



# Mathematical Model of Boundary Layer Flow over a Moving Plate in a Nanofluid with Viscous Dissipation

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## ABSTRACT

In this study, the numerical investigation of boundary layer flow over a moving plate in a nanofluid with viscous dissipation and constant wall temperature is considered. The governing non-linear partial differential equations are first transformed into a system of ordinary differential equations using a similarity transformation. The transformed equations are then solved numerically using the Keller-box method. Numerical solutions are obtained for the Nusselt number, Sherwood number and the skin friction coefficient as well as the concentration and temperature profiles. The features of the flow and heat transfer characteristics for various values of the Prandtl number, plate velocity parameter, Brownian motion and thermophoresis parameters, Eckert number and Lewis number are analyzed and discussed. It is found that the presence of viscous dissipation reduces the range of the plate velocity parameter for which the solution exists. The increase of both Brownian motion and thermophoresis parameters results to the decrease of the Nusselt number, while the Sherwood number increases with the increase of the thermophoresis parameter.

**Keywords:** Constant wall temperature; Moving plate; Nanofluid; Viscous dissipation.

## NOMENCLATURE

$C$	nanoparticle volume fraction	$Shr$	reduced Sherwood number
$C_f$	skin friction coefficient	$T$	temperature inside the boundary layer
$C_{fr}$	reduced skin friction coefficient	$T_w$	wall temperature
$C_p$	specific heat capacity at constant pressure	$T_\infty$	ambient temperature
$C_w$	nanoparticle volume fraction at the surface	$U_\infty$	free stream velocity
$C_\infty$	ambient nanoparticle volume fraction	$u$	velocity components along the $x$ -direction
$D_B$	Brownian diffusion coefficient	$u_w$	plate velocity
$D_T$	thermophoresis diffusion coefficient	$v$	velocity components along the $y$ -direction
$Ec$	Eckert number	$v$	plate velocity parameter
$j_w$	surface mass flux	$Y$	similarity variable
$k$	thermal conductivity	$y_\infty$	boundary layer thickness
$Le$	Lewis number	$\Delta y$	step size
$N_b$	Brownian motion parameter	$\theta$	dimensionless temperature of the fluid
$N_r$	thermophoresis parameter	$\sim$	dynamic viscosity
$Nu_x$	local Nusselt number	$\epsilon$	kinematic viscosity
$Nur$	Reduced Nusselt number	$\rho$	fluid density
$Pr$	Prandtl number	$\dagger$	ratio of the effective heat capacity of the nanoparticle material and the heat capacity of ordinary fluid.
$q_w$	surface heat flux	$\dagger_w$	surface shear stress
$Re_x$	local Reynolds number		

$Sh_x$  Sherwood number

$W$  rescaled nanoparticle volume fraction  
 $\mathcal{E}$  stream function

## 1. INTRODUCTION

Convection boundary layer flow plays an important role in engineering and industrial activities. These configurations are applied in thermal effects managements in many industrial outputs for example in electronic devices, computer power supply and also in engine cooling system such as heat sink in car radiator. Because of the large contributions, this topic has attracted many researchers to study and expand the knowledge so that it could be applied in order to handle the thermal problems produced by these industrial outputs (Pop and Ingham 2001; Salleh *et al.* 2010).

The study of boundary layer flow on a constant speed moving plate was first studied by Sakiadis (1961). Due to entrainment of the ambient fluid, this boundary layer flow is quite different from Blasius flow past a flat plate. Sakiadis' theoretical predictions for Newtonian fluids were later corroborated experimentally by Tsou *et al.* (1967). Karwe and Jaluria (1988) considered the mixed convection from a moving plate in rolling and extrusion processes. Other papers that considered the boundary layer flow over a moving plate are Kumari and Nath (1996), Ali and Al-Yousef (1998), Chen (1999) and Elbashbeshy and Bazid (2000) which introduced the force convection, impulsive motion, suction or injection effects and temperature dependent viscosity, respectively. Bataller (2008) and Ishak *et al.* (2011) observed the radiation effects on the thermal boundary layer flow for Blasius and Sakiadis flows with convective boundary conditions, respectively. It was found that the presence of thermal radiation and convective boundary conditions reduce the heat transfer rate at the surface. Next, the effect of transpiration on the flow and self-similar boundary layer flow over a moving surface was studied by Weidman *et al.* (2006) and Ishak *et al.* (2009). A permeable surface was considered and it was found that dual solutions were obtained in both studies. Weidman *et al.* (2006) reported that the solution is unique for positive values of the plate velocity parameter, while dual solutions exist for its negative values, up to the critical value, beyond which no solution is in existence. The stability analysis revealed that the upper branch solution is stable, while the lower branch solution is unstable.

The investigations involving the flow on a moving plate were also extended to other type of fluids such as viscoelastic fluid, micropolar fluid and nanofluid by many investigators including Abel *et al.* (2005), Anuradha and Krishnambal (2010) and recently by Khan *et al.* (2014) who considered the flow in a viscoelastic fluid. The MHD and buoyancy effects on a moving stretching surface were analysed and discussed. Aman *et al.* (2013) and Kundu *et al.* (2015) investigated the MHD effects in micropolar

fluid while Sandeep and Sugunamma (2014) considered the unsteady hydromagnetic flow past an impulsively moving vertical plate in a porous medium. Furthermore, Bachok *et al.* (2010) and Ro ca and Pop (2014) investigated the steady and unsteady boundary layer flow of a nanofluid past a moving surface in an external uniform free stream, respectively. Both studies considered the Bongiorno-Darcy nanofluid model. The effects of the Prandtl number, plate velocity parameter and the nanofluid parameter which is Brownian motion parameter, thermophoresis parameter and Lewis number was analysed and discussed. The numerical solutions were obtained by using the Keller-box method and the bvp4c package in Matlab, respectively.

In considering the viscous dissipation effects, from literature study it is found that Gebhart (1962) is probably the first who studied viscous dissipation in free convection flow. The viscous dissipation effects on unsteady free convective flow over a vertical porous plate was then investigated by Soundalgekar (1972). Vajravelu and Hadjinicolaou (1993) then studied the viscous dissipation effects on the flow and heat transfer over a stretching sheet. Chen (2004) observed the heat and mass transfer in MHD free convection from a vertical surface with Ohmic heating and viscous dissipation effects. Next, Partha *et al.* (2005) and Yirga and Shankar (2013) considered this topic on exponentially stretching surface and stagnation point flow in a nanofluid with thermal radiation and magnetohydrodynamic effects, respectively. It was reported that the presence of viscous dissipation which represented by Eckert number has contributed to the increase in velocity and thermal boundary layer thicknesses while the local Nusselt number decreases. Recently, Pal and Mondal (2014) and Gangadhar (2015) investigated the viscous disipation effects on a stretching plate, Blasius and Sakiadis plate with Soret-Dufour effects, radiation, heat generation, MHD, porous medium and Ohmic heating, respectively.

