A Study on Multi-Agent Systems for Stable Matching

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Abstract

The studies in the multi-agents system (MAS) have attracted many researchers from various fields. The focal point of investigation in MAS has not only surrounded on the development of the control algorithms, but nowadays covers on the practically to use MAS as one of the problem solvers in the other fields of studies.

This dissertation is concerned on solving the Stable Marriage Problem (SMP). The SMP is a combinatorial optimization problem of finding the stable one-to-one pairing between the bipartite sets. The simplest example to illustrate the bipartite sets is to consider the groups of men and women. Each individual in both groups has their own sets of preference lists. In these lists, they contain the information of the preferred partners ranked in the orderly manner. The main objective in solving SMP is to attain pairs between these two sets, such that there exists no blocking pairs. Hence, if this is achieved then the matching is called a stable matching.

David Gale and Llyod Shapley were the first to introduce the SMP theory in year 1962. They proposed sequential algorithm (namely, G-S algorithm) to attain matching between the two sets. But, the results indicted that the optimal matching favors more to the proposers than the receivers. Furthermore, since the proposed algorithm partially utilized the information from preference lists, hence opens for possibility that there exists other methods which yields to better matching.

In this dissertation, we proposed a new method of finding matching between the bipartite sets by the means of MAS. By assigning each agent to represent the individuals in the sets (i.e., to denote agents as man and woman), we designed the

suitable control laws to steer the motions of the agents such that their final positions correspond to the final matching.

To achieve this objective, we utilized the same preference lists as used in G-S algorithm to find matching. We introduced a global Lyapunov function that make use of all the available information. There were two control laws proposed in this dissertation. The first control law considered full communication topology between all agents. However, there was drawback such that the amplitude of the calculated control signals became large. To overcome this phenomenon, we proposed the second control law, where it is almost identical to the previous one. Here, the communication between agents were now limited to exist only among their neighbors. Consequently, the amplitude of the calculated control signals were reduced.

By executing the proposed control laws, the final trajectories and positions of the agents gave us the information on the agents' final pairs. It was also found that the steering motions of the agents corresponded to the minimization of the global Lyapunov function. The total system of MAS is Lyapunov stable, but the final matching still indicated the existence of the blocking pairs.

In conclusion, the proposed control algorithms can guarantee that each of the agents to be matched with his or her stable partners. It was also observed that this matching yielded to the total system to be Lyapunov stable, despite the existence of some blocking pairs.

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Nomenclature

$\succ_y$	Order of preference of y
$\mathcal{A}$	Adjacency matrix
$\mathcal{D}$	Degree matrix
$\mathcal{E}$	Edge set
$\mathbb{G}$	Graph
$\mathcal{J}_i$	Set of same gender agents
$\mathcal{K}_i$	Set of different gender agents
$\mathcal{L}$	Laplacian matrix
$\mathcal{M}$	Set of men
$\mathcal{W}$	Set of women
$\cap$	Intersect
μ	n dimensional Lebesque measure
$\mathcal{N}$	Neighborhood set
$\Omega = \{x : V(x) \leq \gamma\}$	Set of states with bounded Lyapunov trajectory
Ω_m	Weakly invariant set

$\otimes$ Kronecker product

$\overline{\text{co}}$ Closed convex hull

$\mathbb{R}$ Real number

$\mathbb{R}^n$ Real vector of n-dimension

$\subset$ Subset

$\mathcal{X}$ Set of agents

$a.e.$ "almost everywhere"

$B(x, \delta)$ Closed δ neighborhood of x

$K\{f(x, t)\}$ Filippov's differential inclusion

$p_M(\cdot)$ Partner

Chapter 1

Introduction

1.1 Motivation and objectives

The multi-agents system (MAS) is composed of multiple interacting intelligent agents that share information among them to accomplish a certain task. Substantial platforms of researches related to MAS can be found in many areas[1]. Well-established and on-going discussion topics related to MAS vary from formation control (i.e., leader-follower[2]), consensus[3] , and pursuit[4]) to wireless sensor network[5], and system tracking[6].

On the other hand, the stable marriage problem (SMP) is one of the combinatorial optimization problems. Given the groups of men and women, where each of the individuals has their own sets of preference lists, such that they rank orderly their preferred partners. With this information, the aim of solving the matching problem is to establish a stable partnership between these two sets without the blocking pairs.

