CONTROL OF A QUADCOPTER

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Report submitted in partial fulfillment of the requirements for the award of the degree of Bachelor of Mechatronics Engineering

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February 2016

ABSTRACT

This thesis is about the development of a quadcopter project from the modelling and simulation to the construction of the real testbed. Objective of this project is to evaluate the reliability of the mathematical model of the quadcopter on the real platform, mainly for when performing hover. For the modelling part of this thesis it discusses the application of Euler-Newton method and how the simulation in MATLAB Simulink is built based on the equations derived from it. Analysis of the simulation includes comparison of different values of tuning gain together with their effect on the performance of the quadcopter. Then a real quadcopter is built from scratch as a prototype model to test the controller obtained from the simulation.

ABSTRAK

Tesis ini ialah berkenaan projek untuk membina sebuah quadkopter melibatkan kerja dari pemodelan dan simulasi sehinggalah ke pembinaan sebenar model untuk diuji. Objektif projek ini ialah untuk melihat bagaimana model matematik boleh diguna pakai ke atas quadkopter sebenar, khususnya untuk melakukan pengapungan di udara. Untuk bahagian permodelan quadkopter, tesis ini membincangkan penggunaan kaedah Euler-Newton dalam menafsirkan sistem dinamik sesebuah quadkopter dan bagaimana simulasi yang dibina menggunakan MATLAB Simulink berdasarkan semua persamaan yang dihasilkan dari kaedah ini. Analisis untuk simulasi ini merangkumi perbandingan prestasi quadkopter menggunakan nilai yang berbeza di dalam sistem pengawalan. Selepas pemodelan dan simulasi telah dijalankan dan dijustifikasikan, sebuah quadkopter sebenar dibina sebagai model prototype untuk menguji sistem pengawalan yang didapati dari simulasi itu dalam dunia sebenar.

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τ	Motor torque
K_t	Torque proportionality constant
I	Current
V	Voltage drop across motor
R_m	Motor resistance
ω	Angular velocity
K_v	Proportionality constant between the angular velocity and the generated back-EMF
T	Thrust
P	Power
v_{air}	Air velocity
ρ	Surrounding air density
K_τ	Proportionality constant of torque to thrust
c_T	Thrust coefficient
τ_z	Torque acting on a single propeller of the quadcopter,
I	Moment of inertia
τ_D	Drag torque
F_D	Frictional force
P	Rate of roll angle
Φ	Roll angle
Ψ	Yaw angle
ϑ	Pitch angle
P	Rate of roll angle
Q	Rate of pitch angle
R	Rate of yaw angle
U	Linear velocity in x direction
V	Linear velocity in y direction
W	Linear velocity in z direction

Chapter 1

Introduction

Quadcopter is a type of multi-rotors helicopter. The word "quad" originated from a Latin word "quattro" which means four to indicate the number of rotors that give it the thrust it needs to move. They are divided to two sets of identical, fixed pitch propellers- two spin clockwise and the other two, counter-clockwise. The control of the craft is achieved by using remote control transmitters to change the speed of the rotors. Among the multi-rotors group, quadcopter is actually the most famous one and is widely accepted by the R/C hobbyist. Quadcopter is normally referred as an unmanned aerial vehicle and better known in a small scale size but back in the 1920s and 1930s, there were many manned versions but did not gain popularity because of their poor performance and were very hard to navigate as the pilot on board had to do a lot of works to control all four rotors to achieve stability even when only hovering. There were too much decision to be made by the pilot. The only solution to that is with the help of electronic controller and sensors and that is why the electronic parts of the quadcopter are as important as the mechanical parts. The harmony collaboration of mechanical and electronic part made navigating the quadcopter much easier. And the rapid development of computers microelectronics technology completed the puzzle that made quadcopter as a reliable rotorcraft.

Nowadays there are already multiple choices of commercial quadcopters can be found. Among the famous one are X4-Flyer, STARMAC II, Draganfly XPro, AR. Drone, Parrot Rolling Spider and DJI Phantom 2 Vision+. Some are used only for acrobatic purpose and some are for photography or both. Currently trending quadcopters among the enthusiasts are the type of First Person View (FPV) as the pilot can experience on-board view of the quadcopter by wearing the virtual reality goggles.