Motivated by the above factor and contributions, the objective of the present paper is to investigate the boundary layer flow over a moving flat plate in a nanofluid with viscous dissipation. It is known that nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. This type of fluid is believed may enhance thermal conductivity, viscosity, thermal diffusivity and convective heat transfer compared to those base fluids like water and oil. Many investigations on the nanofluid have been done such as Vajravelu *et al.* (2011), Anwar *et al.* (2012), Bachok *et al.* (2012) and recently by Kameswaran *et al.* (2013), Rashad *et al.* (2013), Makinde *et al.* (2013) and Sreenivasulu and Bhaskar (2015). To the best of our knowledge, the present study is never been considered before, so that the reported results are new.

## 2. MATHEMATICAL FORMULATIONS

A steady two-dimensional boundary layer flow over a moving plate immersed in a nanofluid of ambient temperature  $T_\infty$  is considered. It is assumed that  $T$  be the temperature inside the boundary layer,  $T_w$  be the wall temperature,  $U_\infty$  be the free stream velocity and  $u_w(x) = vU_\infty$  be the plate velocity where  $v$  is the plate velocity parameter (see Weidman *et al.* 2006). Furthermore,  $C$  is the nanoparticle volume fraction,  $C_w$  is the nanoparticle volume fraction at the surface and  $C_\infty$  is the ambient nanoparticle volume fraction. The physical model and coordinate system of this problem is shown in Fig. 1. The governing boundary layer equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \epsilon \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \dagger \left[ D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right] + \frac{\tilde{\nu}}{\rho C_p} \left( \frac{\partial u}{\partial y} \right)^2, \tag{3}$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2}, \tag{4}$$

subject to the boundary conditions

$$\begin{aligned} u = u_w(x) = vU_\infty, v = 0, T = T_w, C = C_w \text{ at } y = 0, \\ u = U_\infty, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } y \rightarrow \infty, \end{aligned} \tag{5}$$

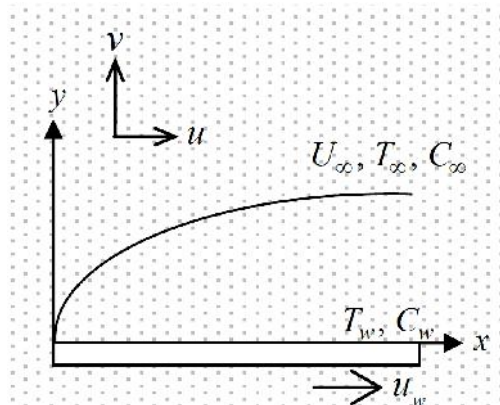


Fig. 1. Physical model of the coordinate system.

where  $u$  and  $v$  are the velocity components along the  $x$  and  $y$  directions, respectively,  $\tilde{\nu}$  is the dynamic viscosity,  $\epsilon$  is the kinematic viscosity,  $\rho$  is the fluid density,  $k$  is the thermal conductivity

and  $C_p$  is the specific heat capacity at constant pressure. Next,  $D_B$  is the Brownian diffusion coefficient,  $D_T$  is the thermophoresis diffusion coefficient,  $\dagger$  is the ratio of the effective heat capacity of the nanoparticle material and the heat capacity of the ordinary fluid.

The set of similarity transformation for Eqs. (1) to (4) subjected to the boundary conditions (5) is (Weidman *et al.* 2006; Bachok *et al.* 2010)

$$\begin{aligned} y = \left( \frac{U_\infty}{2\epsilon x} \right)^{1/2} \mathbb{E}, \mathbb{E} = (2U_\infty \epsilon x)^{1/2} f(\mathbb{Y}), \\ \mathbb{W}(\mathbb{Y}) = \frac{T - T_\infty}{T_w - T_\infty}, w(\mathbb{Y}) = \frac{C - C_\infty}{C_w - C_\infty}, \end{aligned} \tag{6}$$

where  $W$  and  $w$  are the rescaled nanoparticle volume fraction and dimensionless temperature of the fluid, respectively.  $\mathbb{E}$  is the stream function defined as  $u = \frac{\partial \mathbb{E}}{\partial y}$  and  $v = -\frac{\partial \mathbb{E}}{\partial x}$  which identically satisfy Eq. (1). Then,  $u$  and  $v$  can be derived as

$$\begin{aligned} u = U_\infty f'(\mathbb{Y}), \\ v = -\left( \frac{U_\infty \epsilon}{2x} \right)^{1/2} f(\mathbb{Y}) + \frac{U_\infty y}{2x} f'(\mathbb{Y}), \end{aligned} \tag{7}$$

By substituting Eqs. (6) and (7) into Eqs. (2) to (4), we have

$$f''' + ff'' = 0 \tag{8}$$

$$\frac{1}{Pr} \mathbb{W}'' + f \mathbb{W}' + N_b \mathbb{W}' + N_t \mathbb{W}'' + Ec f''^2 = 0, \tag{9}$$

$$w'' + \frac{N_t}{N_b} \mathbb{W}'' + Le f w' = 0 \tag{10}$$

Where  $Pr = \frac{\epsilon \rho C_p}{k}$  is the Prandtl number,

$N_b = \frac{\dagger D_B (C_w - C_\infty)}{\epsilon}$  is the Brownian motion

parameter,  $N_t = \frac{\dagger D_T (T_w - T_\infty)}{T_\infty \epsilon}$  is the

thermophoresis parameters,  $Ec = \frac{(U_\infty)^2}{C_p (T_w - T_\infty)}$  is the

Eckert number and  $Le = \frac{\epsilon}{D_B}$  is the Lewis number.

The boundary conditions (5) become

$$\begin{aligned} f(0) = 0, f'(0) = v, \mathbb{W}(0) = 1, w(0) = 1, \\ f'(\mathbb{Y}) \rightarrow 1, \mathbb{W}(\mathbb{Y}) \rightarrow 0, w(\mathbb{Y}) \rightarrow 0, \text{ as } \mathbb{Y} \rightarrow \infty \end{aligned} \tag{11}$$

Note that  $v > 0$  corresponds to downstream movement of the plate from the origin (Weidman *et al.* 2006). The physical quantities of interest are the skin friction coefficient  $C_f$  the local Nusselt number  $Nu_x$  and the local Sherwood number  $Sh_x$

which are given by

$$C_f = \frac{\tau_w}{\rho u_e^2}, Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \tag{12}$$

$$Sh_x = \frac{xj_w}{D_B(C_w - C_\infty)}.$$

The surface shear stress  $\tau_w$ , the surface heat flux  $q_w$  and the surface mass flux  $j_w$  are given by

$$\tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}, \tag{13}$$

$$j_w = -D_B \left( \frac{\partial C}{\partial y} \right)_{y=0}$$

with  $\mu = \rho \epsilon$  being the dynamic viscosity. Using the similarity variables in Eq. (6) give

$$C_f (2Re_x)^{1/2} = f''(0), Nu_x \left( \frac{Re_x}{2} \right)^{-1/2} = -\theta'(0), \tag{14}$$

$$Sh_x \left( \frac{Re_x}{2} \right)^{-1/2} = -w'(0)$$

where  $Re_x = \frac{U_\infty x}{\epsilon}$  is the local Reynolds number. It is worth mentioning that the physical quantities of interest in the present context are

$$C_f (2Re_x)^{1/2}, Nu_x \left( \frac{Re_x}{2} \right)^{-1/2} \text{ and } Sh_x \left( \frac{Re_x}{2} \right)^{-1/2}$$

which are referred as the reduced skin friction coefficient, the reduced Nusselt number and the reduced Sherwood number and can be denoted as  $C_{fr}$ ,  $Nur$  and  $Shr$  which are represented by  $f''(0)$ ,  $-\theta'(0)$  and  $-w'(0)$ , respectively.