The terminology of SMP was first coined in year 1962 by David Gale and Lyod Shapley in their seminal work of [7]. They introduced a sequential algorithm (as from now on we refer it as G-S algorithm) to establish matching between bipartite sets, in consideration of matching students with their appropriate colleges, or in the marriage situation per se. The algorithm works by one side of the divides to make the proposal, while the other evaluates either to accept or reject the said proposal. The proposing iteration continues as long as there exists proposer which has yet to be matched with his/her stable partners.

The G-S algorithm works flawlessly to attain stable matching between these two sets without the blocking pairs. However, it only uses partial information from the preference lists where the matching iteration stop once all the proposers found their matches. Therefore, there are possibilities that other probable matching might yield to better matching results suppose full information in the preference lists is used. Moreover, the G-S algorithm favors proposer than the receiver. Hence, the attained final matching will be optimal to the men (if they are the proposer) over the women. This phenomenon is commonly known as the *men-optimal, women-pessimal* situation.

The way this algorithm works in sequential manner limits the usage of all the information available in the preference lists. Therefore, in this dissertation we aim to emulate the strategies given by the G-S algorithm, but to introduce dynamical approach in attaining stable matching between the bipartite sets. In the following, we state the main objectives which motivate us in our investigation of the stable marriage problem.

The objectives of this work are as follows:

- To investigate the potential of treating a discrete optimization problem through solving the sets of differential equations in the multi-agents formulation.

- To identify and formulate suitable cost function that best interpret the preference lists of both men and women.

- To attain dynamically stable matching between men and women.

1.2 Surveys on past and related researches

The studies on SMP have attracted many researchers in various fields particularly in discrete mathematics and economics. One of the famous examples which thoroughly utilizes the matching theory is the National Resident Matching Program (NRMP) of the United States, that focus on assigning the medical school students to their appropriate residency program[8, 9]. Apart from NRMP, the SMP theory has also been applied in computer science[10] and engineering[11, 12].

Vast approaches in finding optimal matching in the case of SMP can be found in the literature. As the matching theory was first introduced by Gale & Shapley[7], it was observed that the pattern in the earlier works of SMP revolved around the key concepts of utilizing the discrete or sequential algorithms such as in [13, 14, 15, 16]. In the later years, more optimization techniques came into existence. In [17], a linear programming technique was proposed to find the stable matching. Further, the applications of linear programming was extended to the case of weighted graph matching problem[18], and as part of a dual ascent algorithm[19]

Another approach of solving SMP is by utilizing the biologicaly-inspired solutions. In Nakamura *et al.*[20], they proposed the Genetic Algorithm(GA) in the sex-fair matching. They treated the SMP as the graph problems for effective application of GA. In addition, Vien & Chung [21] also adopted GA. In contrast to [20], they proposed the conversion of the preference list into multi-objective fitness function, and then sought the optimal solution by GA. In [22], the authors considered the multi-attribute bilateral matching problem and employed the Ant Colony algorithm to seek the optimal matching.

The SMP has also been investigated through graphical elucidation. In [23], a network visualization of stable matching in SMP has been presented. The authors introduced the network consisting of nodes which represents matching, and achieved stability by exchanging a partner between two pairs. In the extended work[24], the authors proposed the case of SMP represented by the multi-edges bipartite graph. In this approach, the classification of all matching instances could be identified by a diagram that contains several constraints.

In this dissertation, we aim to investigate the potential of attaining stable matching between the bipartite sets of men and women by dynamical approach. In similar fashion to our main objective, Hata & Ishida[25] proposed the prey-predator strategy based on the Lotka-Volterra model to dynamically attain stable matching. A set of differential equations representing the potential pairs in the matching were formulated from the preference lists. In contrast to [25], our proposal focus on designing the suitable control action to each of the agents that represents the individual in both men and women sets. The proposed control laws are derived based on the Lyapunov function minimization formulated from the preference lists. By steering the agents' motions to the intended common position, we dynamically attain stable matching.

1.3 Applications overview

The implementation of the matching theory can be found in the broad spectrum of applications ranging from economics to engineering. To mention a few works in economics, in [26, 27], the dynamics of the two-sided matching has been investigated. Meanwhile in [28], Chen & Song studied the matching between banks and firms in the loan market.

