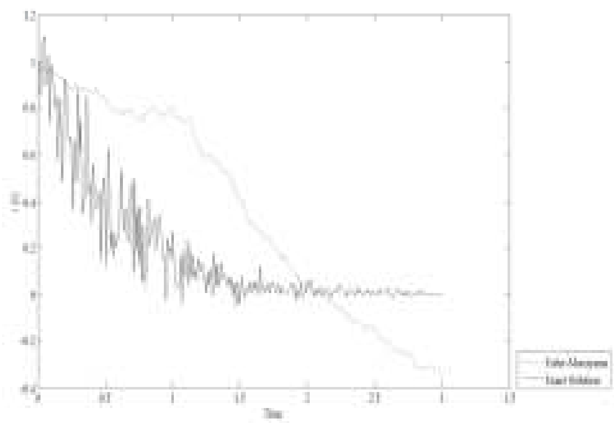

Figure 1 Strong approximation of SDDEs via Euler-Maruyama

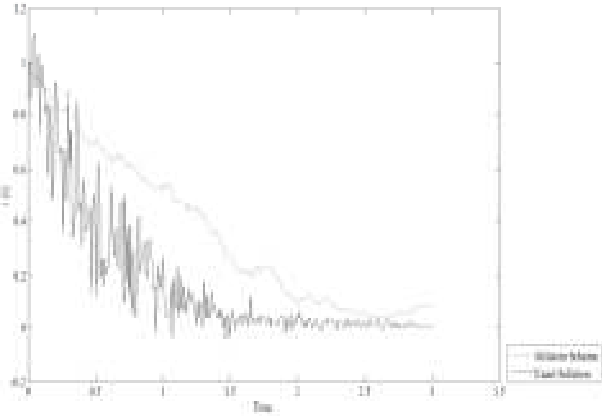

Figure 2 Strong approximation of SDDEs via Misltein scheme

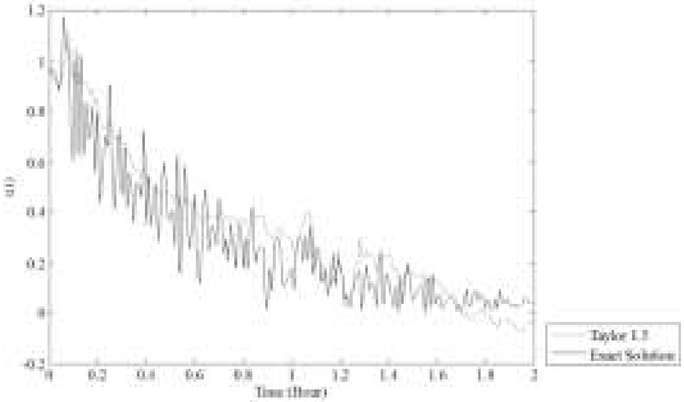

Figure 3 Strong approximation of SDDEs via Taylor method of order 1.5

A glance at Figure 1, Figure 2 and Figure 3 reveals that the result illustrated by Figure 3 shows better performance than the results display in Figure 1 and Figure 2. Next, mean-square error between simulated solution and exact solution are calculated. The results were shown in Table 1.

Table 1 Mean-Square Error of Numerical Solution and Exact Solution.

Numerical Scheme	Euler-Maruyama	Milstein	1.5 Stochastic Taylor Method
MSE	0.1341	0.0581	0.0124

This example visually demonstrates that higher--order method can significantly improve the accuracy of the solution.

6 Conclusion

This paper provides the derivation of numerical schemes of order 0.5, 1.0 and 1.5 from stochastic Taylor expansion to approximate the solution of SDDEs.

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