As mentioned before, history of quadcopter has showed that it is almost impossible for a human to control the rotational speed of four motors simultaneously with precision to balance the quadcopter in the air. The only way it can be done is with the help of flight controllers (FC) which is a small circuit board of varying complexity used to direct the RPM of the motors individually in response to input. The pilot just have to command which direction the quadcopter will go and the command signal is fed into flight controller, which decides how to manipulate the motors accordingly. In other word, the flight controller is the brain of the quadcopter. A suitable analogy for the function of the flight controller would be like a situation when a mother tell her son to go buy groceries at the shop. The pilot is the mother and the quadcopter is her son. She just have to tell him where to go and his son, using his brain, determines which part of his body to move to go to the place as demanded. And just like a human brain, a flight controller would need sensors to help making the right decision. Common sensors used are gyroscopes, magnetometer, barometers and accelerometer.

Quadcopters also rely on sturdy and lightweight hull. It usually has a simple and symmetrical shape as it is where most components will be mounted to and where the center of gravity supposed to be. To avoid damaging the important and expensive components, a frame needs to be both strong and stiff while being light enough to move around in the air easily. Another mechanical parts of the quadcopter are the arms. Usually this part can be found as very cheap and should be easy to replace because it is more preferable to have the arms break in the event of a crash in order to absorb all the impact from reaching the hull. It is like the cyclist helmet that breaks when hitting the ground, absorbing all the impact from the high speed collision and thus protecting the cyclist's head. As the arms are the only part that have direct link with the rotors, all the vibration from the spinning rotors will be most likely passed on through them. So, vibration issue is definitely one of the considerations when choosing the right material for the arm. The scale of a quadcopter is often determined by the diagonal measurement in millimeters from motor to motor through the center of the frame. If a model includes numbers in the title, they probably refer to this measurement. A DJI F450 is around 450 mm across, for example. The F330 is 330 mm, and so on.

No matter how the mechanical setup is, it should be modelled as mathematical expression first before translated into lines of coding into the flight controllers. From the mathematical model, simulation can be done to test the functionality of the controller before real flight. Test flights on a real platform should be done too to measure the accuracy and performance of the controller in the hardware.

Chapter 2

Literature Review

Many papers on quadcopter has been published and the topics ranging from development issues to the various applications of a quadcopter either as a research tool, commercial use, or military usage. As for this project, priorities will be on the papers discussing about the modelling method, simulation, and the electronic hardware of flight controllers.

To solve an engineering problem, it is recommended that the problem should be modelled first as a set of mathematical equations by taking all of the necessary engineering principles such as the Newton's Laws into considerations. A model is a physical, mathematical, or logical representation of a system entity, phenomenon, or process. This is done so that the overview of the problem can be observe clearly and eliminate any irrelevant possibilities that do not affect the system while also making it more analysis-friendly. Moreover, it is then feasible to simulate the problem based on the mathematical model using advanced software that is available today. A simulation is the implementation of a model over time. It brings a model to life and shows how a particular object or phenomenon will behave. It is useful for testing, analysis or training where real-world systems or concepts can be represented by a model. One of it is Matlab which is already a well-known simulation software due to its flexibility to almost any application in engineering field.

2.1 Modelling of a Quadcopter

There are several methods to do the mathematical approach to model the kinematics and dynamics of a quadcopter. According to the majority of the papers written, the traditional way of modelling a quadcopter is by using the Euler angles method. It is a method introduced by Leonhard Euler that uses three angles to describe the orientation of a rigid body in 3-dimensional space.

Andrew Gibiansky, a software and robotics engineer, used Euler Angle ZYZ in his approach to model a quadcopter according to his journal "Quadcopter Dynamics, Simulation, and Control".[1] He claimed that his quadcopter's mathematical modelling to be highly simplified as there are a number of ignored advanced effects that contribute to the dynamics of the quadcopter to be highly nonlinear. His derivation of quadcopter's equations motion started with the voltage-torque relation for the brushless motor and then with the quadcopter kinematics and dynamics. Aerodynamical effects such as flapping blade and non-zero free stream velocity are also ignored and replaced by a linear drag force representing air friction in all directions to complete his version of mathematical modelling of a quadcopter.