Other promising practical applications of SMP are also emerging in the engineering. Nitin & Verma [11] presented evidences that SMP is identical to the problem of Stable Configurations of Multi-Stage Interconnections Network (MINs), and proposed efficient algorithms to solve the dynamic MINs stability problem in [12]. In [29], Leshem *et al.*proposed the use of SMP theory in allocating spectrum of cognitive radio systems. On the other hand, Xu & Li[30] treated the virtual machine (VM) migration problem in cloud computing and proposed the egalitarian approach to find a stable matching fair between VM and servers.

To add further to the list, the SMP has also been implemented to suit the applications in robotics and image processing. In [31], the distributed SMP was formulated on the battery charging planning of autonomous mobile robots. They assumed that there exists n autonomous mobile robots and n charger stations, and the planning was needed to avoid starvation of robots. Next, in [32] the SMP algorithms was proposed for image processing. The author introduced the BZ algorithm to achieve efficient trade-off between the global satisfaction, the fairness and stability. BZ was formulated based on the G-S algorithm in which it scans the marriage table cells to maximize stability and global satisfaction concurrently.

1.4 Main contributions of the dissertation

In this dissertation, we present the approach of attaining stable matching in the multi-agents framework. The dominant aim of this work is to formulate suitable Lyapunov function based on the given preference lists in order to attain stable matching. We claim the following as the contributions derived from our work:

- We introduce the alternative definition of stable matching as given in Definition 3.3.1 as to complement the existing definition of stability in matching theory.
- We propose Theorems 4.1.1 and 5.2.1 to present the suitable control laws executed by each agent, such that the dynamical stable matching can be attained.
- The stability of the matching can be analyzed by investigating the stability of the total system utilizing the Lyapunov stability theory.

1.5 Organization of the dissertation

The narrative of chapters for the rest of dissertation are as follows:

In Chapter 2, we provide the mathematical notations related to our work. We also discuss the definition of stability commonly used in a dynamic system and the definition of stability used in the matching theory.

Next in Chapter 3 , we revisit the G-S algorithm before introducing our problem setting in attaining stable matching by the means of MAS.

Chapters 4 and 5 contains our main results. Here, we formulate the control laws for two different conditions. The stability of the total system formed by the proposed control algorithms are also analyzed.

Then, in Chapter 5, the effectiveness of the proposed methods are illustrated by numerical example.

Finally in Chapter 7, we state our conclusions and suggestions for future work.

Chapter 2

Mathematical Preliminaries

In this chapter, we provide the preliminaries of some mathematical notions and definitions which are crucial in our work. In Section 2.1 we address some basic notations. In Section 2.2, the basic structure related to the graph theory is presented. Then, in Section 2.3 we restate the fundamental definitions related to SMP from [33]. Finally in Section 2.4, the definitions of stability for both Lyapunov and matching theories are discussed.

2.1 Basic notations

The sets of real numbers and vectors with m-dimension are denoted as $\mathbb{R}$ and $\mathbb{R}^m$, respectively. We say that the vector A^T and the matrix B^T as the transpose of $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$. If $B \in \mathbb{R}^{n \times n}$ is non-singular, then its inverse is denoted as B^{-1}. Let $\mathbf{1} = [1, \ldots, 1]^T$ be the vector with all elements of 1. The identity matrix with $n \times n$ dimension is denoted as $\mathbf{I}_n \in \mathbb{R}^{n \times n}$. For $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{l \times k}$, we denote the Kronecker product $A \otimes B \in \mathbb{R}^{nl \times mk}$ as

$$A \otimes B = \begin{bmatrix} a_{11}B & \ldots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ldots & a_{nm}B \end{bmatrix} \tag{2.1}$$

For a given Euclidean norm function, we treat the following lemma as to provide conditions relaxation when we encounter a discontinuous point $x = 0$.

Lemma 2.1.1. Given the Euclidean norm $h(x) := \|x(t)\|$ with $x \in \mathbb{R}^n$, its derivative with respect to t is

$$\frac{d}{dt}\|x\| = (\partial h(x))^T \frac{dx}{dt} \tag{2.2}$$

with subdifferential of $h(x)$ with respect to x

$$\partial h(x) = \begin{cases} \dfrac{x}{\|x\|} & \text{if } x \neq 0 \\ \{g \mid \|g\| \leq 1\} & \text{if } x = 0. \end{cases} \tag{2.3}$$