### 3. NUMERICAL METHOD

The transformed ordinary differential equations subject to the associated boundary conditions are solved numerically using the Keller-box method. As described in the books by Na (1979) and Cebeci and Bradshaw (1988), the solution is obtained in the following four steps: Firstly, reduce Eqs. (8) to (11) to a first-order system. Secondly, write the difference equations using central differences; Thirdly, linearize the resulting algebraic equations by Newton's method, and write them in the matrix-vector form; and finally, solve the linear system by the block tridiagonal elimination technique.

### 4. RESULTS AND DISCUSSION

Six parameters are considered, namely the Prandtl number  $Pr$ , the plate velocity parameter  $v$ , the Brownian motion parameter  $N_b$ , the thermophoresis parameter  $N_t$ , the Eckert number  $Ec$ , and the Lewis number  $Le$ . The step size

$\Delta y = 0.02$  and the boundary layer thickness  $y_\infty$  from 2.5 to 8 were used in obtaining the numerical results. From the literature review, in considering the boundary layer flow over a stretching surface in nanofluids, it was found that  $N_b$  and  $N_t$  effects are practically studied at range between 0.1 and 0.5 while  $Le$  in the range of 1 to 40 (Khan and Pop 2010; Noghrehabadi *et al.* 2012; Nandy and Mahapatra 2013). Therefore, in the present study, the same range is considered in our analysis and discussion.

Table 1 shows the comparison values of  $f''(0)/\sqrt{2}$  with previous results by Ro ca and Pop (2014) for various values of Prandtl number  $Pr$ . It is found that they are in good agreement. We can conclude that this method works efficiently for the present problem.

Table 2 shows the values of  $Nur$ ,  $Shr$  and  $C_{fr}$  for various values of  $v$ . From this table, it is found that the present of plate velocity parameter  $v$  results to the increase of  $Nur$  and  $Shr$ . Meanwhile, the skin friction coefficient  $C_{fr}$  decreases as  $v$  increases.

Note that when  $v = 0$ ,  $C_{fr} = 0.4696$  which is in a very good agreement with Blasius (1908). Next,  $C_{fr} = 0$  for  $v = 1$  due to the plate and the fluid move with the same velocity which results in no velocity gradient, i.e. there is no friction at the fluid-solid interface.

**Table 1 Comparison values of  $f''(0)/\sqrt{2}$  with previous published results for various values of  $Pr$  when  $v = 0, N_b = 0, N_t = 0, Ec = 0, Le = 10$**

Pr	Ro ca and Pop (2014)	Present
0.7	0.29268	0.292680
0.8	0.30691	0.306917
1	0.33205	0.332057
5	0.57668	0.576689
10	0.72814	0.728141

**Table 2. Values of  $Nur$ ,  $Shr$  and  $C_{fr}$  for various values of  $v$  when  $Pr = 7, N_b = 0, N_t = 0, Ec = 0.1$  and  $Le = 10$**

$v$	$Nur$	$Shr$	$C_{fr}$
0	0.3747	1.1672	0.4696
0.1	0.4705	1.3369	0.4625
0.5	0.7875	1.8657	0.3288
1	1.0717	2.3805	0
2	1.2994	3.3099	-1.0191

Table 3 presents the values of the reduced Nusselt number  $Nur$  and Sherwood number  $Shr$  for various values of  $N_b$  and  $N_t$ . From this table, it is

concluded that the increase of both parameters  $N_b$  and  $N_t$  results to the decrease of  $Nur$  while  $Shr$  increases with the increase of  $N_t$ . Physically, it is suggested that the small values of  $N_b$  and  $N_t$  enhance convective heat transfer capabilities while the large values of  $N_b$  and  $N_t$  enhance the convective mass transfer capabilities.

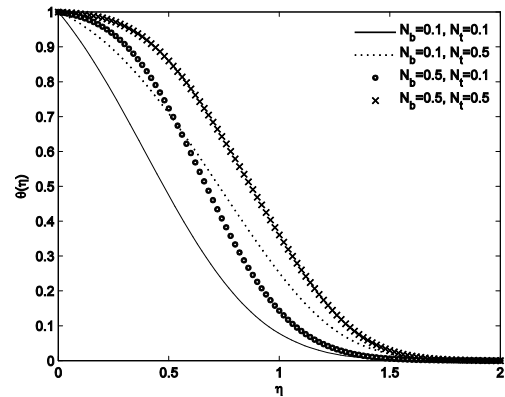
Figures 2 to 4 illustrate the temperature profiles for various values of  $N_b$ ,  $N_t$ ,  $Le$ ,  $Ec$  and  $v$ . These Figs are plotted in order to understand the behavior of the thermal boundary layer thickness for various values of parameter discussed. From Fig. 2, it is found that as  $N_b$  increases, the thermal boundary layer thickness also increases. According to Anwar *et al.* (2013), this is due to the fact that larger values of  $N_b$  have a large extent of fluid hence thickening the thermal boundary layer thickness. The same trends happen for parameter  $N_t$ . This result from the deeper penetration into the fluid which cause to the thickening of thermal boundary layer thickness.

**Table 3 Values of  $Nur$  and  $Shr$  for various values of  $N_b$  and  $N_t$  when  $Pr = 7, Ec = 0.1, Le = 10$  and  $v = 0.5$**

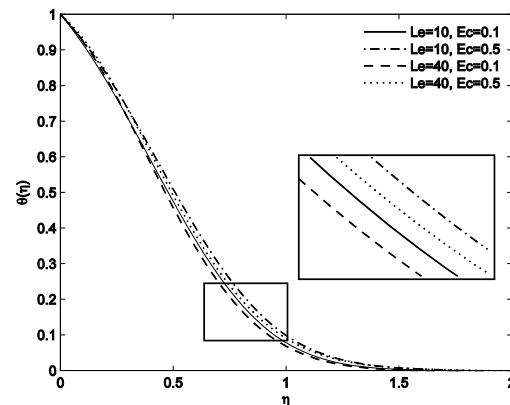
$N_b$	$N_t$	$Nur$	$Shr$
0.1	0.1	0.7875	1.8657
0.2	0.1	0.4923	1.9951
0.3	0.1	0.2959	2.0091
0.4	0.1	0.1712	2.0010
0.5	0.1	0.0952	1.9884
0.1	0.2	0.6047	2.0601
0.1	0.3	0.4708	2.3624
0.1	0.4	0.3722	2.6994
0.1	0.5	0.2990	3.0319

In Fig. 3, it is found that the increase of  $Le$  results to the decrease in thermal boundary layer thickness. The opposite trend occurs for  $Ec$  where the increase of this parameter results to the increase of thermal boundary layer thickness. This may be explained as follows: The increase of  $Ec$  directly proportional to the increase of external velocity (see definition of  $Ec$ ) and this situation reduces the effect of the plate velocity parameter  $v$ , which thickening the thermal boundary layer thickness.