An Italian hardware engineer, Tommaso Bresciani, has his own version of quadcopter model.[2] In his paper he mentioned that he used Newton-Euler method to describe the quadcopter's orientation, which is actually the derivation from Euler Angler. In classical mechanics, the Newton–Euler equations describe the combined translational and rotational dynamics of a rigid body. Traditionally the Newton–Euler equations is the grouping together of Euler's two laws of motion for a rigid body into a single equation with 6 components, using column vectors and matrices. Unlike Gibiansky, aerodynamical effects is not ignored in his modelling as his thesis work was to be implemented on a real platform. And to improve his mathematical modelling, identification process is also used to obtain the necessary physic constants of the system.

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[1] Gibiansky, Andrew. Quadcopter Dynamics, Simulation, and Control. 2010.

[2] Tommaso Bresciani.Modelling, Identification and Control of a Quadrotor Helicopter. Lund Sweden: Department of Automatic Control, 2008.

This Newton-Euler approach has been used more than any other and is considered a fundamental one but with three drawbacks as recognized by Emil Fresk and George Nikolakopoulos.[3] In their paper “Full Quaternion Based Attitude Control for a Quadrotor”, it is mentioned that the first drawback is because of its base is solely Euler angles, which is limited by “gimbal lock” problem. Gimbal lock is the loss of one degree of freedom in a three-dimensional, three-gimbal mechanism that occurs when the axes of two of the three gimbals are driven into a parallel configuration, "locking" the system into rotation in a degenerate two-dimensional space. The second drawback they mentioned is that the calculation of sines and cosines is considered computationally expensive and is not suitable for low cost hardware. The third drawback is when estimating controllers, which includes utilizing the Jacobian if the system states, will have even greater computational cost because all of the matrix elements will have even one or more sines or cosines. This will then overwhelm the system. From these drawbacks, they suggested three solutions. First solution is by keeping the system inside the bounds of functional Euler angles, considering that the quadcopter will never flip due to unknown external disturbances. Next solution is by using only Direction Cosine Matrix (DCM) approach which is free from singularities problem that the Euler angles have, but the calculation will resulted in at least six (nine at most) system coupled differential equations to be solved. The authors preferred the third solution- the quaternion approach. By using quaternion approach, there will only be four coupled differential equations to be solved and flipping a quadcopter should be possible.

[3] Fresk, Emil, and George Nikolakopoulos. "Full quaternion based attitude control for a quadrotor." *Control Conference (ECC), 2013 European*. IEEE, 2013.

So far all three modelling from the mentioned publications are based on two assumptions:

a) the origin of the body-fixed frame is coincident with the center of mass (COM) of the body,
b) b) the axes of the quadcopter's body frame coincide with the body principal axes of inertia.

In other words, all three models is based on the '+' configuration instead of using the 'x' configuration which is used more commonly in commercial quadcopter. Figures2.1 shows both quadcopter configurations. Mokhtari and Benallegue have modelled the vehicle with axes orientation according to the 'x' configuration in a paper entitled "Dynamic Feedback Controller of Euler Angles and Wind parameters estimation for a Quadrotor Unmanned Aerial Vehicle".[4] The difference between these two configuration will be the selections of which rotors' speed to be altered when to change the pitch and roll angle. 'X' configuration is usually used for First Person View (FPV) quadcopter and other application that involves the use of camera.