Proof. First, consider the part when $x \neq 0$. By defining $s(x) := r^2 = \|x\|^2 = x^T x$ and differentiating it by t, we have $\frac{ds}{dt} = 2x^T\dot{x}$. Then, the time derivative of $\|x\|$ is calculated by applying the chain rule such that

$$\frac{d}{dt}\|x\| = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = \frac{1}{2}s^{-\frac{1}{2}}\frac{ds}{dt} = \frac{x^T}{\|x\|}\dot{x}$$

which is valid $\forall x \neq 0$. Next, we consider the part when $x = 0$. Since $h(x)$ is non-smooth at $x = 0$, then we define a subgradient g which is any element of $[-1, 1]$ to ensure the continuity of the $\partial h(x)$, that is

$$\frac{d}{dt}\|x\| = g^T\dot{x}.$$

□

Assumption 2.1.1. The Euclidean norm $h(x)$ defined in Lemma 2.1.1 is a C^0 function and is not differentiable at point $x = 0$. By introducing the subdifferential (2.3) with subgradients $x/\|x\|$ and g, we assume this function becomes smooth at $x = 0$.

2.2 Graph theory

A graph $\mathbb{G}$ of N agents is built upon a finite set called the *vertex/set* represented by $\mathcal{I} = \{1, 2, \ldots, N\}$, and the corresponding positions $\mathcal{X} = \{x_1, \ldots, x_N\}$, with the *edge set* of $\mathcal{E}$. We said $x_i, x_j \in \mathcal{E}$ if there is a communication link between them. We define the neighborhood set of agent i as $\mathcal{N}_i = \{x_j \mid x_i, x_j \in \mathcal{E}, \forall j\}$. In some occasions, if the sensing range R between the agents is specified, then we define the neighborhood set as in Def. 2.2.1

Definition 2.2.1 (Neighbouring Agents). Let R be the distance sensing range of agent i. Agent j is a neighbor of i if j belongs to $\mathcal{N}_i$ where

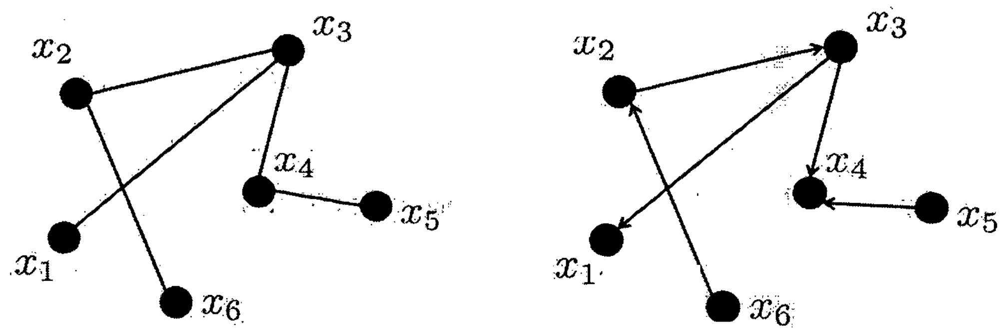
$$\mathcal{N}_i = \left\{ j \in \mathcal{I} \mid r_{ij} = \|x_j - x_i\| \leq R \right\} \tag{2.4}$$

is the neighboring set of each i.

The cardinality of agent x_i is denoted as $|x_i|$ where cardinality is the number of edges connecting x_i to its neighbors. The graph $\mathbb{G}$ is an undirected graph if $\forall x_i \in \mathcal{X}$, there is no arc connecting from x_i to $x_j \in \mathcal{N}_i$. In contrary, the graph $\mathbb{G}$ is said to be a directed graph if such arc exist. The examples of undirected and directed graph topology of six agents system are illustrated in Figs. 2.1(a) and 2.1(b), respectively. Let the adjacency relationship of agent x_i with its neighbors be encoded by the symmetric $N \times N$ *adjacency matrix* $\mathcal{A}(\mathbb{G}) = [a_{ij}] \in \mathbb{R}^{N \times N}$, where for the undirected graph topology

$$a_{ij} := \begin{cases} 1 & \text{if } x_j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

The *degree matrix* $\mathcal{D}(\mathbb{G}) \in \mathbb{R}^{N \times N}$ containing the vertex degree of $\mathbb{G}$ on the diagonal,



(a) Undirected graph (b) Directed graph

Figure 2.1: Examples of graph topology for six agents.

