Compatible Pair Of Nontrivial Action For Finite Cyclic 2-Groups

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Abstract— The nonabelian tensor product for a pair of groups is defined when the action act compatibly on each other. By using the necessary and sufficient conditions of two finite cyclic 2-groups, the compatible pairs of nontrivial actions for two same finite cyclic 2-groups with same order of actions are determined. Then, the general exact number of compatible pairs of nontrivial actions for two same finite cyclic 2-groups with same order of actions is given.

Keywords— compatible actions, cyclic groups, nonabelian tensor product

1. INTRODUCTION

The nonabelian tensor product was introduced by Brown and Loday in 1984 [1] as a continuation the concept exact sequence by Whitehead [2]. It has its origins in the algebraic K-theory and homotopy theory. The nonabelian tensor product is defined in [1] as a group generated by the symbols $g \otimes h$ with relations $gg' \otimes h = \left({}^{g}g' \otimes {}^{g}h\right)\left(g \otimes h\right)$, $g \otimes hh' = \left(g \otimes h\right)\left({}^{h}g \otimes {}^{h}h'\right)$ and satisfy the compatibility conditions for all $g, g' \in G$ and $h, h' \in H$. However, the paper by Brown *et al.*[3] in 1987 became starting point for research of nonabelian tensor product. They provided a list of open problems regarding nonabelian tensor product and nonabelian tensor square. From the given problems by Brown *et al.* [3], many researchers had studied group theoretical condition of the nonabelian tensor product.

This research focused on the concept of compatibility since it is the conditions in determining the nonabelian tensor product. Visscher [4] be the first researcher worked on cyclic groups and he determined the compatible conditions of actions for the finite cyclic group when one of action is trivial and both are trivial. Later, Mohamad [5] gave the new necessary and sufficient conditions of finite cyclic 2-groups which included the order of the action as one of the conditions. Mohamad *et al*. [6] also gave the compatible conditions of actions for finite cyclic group of order p^2 with the order of action is p. In 2015, Sulaiman *et al.* [7] gave some specific actions which compatible for some finite cyclic 2-groups but focused on action of order two and four.

In this research, we let $G = H$ and the actions have same order. Then, the compatible pairs of nontrivial actions for finite cyclic 2-groups are determined. Next, the general exact numbers of compatible pairs of nontrivial actions for two same finite cyclic 2-groups with same order of actions are given.

2. THE PREMILARIRY RESULTS

In this section, all related definitions and previous results on compatible conditions that have been done before are stated. We start with the definition of actions. The definition of an action of a group *G* on a group *H* is given in the following.

Definition 2.1 Action [4]
Let *G* and *H* be groups. An action of *G* on *H* is a mapping, $\Phi: G \to \mathrm{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)\left(\Phi(g')(h)\right)$$

for all $g, g' \in G$ *and* $h \in H$.

In this paper, our consideration case is on cyclic groups. If *G* and *H* are cyclic groups, then the action of group *G* act on group *H* be required to have the property that the identity in *G* acts as the identity mapping on *H*. Thus, all elements in *G* act as automorphism *H* on *H*. The definition of compatible action is given as follows.

Definition 2.2 Compatible Action [1]
Let *G* and *H* be groups which act on each other. These mutual action are said to be compatible with each other and with the actions of *G* and *H* on themselves by conjugation if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

Next, the following corollary show the action of G acts trivially on *H*.

Corollary 2.3 [4]
Let *G* and *H* be groups. Furthermore, let *G* act trivially on *H*. If *G* is abelian, then for any action of *H* on *G* the mutual actions are compatible.

Since our consideration case is on cyclic groups, the following theorem gives all automorphisms for finite cyclic 2-groups.

Theorem 2.4 [5]
Let $G = \langle g \rangle \cong C_{2^n}, n \geq 3$. Then, $\text{Aut}(G) = \langle \tau \rangle \times \langle \rho \rangle$, where $\tau(g) = g^{-1}$ and $\rho(g) = g^5$ and every $\sigma \in \text{Aut}(G)$ can be represented as $\sigma = \tau^i \rho^j$ with $i = 0, 1$ and $j = 0, 1, \ldots, 2^{n-2} - 1$ and $\tau^i \rho^j(g) = g^t$ with $t \equiv (-1)^i \cdot 5^j \bmod 2^n$.

Mohamad [5] gave some necessary and sufficient number theoretical condition for a compatible pair of actions for finite cyclic 2-groups to act compatibly on each other is stated in Theorem 2.5 and Theorem 2.6. Theorem 2.5 gives the condition of compatible pair of actions when one of the actions has order two and Theorem 2.6 gives the conditions of the compatible pair of actions when one of the actions has order greater than two.

Theorem 2.5 [5]
Let $G = \langle x \rangle \cong C_{2^m}$ and $H = \langle y \rangle \cong C_{2^n}$ .Let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2$ and $\sigma' \in \text{Aut}(H)$ $m \geq 1, n \geq 3$.

i. If $\sigma(x) = x^t$ with $t \equiv -1 \bmod 2^m$ or $t \equiv 2^{m-1} - 1 \bmod 2^m$, then σ, σ' is a compatible pair if and only if σ' is trivial automorphism or $|\sigma'| = 2$.
ii. If $G = \langle x \rangle \cong C_{2^m}$ with $t \equiv 2^{m-1} + 1 \bmod 2^m$, then σ, σ' is a compatible pair if and only if $|\sigma'| = 2^{s'}$ with $s' \leq m - 1$, in particular σ is compatible with all $\sigma' \in \text{Aut}(H)$ provided $n \leq m + 1$.