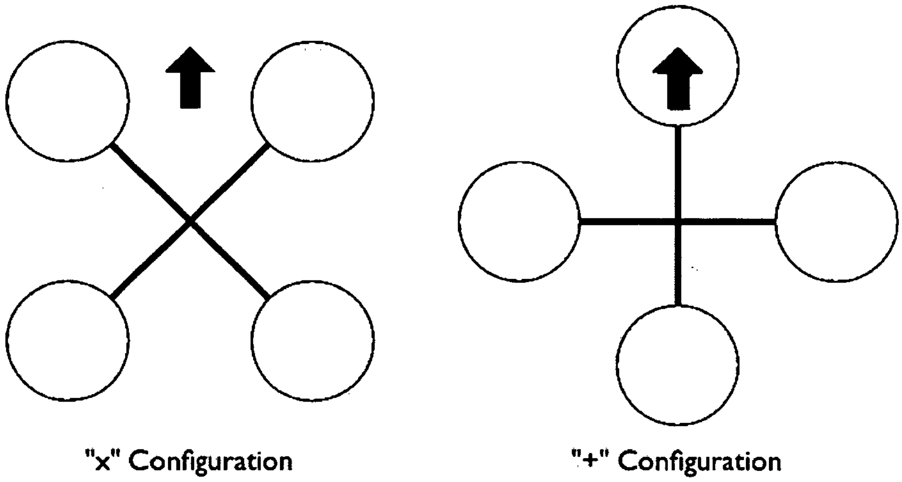

Figure 2.1: 'x' configuration and '+' configuration for a quadcopter

2.2 Control of a Quadcopter

[4] Mokhtari, Abdellah. "Dynamic feedback controller of Euler angles and wind parameters estimation for a quadrotor unmanned aerial vehicle." *Robotics and Automation, 2004. Proceedings. ICRA'04. 2004 IEEE International Conference on*. Vol. 3. IEEE, 2004.

The purpose of deriving a mathematical model of a quadcopter is to assist in developing controllers for physical quadcopters. Andrew Gibiansky first used a PD control for his quadcopter which will only have a gyroscope as its sensor to measure three angle derivatives of yaw, pitch, and roll. These measured values will then be used to calculate the actual error. Based on the hovering simulation of his quadcopter, it is noticed that the angles are not exactly driven to zero. The average steady state error (error after 10 seconds of simulation) is circa 0.3^0 which is an expected problem when using a PD control. He then used a PID control to reduce the error, which is an upgraded version of a PD control. However, PID controls are also known to have a problem of integral windup which causes overshoot and the system to take longer time before reaching steady state. He avoided this by turning off the integral function until the system is near the steady before enabling it. Using the same disturbances for the simulation, the system managed to achieve a much lower error which is approximately 0.06^0 after 10 seconds. But such performance by PID control is achieved after manual tuning of the gain parameters through a number of simulations using different possible disturbances and parameters values. It is more like a trial and error method and it is not a practical method. The solution to this is by using automatic PID tuning through extremum seeking method which resulted in smaller swings, less overshoot, and faster convergence but takes longer time to come to zero for the angular displacement.

Tomasso Bresciani also used a PID control for his quadcopter. The same problem of integral windup is also expected but he used different solution to encounter it. Instead of disabling the integral function and wait the system to almost reaching the steady state before enabling it, he added a saturator after the integral to limit its maximum and minimum values. As for the parameters, he obtained them through laboratory experiments. The result from implementation of the control on a real platform showed roll and pitch angle error always less than one degree which is satisfactory.

Emil Fresk and George Nikolakopoulos used P^2 control for their quaternion based model. The obtained result shows zero error in angular displacement and has phase shift of 0.5 seconds. Flipping the quadcopter to 360^0 is also successful.

There are many other modelling approaches and controlling methods that have been used and suggested.[5,6] Backstepping controller is presented in two different papers which are by E. Altug et al. that used single- and dual-camera visual feedback[7], and the other one by Madani et al. that shows a study on full-state backstepping technique based on Lyapunov stability theory and backstepping sliding mode control.

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[5] Mokhtari, Abdellah. "Dynamic feedback controller of Euler angles and wind parameters estimation for a quadrotor unmanned aerial vehicle." *Robotics and Automation, 2004. Proceedings. ICRA'04. 2004 IEEE International Conference on.* Vol. 3. IEEE, 2004.

[6] B. Bluteau, R. Briand, and O. Patrouix, "Design and control of an outdoor autonomous quadrotor powered by a four strokes RC engine," Proc. of IEEE Industrial Electronics, the 32nd Annual Conference, pp. 4136-4141, 2006.