is denoted by $\mathcal{D}(\mathbb{G}) = \mathbf{diag}\{d_1, \ldots, d_N\} = \mathbf{diag}(\mathcal{A}(\mathbb{G})\mathbf{1})$, where

$$d_i := |x_i| = \sum_{j=1}^{N} a_{ij} \tag{2.6}$$

Then, $\mathcal{L}(\mathbb{G})$ is the Laplacian matrix associated with $\mathbb{G}$ where it satisfies

$$\mathcal{L}(\mathbb{G}) := \mathcal{D}(\mathbb{G}) - \mathcal{A}(\mathbb{G}) \geq 0. \tag{2.7}$$

The real eigenvalues of (2.7) can be sorted to satisfy

$$0 = \lambda_1(\mathbb{G}) \leq \lambda_2(\mathbb{G}) \leq \cdots \leq \lambda_N(\mathbb{G}).$$

The following lemma addresses the connectivity of a given graph $\mathbb{G}$:

Lemma 2.2.1. Suppose $\lambda_2(\mathbb{G}) > 0$, then $\mathbb{G}$ is a connected graph.

Proof. See Theorem 2.8 in [34] for complete proof. □

Besides, given a weighted graph $\mathbb{G}$, we can also define its Laplacian $\mathcal{L}(\mathbb{G}) := (l_{ij})_{n\times n}$ with the associated weight $W_{ij} \geq 0$, for the arc x_i to x_j with

$$l_{ij} = \begin{cases} \sum\limits_{j} W_{ij} & \text{if } j = i \\ -W_{ij} & \text{if } j \neq i. \end{cases} \tag{2.8}$$

Definition 2.2.2. For a graph $\mathbb{G}$, if there always exists path from v_i to v_j, $\forall\ (i, j) \in \mathcal{V}$, then $\mathbb{G}$ is a *strongly connected* graph.

Definition 2.2.3. A graph $\mathbb{G}$ is said to be *balanced* if $deg^i_{in} = deg^i_{out}$, where deg^i_{in} is the number of incoming links into the node v_i, and deg^i_{out} is the number of outgoing link from the node v_i, $\forall i$.

2.3 Important definitions in Stable Marriage Problem

In this section, we restate the definitions adopted from [33], and they are asserted here for clarity.

Definition 2.3.1 (Preferred partner). Suppose the persons x and y are in z's preference list and of the opposite sex, then if and only if x proceeds y in z's list, we say z prefers x than y. We also denote this condition as $x \succ_z y$.

Definition 2.3.2 (Matching). Suppose M is the matching set for a one-to-one correspondence pairing between the men and women sets. For a man m to be called *partner* to a woman w, then $\{m, w\} \in M$. We also denote $m = p_M(w)$, $w = p_M(m)$ to imply $p_M(w)$ and $p_M(w)$ as the M-partners of m and w, respectively.

Definition 2.3.3 (Blocking Pairs). Given two pairs $\{m_x, w_y\}, \{m_y, w_y\} \in M$ such that $m_x = p_M(w_x)$ and $m_y = p_M(w_y)$. Suppose that based on the preference lists of m_x and w_y, we have $w_y \succ_{m_x} w_y$ and $m_x \succ_{w_y} m_y$, then we say that the pair $m_x w_y$ constitute the *blocking pairs* of M, such that $m_x w_y \in \mathcal{BP}$.

Definition 2.3.4 (Stability in SMP). A matching M where $\mathcal{BP} \neq \emptyset$ is called *unstable*, and is otherwise *stable*.

Definition 2.3.5 (Stable Pair). For $m = p_M(w)$, where $\{m, w\} \in M$ such that $\mathcal{BP} = \emptyset$, then m and w constitute a *stable pair* in M.

2.4 Stability theory

An autonomous system can be represented by the first order differential equation of

$$\dot{x}(t) = f(x(t)) \tag{2.9}$$

where $f : D \rightarrow \mathbb{R}^n$ is a locally Lipschitz function and mapped from domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. The system (2.9) is said to be continuous and smooth, since the right hand side is continuous.

On the other hand, to denote the differential equation of discontinuous-right-hand side, it can be represented by a differential inclusion

$$\dot{x}(t) \in^{a.e.} K\{f(x,t)\} \tag{2.10}$$

where $K\{f(x,t)\}$ is the Filippov's differential inclusion. The abbreviation *a.e.* stands for almost everywhere .