Theorem 2.6 [5]
Let $G = \langle x \rangle \cong C_{2^m}$ and $H = \langle y \rangle \cong C_{2^n}$ .Let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2^s$, $s \geq 2$ and $\sigma' \in \text{Aut}(H)$ $m \geq 4, n \geq 1$.

i If $\sigma(x) = x^t$ with $t \equiv (-1)^i 5^j \bmod 2^m$ and *i* =1, then σ, σ' is a compatible pair if and only if $\sigma'(y) = y^{t'}$ with $t \equiv 1 \bmod 2^n$ and $t \equiv 2^{n-1} + 1 \bmod 2^n$.
ii If $\sigma(x) = x^t$ with $t \equiv (-1)^i 5^j \bmod 2^m$ and *i* =0, then σ, σ' is a compatible pair if and only if $|\sigma'| \leq 2^{m-s}$ provided $n \leq m - s + 2$.

In the next section, the compatible pairs of nontrivial actions for finite cyclic 2-groups and the general exact numbers of compatible pairs of nontrivial actions for two same finite cyclic 2-groups with same order of actions are given.

3. THE MAIN RESULTS

In previous section, Mohamad [5] had been done the classification of compatible pair of actions condition and provided the necessary and sufficient conditions of two finite cyclic 2-groups include order as one of the condition. In this section, there are two main proposition are stated. First, the necessary and sufficient conditions of two finite cyclic 2-groups that action compatible on each other when $G = H$ are given in following proposition.

Proposition 3.1
Let $G = H = \langle g \rangle \cong C_{2^m}$. Furthermore, let $\sigma \in \text{Aut}(G)$ and $\sigma' \in \text{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$. σ, σ' is compatible pair of actions if $k = 0, k = 1$ and $k \geq 2$ with $\sigma\ g\ = g^t$ where $t \equiv 5^j \bmod 2^m$.

Proof Let $G = H = \langle g \rangle \cong C_{2^m}$. Furthermore, let $\sigma \in$ Aut(G) and $\sigma' \in \text{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$. We shall consider three cases.

Case 1: Suppose that $k = 0$. Then $|\sigma| = |\sigma'| = 2^0 = 1$. By Corollary 2.3, when one of action is trivial, then the actions are compatible. Thus, σ, σ' is compatible pair of actions if $k = 0$.

Case 2: Suppose that $k = 1$. Then $|\sigma| = |\sigma'| = 2^1 = 2$. By Theorem 2.5, the actions are always compatible when both actions have order two. Thus, σ, σ' is compatible pair of actions if $k = 1$.

Case 3: Suppose that $k \geq 2$. Then $|\sigma| = |\sigma'| = 2^k$ where $k \geq 2$. By Theorem 2.6, σ, σ' is a compatible pair when $\sigma'(y) = y^{t'}$ with $t \equiv 1 \bmod 2^n$ and $t \equiv 2^{n-1} + 1 \bmod 2^n$. Thus, σ, σ' is compatible pair of actions if $k \geq 2$.

As conclusion, σ, σ' is compatible pair of actions if $k = 0, k = 1$ and $k \geq 2$ with $\sigma\ g\ = g^t$ where $t \equiv 5^j \bmod 2^m$. □

Next, the general exact number of compatible pairs of nontrivial actions for two same finite cyclic 2-groups with same order of actions is given in Proposition 3.2.

Proposition 3.2
Let $G = H = \langle g \rangle \cong C_{2^m}$. Furthermore, let $\sigma \in \text{Aut}(G)$ and $\sigma' \in \text{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$. Then,

i if $k = 0$, there is one compatible pair of actions.
ii if $k = 1$, there are 9 compatible pair of actions.
iii if $k \geq 2$, there are 2^{2k-2} compatible pair of actions.

Proof Let $G = H = \langle g \rangle \cong C_{2^m}$. Furthermore, let $\sigma \in \text{Aut}(G)$ and $\sigma' \in \text{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$.

i Let $k = 0$, then there is one automorphism of order one for each σ and σ'. Thus, there is one compatible pair of actions only.
ii Let $k = 1$. There are 3 actions have order 2 and by Proposition 3.1, all actions are always compatible with both actions have order two. Thus, there are 9 compatible pair of actions.
iii Let $k \geq 2$. There are 2^k automorphism of order 2^k and by Theorem 2.6 and Proposition 3.1, there only half of automorphism are compatible when $|\sigma| = |\sigma'| = 2^k$. Thus, the number of compatible pair of actions is $2^{k-1}\ \ 2^{k-1}\ = 2^{2k-2}$. □

4. CONCLUSION

As a conclusion of this research, the compatible pairs of nontrivial actions for finite cyclic 2-groups are determined. There are 1 compatible pair if both actions have order 1, 9 compatible pairs of both actions have order 2 and 2^{2k-2} compatible pairs if both actions have order 2^k when $k \geq 2$ for two same finite cyclic 2-groups with same order *k*.

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