[7] E. Altug, J. P. Ostrowski, and R. Mahony, "Control of a quadrotor helocopter using visual feedback," Proc. of the IEEE International Conference on Robotics and Automation, vol. 1, pp. 72-77. 2002.

Chapter 3

Modelling and Simulation

Modelling is a very useful approach to understanding an object or a system without the need of having it presents in real life. Engineers used it frequently as basis for their simulations of almost everything they have stumbled upon as it is indeed the best way to properly solve a problem. Modelling is where all the theories and principles taught in classes became useful, not in the form of sentences of words, but as their mathematical equations. Each case will require a different set of theories and assumptions in order to come out with the best mathematical modelling. A good modelling will give a simulation results almost similar to the real life data, while a bad one with a false applied theory can cause a catastrophe. And as the failure to model a small UAV has the least damage, it has been the lab rat for many researchers and engineers from different field of expertise, but mostly aeronautics, to run their simulation on real platform.

3.1 Methodology

There are already quite a number of modelling and simulation examples available in the internet but they always find their way to differ from each other like those explained earlier in literature review. Even though it is best to include all the theories mentioned in the journals, one should always remember to set the constraints of the system so that the modelling will not be a pain in the neck. Imagine an undergraduate mechatronics student, who has zero experience in flying a quadcopter, tries to model it for the first time –it should be a system with the least disturbances. The same goes for this project. This project will cover on the modelling of an **indoor flight only**. Indoor flight means that the observed system will have wind disturbance at the lowest value and therefor can be neglected. Another reason that favors the selection of indoor flight is the higher risk of injuring people as there is a possibility that the quadcopter might "fly away". This problem is a well-known problem that happens when a UAV does not receive any control input from the pilot. There is no downside in having the constraint of an indoor flight, only that when implementing on real platform, the space will be too small for a vehicle with six degree of freedoms to move around.

Just like a helicopter, quadcopter can hover, aileron, elevate, and yaw. These movements can be combined to perform some crazy acrobatic moves and are the reason why flying UAV is such an exciting hobby. However, it is not recommended to do all that when flying indoor as the walls are the only destination the quadcopter will be heading to before sending each of its part to all corner of the room separately. Knowing this means that the quadcopter is allowed to perform **hovering only** and that is what the simulation will be focusing on. Other movements can still be simulated but will be of no use as there is no real life data to compare with. But this does not imply that the quadcopter will have only one state, which is the throttle state, because it will still need to play with the pitch and roll angle in order to stabilize the hovering. Even without wind disturbance, instability still exists as each of the motors will have different rotating speed which will produce different amount of thrust at each end of the arms.

Based on the reviewed papers, most of them use the same mathematical approach. The equations of motion are described using ordinary differential equations (ODE) derived using Newton-Euler method. In robotics, there are several other methods for the derivation and besides Newton-Euler, Kane's and Lagrangian methods are also among the most common ones. Applying the Newton-Euler method requires that force and moment balances be applied for each body taking in consideration every interactive and constraint force. It is indeed comprehensive as a complete solution for all the forces and kinematic variables are obtained but can also become inefficient when only a few of the system's forces are needed. Lagrangian formulation or Lagrange's Equations is a method that only concerns on the forces that perform work. The parents for the equation of motion are the kinetic and potential energy of each link of the system. From there, the scalar Lagrangian must be obtained by differentiating the scalar of the energy function and this is the part of the method where things get more complicated for a large multibody systems. In the other hand, Kane's method combines all the good features from both Newton-Euler and Lagrange methods without the disadvantages. With the use of generalized forces, the need for examining interactive and constraint forces between bodies is eliminated. Since Kane's method does not employ the use of energy functions, differentiating is not a problem. The differentiating required to compute velocities and accelerations can be obtained through the use of algorithms based on vector products. Kane's method is proven an elegant solution to develop the dynamics equations for multibody systems when it is used by Ronald L. Huston in 1990 many of his researches on large multibody dynamics systems[8]. But a quadcopter is always considered as a small multibody system. The propellers are fixed pitch, the motors are fixed mounted on the body, and there is no other moving parts except the four rotating propellers. Hence, just like all the modelling papers on quadcopter, this project will also be using the **Newton-Euler approach**.