Definition 2.4.1 (Filippov solution [35, 36]). A vector function $x(\cdot)$ is a solution of (2.10) on $[t_0, t_1]$ if

- $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$, and
- for almost all $t \in [t_0, t_1]$

$$\dot{x}(t) \in K\{f(x,t)\}. \tag{2.11}$$

with

$$K\{f(x,t)\} = \bigcap_{\delta>0} \bigcap_{\mu(D)=0} \overline{\text{co}} f(B(x,\delta) - D, t) \tag{2.12}$$

where co means the closed convex hull; $B(x,\delta)$ is a closed δ neighborhood of x; D is an arbitrary set in R^n; μ is n-dimensional Lebesgue measure; $\bigcap_{\mu(N)=0}$

means the intersection over all sets D of Lebesgue measure zero.

Definition 2.4.2 (Stability of the equilibrium point[37, 38]). Let $x_0 \in D$ be the equilibrium points of either (2.9) or (2.10). Then, the equilibrium point $x = x_0$ is

(i) stable if, for each $\epsilon > 0 \in \mathbb{R}$, there exists $\delta = \delta(\epsilon) > 0$ to satisfy

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall \, t \geq 0,$$

(ii) asymptotically stable, if condition (i) is satisfied, and there exists $\delta \in \mathbb{R}$ such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = x_0.$$

2.4.1 Lyapunov stability

The stability of a given dynamical system can be analyzed by checking the stability of the solutions of its differential equation near to a point of equilibrium. Hence, by ensuring that the stability conditions are satisfied, then we can conclude about the behavior of those system at this point. The general description of the stability is given in Def. 2.4.2. However, the following lemmas provide us with essential tools in analyzing the stability of the dynamical system near its equilibrium points.

Lemma 2.4.1 (Lyapunov's second method for stability). Assume that the system (2.9) has an equilibrium point at $x = x_0$. Then, the system is *asymptotically stable*, if and only if there exist a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ satisfying

- $V(x) > 0$ for all $x \neq x_0$,
- $\dot{V}(x) < 0$ for all $x \neq x_0$, and
- $V(x_0) = \dot{V}(x_0) = 0$.

Suppose the right-hand-side of (2.9) is non-smooth (i.e., equation (2.10)), then the following lemma provides the generalized form of Lemma 2.4.1 to discuss the stability of the non-smooth dynamic.

Lemma 2.4.2 (Generalized Lyapunov Theorem). Given that (2.10) is discontinuous on the right-hand-side, and has an equilibrium point $x = x_0$. Then, if there exists

- a $V : \mathbb{R}^n \to \mathbb{R}$, $V(x_0) = 0$, $V(x) > 0, \forall x \neq x_0$, such that $V(x(t))$ is absolutely continuous on $[t, \infty)$, and
- $\frac{d}{dt}[V(x(t))] < -\epsilon \leq 0$ a.e. on $\{t \mid x(t) \neq x_0\}$,

then x converge to x_0 in finite time. Thus, the system (2.10) is *generally asymptotically stable* in the sense of Lyapunov.

Proof. See Theorem 2 in [36] for the complete proof. □

2.4.2 Stability in matching theory

The stability of a given matching is analyzed by identifying the existence of the blocking pairs. For any given matching, it is said unstable if there exist blocking pairs, and otherwise stable. To clearly comprehend this concept, let's consider two examples of three pairs SMP problem given in Table A.2 of Appendix A .

Example 2.4.1. Let $M_1 = \{m_1w_1,\ m_2w_2,\ m_3w_3\}$ is said to be the matching pairs for men and women based on the preference list in Table A.2. First, consider the pair $m_1w_1 \in M_1$. Here, we can see that m_1 prefers w_2 than his current partner, w_1. Similarly, w_2 also prefers m_1 than her current partner, m_2. Hence, $m_1w_2 \in \mathcal{BP}$ forms the first blocking pair in M_1. Next, for the pair $m_2w_2 \in M_1$, there is no blocking pair associated with it. However, for the pair $m_3w_3 \in M_1$, we can see that m_3 prefers w_2 than this current partner w_3, and w_2 prefers m_3 than her current partner m_2. So, we deduce $m_3w_2 \in \mathcal{BP}$ Therefore, since there exist blocking pairs in M_1 where $\mathcal{BP} = \{m_1w_2,\ m_3w_2\}$, then M_1 is an unstable matching.