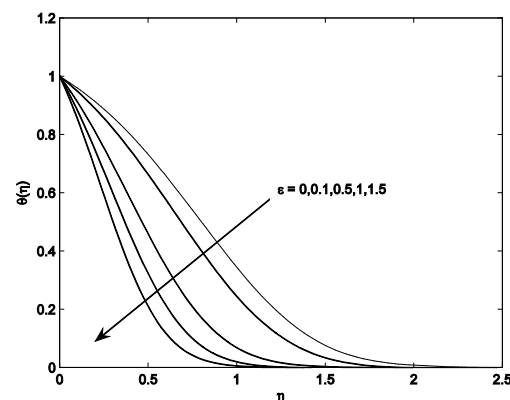
Figure 4 shows the temperature profiles for various values of  $v$ . It is found that the present of plate velocity parameter ( $v > 0$ ) reduces the thermal boundary layer thickness. Physically, the increase of  $v$  results to the increase of ratio velocity differences between the plate and the fluid which enhance the fluid to move away from the plate region rapidly. This situation reduces the thermal diffusivity and thinning the thermal boundary layer thicknesses.



**Fig. 2. Temperature profiles  $\theta(\eta)$  for various values of  $N_b$  and  $N_t$  when  $Pr = 7, Ec = 0.1, Le = 10$  and  $v = 0.5$ .**



**Fig. 3. Temperature profiles  $\theta(\eta)$  for various values of  $Le$  and  $Ec$  when  $N_b = N_t = 0.1, Pr = 7$  and  $v = 0.5$ .**

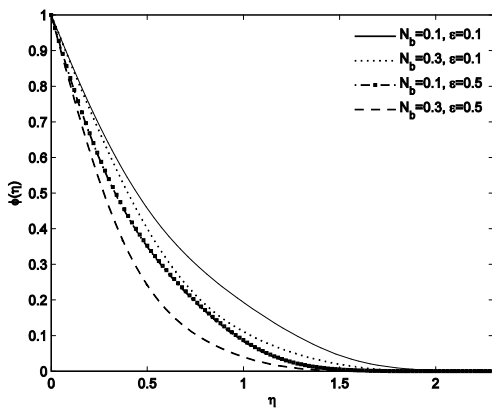


**Fig. 4. Temperature profiles  $\theta(\eta)$  for various values of  $v$  when  $N_b = N_t = 0.1, Pr = 7, Le = 10$  and  $Ec = 0.1$ .**

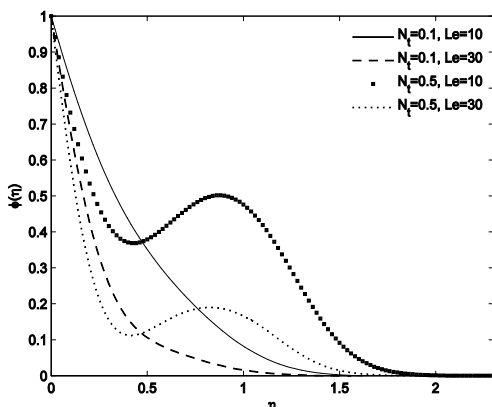
Figures 5 to 7 show the variation of nanoparticle volume fraction  $w(y)$  for various values of parameters. The increase of  $N_b$  in Fig. 5 results to

the decrease of  $w(y)$ . Also, the effects of  $V$  is similar as in Fig. 4. The moving effects reduces the nanoparticle volume fraction  $w(y)$ .

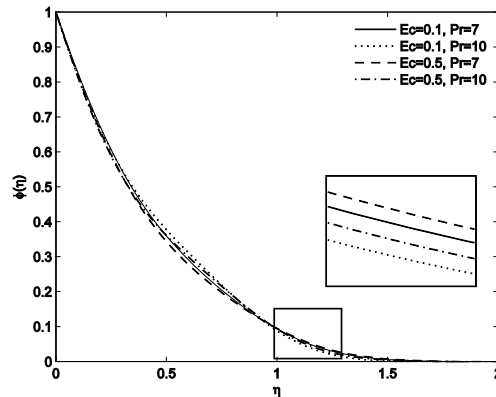
The effects of  $N_t$  and  $Le$  are shown in Fig. 6. Both parameters consume the contra trend where the increase of  $Le$  is to the decrease the nanoparticle volume fraction  $w(y)$  while  $w(y)$  increases as  $N_t$  increases. Further, the nanoparticle volume fraction  $w(y)$  increases monotonically as  $Pr$  decreases. Meanwhile, the Eckert number  $Ec$  shows the contradict trends. Large values of  $Ec$  means large viscous dissipation effect and this enhance the nanoparticle volume fraction  $w(y)$ . The variations of the nanoparticle volume fraction for both parameters  $Pr$  and  $Ec$  are plotted in Fig. 7.



**Fig. 5.** Variation of nanoparticle volume fraction  $w(y)$ : for various values of  $N_b$  and  $v$  when  $Pr = 7$ ,  $N_t = 0.1$  and  $Le = 10$ .



**Fig. 6.** Variation of nanoparticle volume fraction  $w(y)$ : for different values of  $N_t$  and  $Le$  when  $Pr = 7$ ,  $N_b = 0.1$  and  $v = 0.5$ .

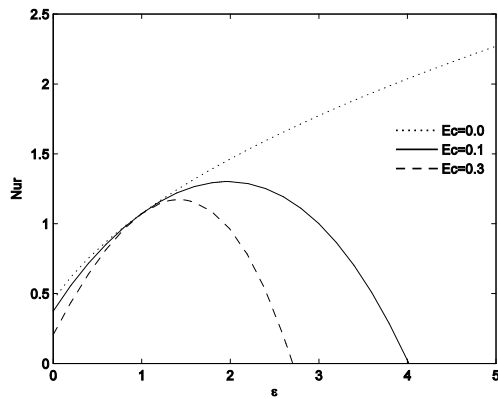


**Fig. 7.** Variation of nanoparticle volume fraction  $w(y)$ : for different values of  $Pr$  and  $Ec$  when  $N_b = 0.1$ ,  $Le = 10$  and  $v = 0.5$ .