Once the mathematical modelling is done, the next step would be the simulation. All of the derived equations would be of no good staying on papers only because they are not the end product. There are still a lot of parameters to be defined and it should already be considered as a millennial sin if one tries to simulate it manually using just a pen, calculator, and graph papers as it would be too tedious, time-wasting, and opened up to all sort of error that could possibly happen. Only through computer simulation can the system be observed and analyzed properly. To do this, the mathematical modelling must first be translated into programming language before running it on a simulation software. And today there are already many simulation software available such as MATLAB, ANSYS, CATIA, SOLIDWORKS, LabVIEW and etc. that are for different types of study. ANSYS, CATIA and SOLIDWORKS are famous among mechanical engineers as they are usually used for precise Finite Element Analysis (FEA) as well as Computational Fluid Dynamics (CFD). But normally these kind of software have high system requirement as to comply with their fine

[8] Huston, R. L. "Multibody dynamics: Modeling and analysis methods." *Appl. Mech. Rev* 44.3 (1991): 109-117.

graphic which is not that important. There is no reason to view mathematics equations or plotted graphs in high-definition. LabVIEW on the other hand is used frequently as electronics simulators. And the one that is most suitable to simulate this project would be **MATLAB Simulink**. Besides from being flexible and light, it is a good platform to simulate the dynamics of the quadcopter, which is what the mathematical modelling is all about. Simulink is a better platform because instead of having just hundreds of lines of coding which are indeed not a sight for sore eyes, it uses block diagram that is more user-friendly, easier to troubleshoot, as well as more presentable.

One of the reasons to perform the simulation is to test the reliability of a controller in the control system of the quadcopter. Controller plays a big role in any control system and in this project, a bad controller would very much likely cause the quadcopter to crash. A good controller should be able to improve steady state accuracy by minimizing the steady states error, and as a result, it will fly more stably. Other features that can be controlled with the help of controllers are overshoot, noise signals produced, and also the time response of the system. Most common controller, the conventional proportional-integral-derivative controller **(PID controller)** can be found being applied widely in the industry and is about to applied in this project too. Despite being well-known to be giving difficulties in the tuning phase, this controller is still the first choice for analytical advantages as most of the reviewed papers used the same controller with the variation of P, PI, and PID. Moreover, the commercial flight controllers mostly use PID controller too and an experienced pilot can manually tune the PID parameters. This gives another advantage that this project would not be an alien topic to be brought in discussion with fellow quadcopter hobbyists and will be very helpful in gaining tips and advices.

3.2 Mathematical Modelling

A system with six degree of freedoms is usually considered to be complex and so the first step to coping with such system is by defining the space frames that will be used. There are two frames –the inertial frame, which is defined by the ground, taking that the positive z-direction pointing against the gravity, and the body frame, which is defined by the orientation of the quadcopter. In this project, the quadcopter will be using '+' configuration which means that

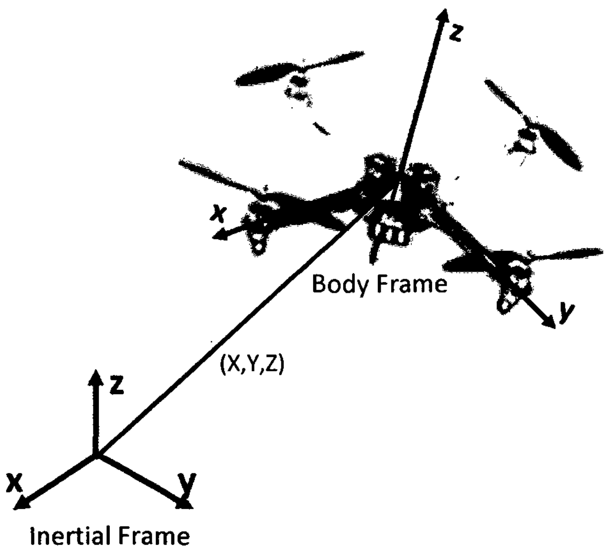

Figure 3.1: Quadcopter Body Frame and Inertial Frame.

the arms are pointing in x- and y-directions, while the rotors are pointing in the positive z-direction. Positive x-direction is defined as the positive forward direction for vehicle movement.