Chapter 1

Introduction

1.1 Motivation and objectives

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- To investigate the potential of treating a discrete optimization problem through solving the sets of differential equations in the multi-agents formulation.
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The SMP has also been investigated through graphical elucidation. In [23], a network visualization of stable matching in SMP has been presented. The authors introduced the network consisting of nodes which represents matching, and achieved stability by exchanging a partner between two pairs. In the extended work[24], the authors proposed the case of SMP represented by the multi-edges

Chapter 3

Problem Formulation

In Section 3.1, we revisit the algorithm proposed by Gale & Shapley[7] to attain stable matching for a given bipartite sets of men and women. Then in Section 3.2, we introduce the dynamic of MAS to represent the individual of man and woman, and our problem formulation. Next, we introduce the new definition of Dynamical Stable Matching in Section 3.3 . We discuss the formulation of the Lyapunov function based on the preference lists in Section 3.4. The chapter is summarized in Section 3.5

3.1 Gale and Shapley algorithm

The terminology of SMP in the matching theory was first coined in year 1962 in the seminal work of Gale and Shapley[7]. The original optimization problem involved on finding the most stable partners for a given sets of men and women such that the established partnerships do not constitute any blocking pairs. Refer Defs. 2.3.3 and 2.3.4 on the description of *blocking pair* and *stability* in terms of SMP framework.

We assume that there exists bipartite sets of $\mathcal{M} := \{m_1, m_2, \ldots, m_P\}$ and $\mathcal{W} := \{w_1, w_2, \ldots, w_P\}$, where $\mathcal{M}$ and $\mathcal{W}$ stand for the men and women sets, respectively. Meanwhile, P is the number of pairs. Each of the individuals in these two sets rank orderly their preferred partners in the preference lists. We denote m as the current man to propose to the woman w in his list. If the proposed woman w is already engaged, then $\bar{m}$ is to denote her current partner. On the hand, we use $\bar{w}$ as the next-woman to be proposed from of the m's preference list. The procedure of the G-S algorithm [7] is presented in Algorithm 1.

Algorithm 1 Gale & Shapley Algorithm (Men-proposer)

procedure StableMatching(m, w)
    Initialization: set all men and women to be free
    while $\exists$ free m **do**
        assign $w = m$'s not-yet-proposed woman of the highest rank
        if w is free **then**
            $m = p_M(w)$ & $w = p_M(m)$
            $(m, w) \rightarrow M$ ▹ m, w become *partner*
        else if w is already engaged with $\bar{m}$ **then**
            if m precedes $\bar{m}$ in w's preference list **then**
                $w = p_M(m)$
                $(m, w) \rightarrow M$ ▹ w choose new *partner*
                $\bar{m}$ becomes free
            else
                $w = p_M(\bar{m})$ remain engaged
                $(\bar{m}, w) \rightarrow M$ ▹ $(\bar{m}, w)$ remain *partner*
                m proposes to $\bar{w}$ in his preference list
            end if
        end if
    end while
    Final matching is established.
end procedure

This algorithm is guaranteed to terminate at $O(P \log P)$ iterations[13], and upon termination, stable pairs M will be established. The stable pairs established in M is said to be *Men-Optimal, Women-Pessimal* which favors men over women, since men are the proposer and women are the receiver. The results will be the opposite such that to favors women over men, if women is the first party to give the proposal [7, 33].

Notice that G-S algorithm utilizes the sequential steps of optimizing in seeking for the stable pairs between the bipartite sets. In the following chapters, we attempt to address the same optimization problem as before, but utilize the multi-agent system to achieve the objective.

3.2 Agents dynamics

We consider MAS consists of N number of agents that move on n-dimensional Euclidean space. Each of the agents is described by a single integrator as

$$\dot{x}_i = u_i, \tag{3.1}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^n$ are the position vector and (velocity) control input to be designed, respectively. To represent the dynamic of the total system, we express the total state and control input vectors as

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{nN} \text{ and } u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^{nN} \tag{3.2}$$

respectively. Therefore, we may further rewrite the total system dynamic in the form of

$$\dot{x} = u. \tag{3.3}$$

By taking the average positions of all agents, the group's center of formation can be determined, such that

$$x_c = \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{3.4}$$

On the other hand, it is also assumed that the desired state trajectory $x_d \in \mathbb{R}^n$ for the group center x_c can be achieved by $u_d \in \mathbb{R}^n$, that is

$$\dot{x}_d = u_d. \tag{3.5}$$