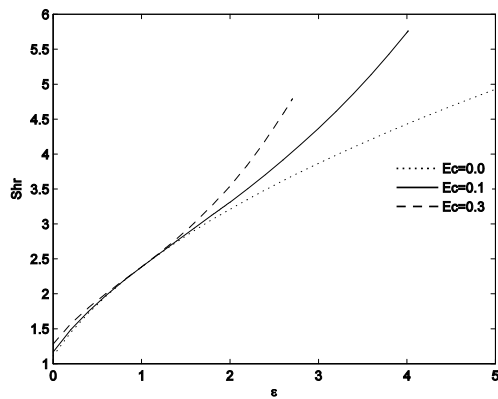
Figures 8 and 9 illustrate the variation of the reduced Nusselt number  $Nur$  and the reduced Sherwood number  $Shr$  with  $V$  for various values of  $Ec$ . From Fig. 8, it is found that the presence of viscous dissipation ( $Ec \neq 0$ ) changes the variation of  $Nur$  curve to a quadratic curve. Also, the presence of  $Ec$  reduces the range of  $v$  for which the solution exists. It is clearly shown when  $Ec = 0.1$ , the physical acceptable solution occurs until  $v = 4.0189$  while when the value of  $Ec$  increases ( $Ec = 0.3$ ),  $Nur$  stops at  $v = 2.7153$ . Since  $Nu \cong 0$ , this means that there is no convection occurs beyond this value, which leads to pure conduction heat transfer. Furthermore, it is noticed that when  $v = 1$ , which results from the equivalent of stretching and external velocity at the plate surface,  $Ec$  does not affect the values of  $Nur$ .

In Fig. 9, it is noticed that the physical acceptable solution for  $v$  has a similar trend as in Fig. 8. Further, it is found that, at large values of  $V$ , larger values of  $Ec$  produce large values of  $Shr$  which physically means large convective mass transfer. This is due to the increase of the ratio of kinetic energy over enthalpy in increasing  $Ec$ . Also, it is noticed that the variation of  $Shr$  is unique for all  $Ec$  when  $v = 1$  as in Fig. 8.

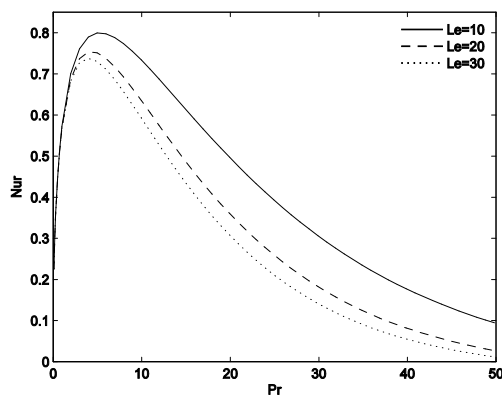
Figures 10 and 11 show the variation of  $Nur$  and  $Shr$  with  $Pr$  for various values of  $Le$ . From Fig. 10, it is found that small values of  $Pr$  such as liquid metal ( $Pr \ll 1$ ), produce very small values of  $Nur$  which imply no convection occurs or the heat transfer is in a pure conduction situation. It is realistic since liquid metals have high thermal conductivity but low viscosity. For fixed values of  $Pr$ , increasing  $Le$  results in the decrease of  $Nur$  but an increase of  $Shr$ .



**Fig. 8.** Variation of reduced Nusselt number with  $v$  for various values of  $Ec$  when  $Pr = 7$ ,  $Le = 10$  and  $N_b = N_t = 0.1$ .



**Fig. 9.** Variation of reduced Sherwood number with  $v$  for various values of  $Ec$  when  $Pr = 7$ ,  $Le = 10$  and  $N_b = N_t = 0.1$ .

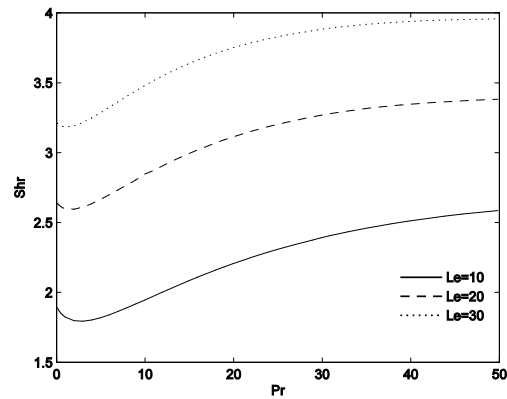


**Fig. 10.** Variation of reduced Nusselt number with  $Pr$  for various values of  $Le$  when  $N_b = N_t = Ec = 0.1$  and  $v = 0.5$ .

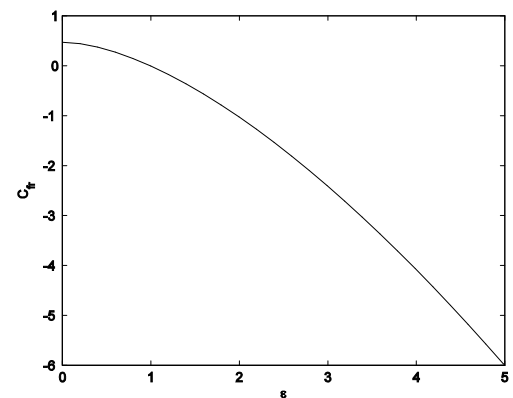
Lastly, Fig. 12 presents the variation of the reduced skin friction coefficient  $C_{fr}$  for various values of  $v$  which produce  $f'(0) = v$  and  $f'(y) = 1$  as  $y \rightarrow \infty$ . From this figure, it is found that the value of  $C_{fr}$  is

positive for  $v < 1$ , and  $C_{fr}$  decreases to 0 as  $v$  approach 1. The value  $C_{fr} = 0$  at  $v = 1$  is due to the plate moves in the same velocity with the fluid. Further, from this figure, we can conclude that  $C_{fr}$  decreases as  $v$  increases.

From this study it is worth mentioning that only moving parameter affects the velocity profiles and the value of the skin friction coefficient. It is clear from the ordinary differential Eqs. (8) to (10) and boundary conditions (11).



**Fig. 11.** Variation of reduced Sherwood number with  $Pr$  for various values of  $Le$  when  $N_b = N_t = Ec = 0.1$  and  $v = 0.5$ .



**Fig. 12.** Reduced skin friction coefficient for various values of  $v$ .

## 5. CONCLUSION

In this paper, the boundary layer flow over a moving plate in a nanofluid in the presence of viscous dissipation was numerically studied. It was shown how the Prandtl number  $Pr$ , plate velocity parameter  $v$ , Brownian motion parameter  $N_b$ , thermophoresis parameter  $N_t$ , Eckert number  $Ec$  and Lewis number  $Le$  affect the values of the reduced Nusselt and Sherwood numbers and the skin friction coefficient as well as the concentration and temperature profiles. As a conclusion, the thermal boundary layer thickness depends strongly on these parameters. It was found that in the

presence of viscous dissipation, the range of the plate velocity parameter for which the solution exists reduces, which physically leads to pure conduction to occur. Further, the same situation occurs to the nanofluid with low Prandtl number. This may be explained as the low Prandtl value means the fluid is low in viscosity but high in thermal conductivity such as liquid metal. The increase of  $N_b$ ,  $N_t$  and  $Ec$  results to the increase in thermal boundary layer thickness. Meanwhile, the increase of  $Pr$ ,  $\nu$  and  $Le$  reduced the thermal boundary layer thickness.