As for the rotation conventions, based on the forward vehicle movement, roll will be about x-axis, pitch about y-axis, and yaw about z-axis. The positive angle direction is determined using right-hand rule and for a better clarity, all the rotation angles are shown in Figure 3.2.

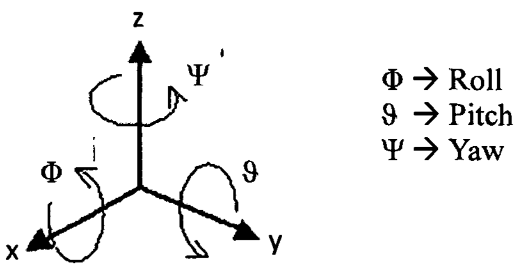

Figure 3.1: Rotation labels in the body frame.

Another important part that need to be set before jumping into the mathematical modelling is the variables and notations. In this project, the variables will be equipped with a notation system to better describe them and avoid confusion. Shown below is the example of the notation system that will be used throughout this chapter:

$$^{b}\dot{\mathrm{v}}^{b}_{CM|i}$$

Linear acceleration, $\dot{\mathrm{v}}$ from above is the base variable with two superscripts and one subscript. Top left superscript, b, represents that the derivative was perform in body frame of reference, while the top right superscript, b, determines that the $\dot{\mathrm{v}}$ is in term of body frame vector components. The subscript, $CM|i$, is telling that this $\dot{\mathrm{v}}$ is referencing the center of mass with respect to inertial frame. Hopefully, with the help of this system, the mathematical model can be easily understand.

3.2.1 Motors

Motors are the actuator of the quadcopter, where the power from the battery is consumed to produce torque. This torque will rotate the propeller and that is when some fluid dynamics and aerodynamics stuff starts happening and produce the thrust that lift the quadcopter. Just like a chain, every point mentioned must connect with each other for it to work, and so to get the model of the thrust, it has to start from modelling the motors. According to Andrew Gibiansky in his journal[9], as the brushless motors are electric motors, the torque is related with the input current and can be described as

$$\tau = K_t(I - I_0)$$

$$I = \frac{\tau}{K_t} + I_0$$

where τ is the motor torque, I is the input current, and I_0 is the current when there is no load on the motor. K_t is the torque proportionality constant. But I_0 is negligible as it is considerably very small to give a simpler equation:

$$I = \frac{\tau}{K_t} \qquad (1)$$

[9] Gibiansky, Andrew. "Quadcopter dynamics and simulation." *Andrew Gibiansky: Math-Code. Np* 23 (2012).

And the voltage across the motor is the sum of resistive loss and back-EMF, written as follows:

$$V = IR_m + K_v\omega$$

where V is the voltage drop across the motor, R_m is the motor resistance, and ω is the angular velocity of the motor. K_v is a proportionality constant between the angular velocity and the generated back-EMF. To simplify the model, the motor resistance is considered negligible:

$$V = K_v\omega \qquad (2)$$

With both current and voltage already available, equation for power, P consumed by the motors can be obtained:

$$P = IV = \frac{K_v}{K_t}\omega\tau \qquad (3)$$

3.2.2 Thrust and Torque

Thrust

The idea is that the power consumed by the motors should be equal to the amount of power consumed by the rotating propeller. By conservation of energy, this is true considering there is zero loss during the process. Power consumed by the propeller is in the terms of thrust, T and the displaced air velocity, v_{air}:

$$P = T \cdot v_{air} \qquad (4)$$

And based on blade element theory[10], the displaced air velocity can be written as a function of thrust:

$$v_{air} = \sqrt{\frac{T}{2\rho A}} \qquad (5)$$

where ρ is the density of surrounding air and A is the area swept out by the rotor which is π times the square of the length of half the propeller. By inserting equation 5 into 4, the power consumed by propeller can be rewritten as

[10] Froude, William, et al. *The Papers of William Froude, MA, LL. D., FRS, 1810-1879.* Institution of Naval Architects, 1955.