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# Mathematical Model of Boundary Layer Flow over a Moving Plate in a Nanofluid with Viscous Dissipation

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## ABSTRACT

In this study, the numerical investigation of boundary layer flow over a moving plate in a nanofluid with viscous dissipation and constant wall temperature is considered. The governing non-linear partial differential equations are first transformed into a system of ordinary differential equations using a similarity transformation. The transformed equations are then solved numerically using the Keller-box method. Numerical solutions are obtained for the Nusselt number, Sherwood number and the skin friction coefficient as well as the concentration and temperature profiles. The features of the flow and heat transfer characteristics for various values of the Prandtl number, plate velocity parameter, Brownian motion and thermophoresis parameters, Eckert number and Lewis number are analyzed and discussed. It is found that the presence of viscous dissipation reduces the range of the plate velocity parameter for which the solution exists. The increase of both Brownian motion and thermophoresis parameters results to the decrease of the Nusselt number, while the Sherwood number increases with the increase of the thermophoresis parameter.

**Keywords:** Constant wall temperature; Moving plate; Nanofluid; Viscous dissipation.

## NOMENCLATURE

$C$	nanoparticle volume fraction	$Shr$	reduced Sherwood number
$C_f$	skin friction coefficient	$T$	temperature inside the boundary layer
$C_{fr}$	reduced skin friction coefficient	$T_w$	wall temperature
$C_p$	specific heat capacity at constant pressure	$T_\infty$	ambient temperature
$C_w$	nanoparticle volume fraction at the surface	$U_\infty$	free stream velocity
$C_\infty$	ambient nanoparticle volume fraction	$u$	velocity components along the $x$ -direction
$D_B$	Brownian diffusion coefficient	$u_w$	plate velocity
$D_T$	thermophoresis diffusion coefficient	$v$	velocity components along the $y$ -direction
$Ec$	Eckert number	$v$	plate velocity parameter
$j_w$	surface mass flux	$Y$	similarity variable
$k$	thermal conductivity	$y_\infty$	boundary layer thickness
$Le$	Lewis number	$\Delta y$	step size
$N_b$	Brownian motion parameter	$\theta$	dimensionless temperature of the fluid
$N_r$	thermophoresis parameter	$\sim$	dynamic viscosity
$Nu_x$	local Nusselt number	$\epsilon$	kinematic viscosity
$Nur$	Reduced Nusselt number	$\rho$	fluid density
$Pr$	Prandtl number	$\dagger$	ratio of the effective heat capacity of the nanoparticle material and the heat capacity of ordinary fluid.
$q_w$	surface heat flux	$\dagger_w$	surface shear stress
$Re_x$	local Reynolds number		

$Sh_x$  Sherwood number

$W$  rescaled nanoparticle volume fraction  
 $\mathcal{E}$  stream function

## 1. INTRODUCTION

Convection boundary layer flow plays an important role in engineering and industrial activities. These configurations are applied in thermal effects managements in many industrial outputs for example in electronic devices, computer power supply and also in engine cooling system such as heat sink in car radiator. Because of the large contributions, this topic has attracted many researchers to study and expand the knowledge so that it could be applied in order to handle the thermal problems produced by these industrial outputs (Pop and Ingham 2001; Salleh *et al.* 2010).

The study of boundary layer flow on a constant speed moving plate was first studied by Sakiadis (1961). Due to entrainment of the ambient fluid, this boundary layer flow is quite different from Blasius flow past a flat plate. Sakiadis' theoretical predictions for Newtonian fluids were later corroborated experimentally by Tsou *et al.* (1967). Karwe and Jaluria (1988) considered the mixed convection from a moving plate in rolling and extrusion processes. Other papers that considered the boundary layer flow over a moving plate are Kumari and Nath (1996), Ali and Al-Yousef (1998), Chen (1999) and Elbashbeshy and Bazid (2000) which introduced the force convection, impulsive motion, suction or injection effects and temperature dependent viscosity, respectively. Bataller (2008) and Ishak *et al.* (2011) observed the radiation effects on the thermal boundary layer flow for Blasius and Sakiadis flows with convective boundary conditions, respectively. It was found that the presence of thermal radiation and convective boundary conditions reduce the heat transfer rate at the surface. Next, the effect of transpiration on the flow and self-similar boundary layer flow over a moving surface was studied by Weidman *et al.* (2006) and Ishak *et al.* (2009). A permeable surface was considered and it was found that dual solutions were obtained in both studies. Weidman *et al.* (2006) reported that the solution is unique for positive values of the plate velocity parameter, while dual solutions exist for its negative values, up to the critical value, beyond which no solution is in existence. The stability analysis revealed that the upper branch solution is stable, while the lower branch solution is unstable.

The investigations involving the flow on a moving plate were also extended to other type of fluids such as viscoelastic fluid, micropolar fluid and nanofluid by many investigators including Abel *et al.* (2005), Anuradha and Krishnambal (2010) and recently by Khan *et al.* (2014) who considered the flow in a viscoelastic fluid. The MHD and buoyancy effects on a moving stretching surface were analysed and discussed. Aman *et al.* (2013) and Kundu *et al.* (2015) investigated the MHD effects in micropolar

fluid while Sandeep and Sugunamma (2014) considered the unsteady hydromagnetic flow past an impulsively moving vertical plate in a porous medium. Furthermore, Bachok *et al.* (2010) and Ro ca and Pop (2014) investigated the steady and unsteady boundary layer flow of a nanofluid past a moving surface in an external uniform free stream, respectively. Both studies considered the Bongiorno-Darcy nanofluid model. The effects of the Prandtl number, plate velocity parameter and the nanofluid parameter which is Brownian motion parameter, thermophoresis parameter and Lewis number was analysed and discussed. The numerical solutions were obtained by using the Keller-box method and the bvp4c package in Matlab, respectively.

In considering the viscous dissipation effects, from literature study it is found that Gebhart (1962) is probably the first who studied viscous dissipation in free convection flow. The viscous dissipation effects on unsteady free convective flow over a vertical porous plate was then investigated by Soundalgekar (1972). Vajravelu and Hadjinicolaou (1993) then studied the viscous dissipation effects on the flow and heat transfer over a stretching sheet. Chen (2004) observed the heat and mass transfer in MHD free convection from a vertical surface with Ohmic heating and viscous dissipation effects. Next, Partha *et al.* (2005) and Yirga and Shankar (2013) considered this topic on exponentially stretching surface and stagnation point flow in a nanofluid with thermal radiation and magnetohydrodynamic effects, respectively. It was reported that the presence of viscous dissipation which represented by Eckert number has contributed to the increase in velocity and thermal boundary layer thicknesses while the local Nusselt number decreases. Recently, Pal and Mondal (2014) and Gangadhar (2015) investigated the viscous disipation effects on a stretching plate, Blasius and Sakiadis plate with Soret-Dufour effects, radiation, heat generation, MHD, porous medium and Ohmic heating, respectively.

Motivated by the above factor and contributions, the objective of the present paper is to investigate the boundary layer flow over a moving flat plate in a nanofluid with viscous dissipation. It is known that nanofluid is a fluid containing nanometer-sized particles, called nanoparticles. This type of fluid is believed may enhance thermal conductivity, viscosity, thermal diffusivity and convective heat transfer compared to those base fluids like water and oil. Many investigations on the nanofluid have been done such as Vajravelu *et al.* (2011), Anwar *et al.* (2012), Bachok *et al.* (2012) and recently by Kameswaran *et al.* (2013), Rashad *et al.* (2013), Makinde *et al.* (2013) and Sreenivasulu and Bhaskar (2015). To the best of our knowledge, the present study is never been considered before, so that the reported results are new.

## 2. MATHEMATICAL FORMULATIONS

A steady two-dimensional boundary layer flow over a moving plate immersed in a nanofluid of ambient temperature  $T_\infty$  is considered. It is assumed that  $T$  be the temperature inside the boundary layer,  $T_w$  be the wall temperature,  $U_\infty$  be the free stream velocity and  $u_w(x) = vU_\infty$  be the plate velocity where  $v$  is the plate velocity parameter (see Weidman *et al.* 2006). Furthermore,  $C$  is the nanoparticle volume fraction,  $C_w$  is the nanoparticle volume fraction at the surface and  $C_\infty$  is the ambient nanoparticle volume fraction. The physical model and coordinate system of this problem is shown in Fig. 1. The governing boundary layer equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \epsilon \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \dagger \left[ D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 \right] + \frac{\tilde{\nu}}{\rho C_p} \left( \frac{\partial u}{\partial y} \right)^2, \tag{3}$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2}, \tag{4}$$

subject to the boundary conditions

$$\begin{aligned} u = u_w(x) = vU_\infty, v = 0, T = T_w, C = C_w \text{ at } y = 0, \\ u = U_\infty, T \rightarrow T_\infty, C \rightarrow C_\infty \text{ as } y \rightarrow \infty, \end{aligned} \tag{5}$$

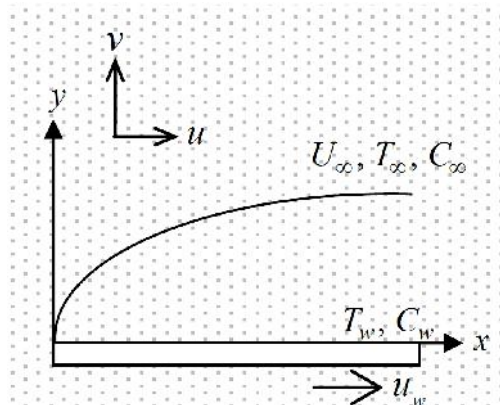


Fig. 1. Physical model of the coordinate system.

where  $u$  and  $v$  are the velocity components along the  $x$  and  $y$  directions, respectively,  $\tilde{\nu}$  is the dynamic viscosity,  $\epsilon$  is the kinematic viscosity,  $\rho$  is the fluid density,  $k$  is the thermal conductivity

and  $C_p$  is the specific heat capacity at constant pressure. Next,  $D_B$  is the Brownian diffusion coefficient,  $D_T$  is the thermophoresis diffusion coefficient,  $\dagger$  is the ratio of the effective heat capacity of the nanoparticle material and the heat capacity of the ordinary fluid.

The set of similarity transformation for Eqs. (1) to (4) subjected to the boundary conditions (5) is (Weidman *et al.* 2006; Bachok *et al.* 2010)

$$\begin{aligned} y = \left( \frac{U_\infty}{2\epsilon x} \right)^{1/2} \mathfrak{E}, \quad \mathfrak{E} = (2U_\infty \epsilon x)^{1/2} f(\mathfrak{Y}), \\ \mathfrak{W}(\mathfrak{Y}) = \frac{T - T_\infty}{T_w - T_\infty}, \quad w(\mathfrak{Y}) = \frac{C - C_\infty}{C_w - C_\infty}, \end{aligned} \tag{6}$$

where  $W$  and  $w$  are the rescaled nanoparticle volume fraction and dimensionless temperature of the fluid, respectively.  $\mathfrak{E}$  is the stream function defined as  $u = \frac{\partial \mathfrak{E}}{\partial y}$  and  $v = -\frac{\partial \mathfrak{E}}{\partial x}$  which identically satisfy Eq. (1). Then,  $u$  and  $v$  can be derived as

$$\begin{aligned} u = U_\infty f'(\mathfrak{Y}), \\ v = -\left( \frac{U_\infty \epsilon}{2x} \right)^{1/2} f(\mathfrak{Y}) + \frac{U_\infty y}{2x} f'(\mathfrak{Y}), \end{aligned} \tag{7}$$

By substituting Eqs. (6) and (7) into Eqs. (2) to (4), we have

$$f''' + ff'' = 0 \tag{8}$$

$$\frac{1}{Pr} \mathfrak{W}'' + f \mathfrak{W}' + N_b \mathfrak{W}' + N_t \mathfrak{W}^2 + Ec f'^2 = 0, \tag{9}$$

$$w'' + \frac{N_t}{N_b} \mathfrak{W}'' + Le f w' = 0 \tag{10}$$

Where  $Pr = \frac{\epsilon \rho C_p}{k}$  is the Prandtl number,

$N_b = \frac{\dagger D_B (C_w - C_\infty)}{\epsilon}$  is the Brownian motion

parameter,  $N_t = \frac{\dagger D_T (T_w - T_\infty)}{T_\infty \epsilon}$  is the

thermophoresis parameters,  $Ec = \frac{(U_\infty)^2}{C_p (T_w - T_\infty)}$  is the

Eckert number and  $Le = \frac{\epsilon}{D_B}$  is the Lewis number.

The boundary conditions (5) become

$$\begin{aligned} f(0) = 0, f'(0) = v, \mathfrak{W}(0) = 1, w(0) = 1, \\ f'(\mathfrak{Y}) \rightarrow 1, \mathfrak{W}(\mathfrak{Y}) \rightarrow 0, w(\mathfrak{Y}) \rightarrow 0, \text{ as } \mathfrak{Y} \rightarrow \infty \end{aligned} \tag{11}$$

Note that  $v > 0$  corresponds to downstream movement of the plate from the origin (Weidman *et al.* 2006). The physical quantities of interest are the skin friction coefficient  $C_f$  the local Nusselt number  $Nu_x$  and the local Sherwood number  $Sh_x$

which are given by

$$C_f = \frac{\dagger_w}{\dots u_e^2}, Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \tag{12}$$

$$Sh_x = \frac{xj_w}{D_B(C_w - C_\infty)}.$$

The surface shear stress  $\dagger_w$ , the surface heat flux  $q_w$  and the surface mass flux  $j_w$  are given by

$$\tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}, \tag{13}$$

$$j_w = -D_B \left( \frac{\partial C}{\partial y} \right)_{y=0}$$

with  $\sim = \dots \epsilon$  being the dynamic viscosity. Using the similarity variables in Eq. (6) give

$$C_f (2Re_x)^{1/2} = f''(0), Nu_x \left( \frac{Re_x}{2} \right)^{-1/2} = -w'(0), \tag{14}$$

$$Sh_x \left( \frac{Re_x}{2} \right)^{-1/2} = -w'(0)$$

where  $Re_x = \frac{U_\infty x}{\epsilon}$  is the local Reynolds number. It is worth mentioning that the physical quantities of interest in the present context are

$$C_f (2Re_x)^{1/2}, Nu_x \left( \frac{Re_x}{2} \right)^{-1/2} \text{ and } Sh_x \left( \frac{Re_x}{2} \right)^{-1/2}$$

which are referred as the reduced skin friction coefficient, the reduced Nusselt number and the reduced Sherwood number and can be denoted as  $C_{fr}$ ,  $Nur$  and  $Shr$  which are represented by  $f''(0)$ ,  $-w'(0)$  and  $-w'(0)$ , respectively.

### 3. NUMERICAL METHOD

The transformed ordinary differential equations subject to the associated boundary conditions are solved numerically using the Keller-box method. As described in the books by Na (1979) and Cebeci and Bradshaw (1988), the solution is obtained in the following four steps: Firstly, reduce Eqs. (8) to (11) to a first-order system. Secondly, write the difference equations using central differences; Thirdly, linearize the resulting algebraic equations by Newton's method, and write them in the matrix-vector form; and finally, solve the linear system by the block tridiagonal elimination technique.

### 4. RESULTS AND DISCUSSION

Six parameters are considered, namely the Prandtl number  $Pr$ , the plate velocity parameter  $v$ , the Brownian motion parameter  $N_b$ , the thermophoresis parameter  $N_t$ , the Eckert number  $Ec$ , and the Lewis number  $Le$ . The step size

$\Delta y = 0.02$  and the boundary layer thickness  $y_\infty$  from 2.5 to 8 were used in obtaining the numerical results. From the literature review, in considering the boundary layer flow over a stretching surface in nanofluids, it was found that  $N_b$  and  $N_t$  effects are practically studied at range between 0.1 and 0.5 while  $Le$  in the range of 1 to 40 (Khan and Pop 2010; Noghrehabadi *et al.* 2012; Nandy and Mahapatra 2013). Therefore, in the present study, the same range is considered in our analysis and discussion.

Table 1 shows the comparison values of  $-w'(0)/\sqrt{2}$  with previous results by Ro ca and Pop (2014) for various values of Prandtl number  $Pr$ . It is found that they are in good agreement. We can conclude that this method works efficiently for the present problem.

Table 2 shows the values of  $Nur$ ,  $Shr$  and  $C_{fr}$  for various values of  $v$ . From this table, it is found that the present of plate velocity parameter  $v$  results to the increase of  $Nur$  and  $Shr$ . Meanwhile, the skin friction coefficient  $C_{fr}$  decreases as  $v$  increases.

Note that when  $v = 0$ ,  $C_{fr} = 0.4696$  which is in a very good agreement with Blasius (1908). Next,  $C_{fr} = 0$  for  $v = 1$  due to the plate and the fluid move with the same velocity which results in no velocity gradient, i.e. there is no friction at the fluid-solid interface.

**Table 1 Comparison values of  $-w'(0)/\sqrt{2}$  with previous published results for various values of  $Pr$  when  $v = 0, N_b = 0, N_t = 0, Ec = 0, Le = 10$**

Pr	Ro ca and Pop (2014)	Present
0.7	0.29268	0.292680
0.8	0.30691	0.306917
1	0.33205	0.332057
5	0.57668	0.576689
10	0.72814	0.728141

**Table 2 Values of  $Nur$ ,  $Shr$  and  $C_{fr}$  for various values of  $v$  when  $Pr = 7, N_b = 0, N_t = 0, Ec = 0.1$  and  $Le = 10$**

$v$	$Nur$	$Shr$	$C_{fr}$
0	0.3747	1.1672	0.4696
0.1	0.4705	1.3369	0.4625
0.5	0.7875	1.8657	0.3288
1	1.0717	2.3805	0
2	1.2994	3.3099	-1.0191

Table 3 presents the values of the reduced Nusselt number  $Nur$  and Sherwood number  $Shr$  for various values of  $N_b$  and  $N_t$ . From this table, it is

concluded that the increase of both parameters  $N_b$  and  $N_t$  results to the decrease of  $Nur$  while  $Shr$  increases with the increase of  $N_t$ . Physically, it is suggested that the small values of  $N_b$  and  $N_t$  enhance convective heat transfer capabilities while the large values of  $N_b$  and  $N_t$  enhance the convective mass transfer capabilities.

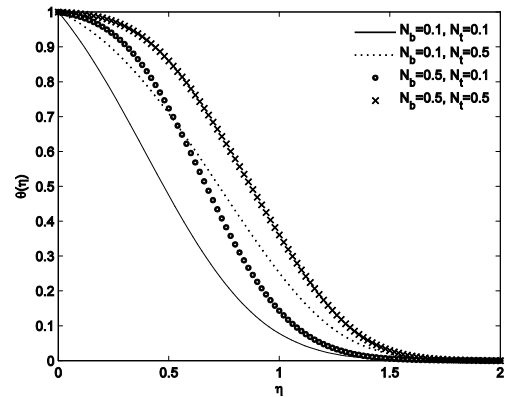
Figures 2 to 4 illustrate the temperature profiles for various values of  $N_b$ ,  $N_t$ ,  $Le$ ,  $Ec$  and  $v$ . These Figs are plotted in order to understand the behavior of the thermal boundary layer thickness for various values of parameter discussed. From Fig. 2, it is found that as  $N_b$  increases, the thermal boundary layer thickness also increases. According to Anwar *et al.* (2013), this is due to the fact that larger values of  $N_b$  have a large extent of fluid hence thickening the thermal boundary layer thickness. The same trends happen for parameter  $N_t$ . This result from the deeper penetration into the fluid which cause to the thickening of thermal boundary layer thickness.

**Table 3 Values of  $Nur$  and  $Shr$  for various values of  $N_b$  and  $N_t$  when  $Pr = 7, Ec = 0.1, Le = 10$  and  $v = 0.5$**

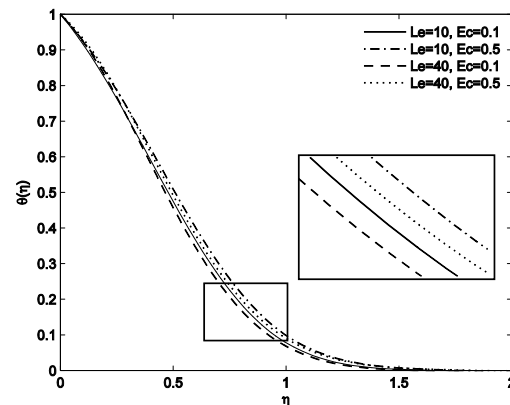
$N_b$	$N_t$	$Nur$	$Shr$
0.1	0.1	0.7875	1.8657
0.2	0.1	0.4923	1.9951
0.3	0.1	0.2959	2.0091
0.4	0.1	0.1712	2.0010
0.5	0.1	0.0952	1.9884
0.1	0.2	0.6047	2.0601
0.1	0.3	0.4708	2.3624
0.1	0.4	0.3722	2.6994
0.1	0.5	0.2990	3.0319

In Fig. 3, it is found that the increase of  $Le$  results to the decrease in thermal boundary layer thickness. The opposite trend occurs for  $Ec$  where the increase of this parameter results to the increase of thermal boundary layer thickness. This may be explained as follows: The increase of  $Ec$  directly proportional to the increase of external velocity (see definition of  $Ec$ ) and this situation reduces the effect of the plate velocity parameter  $v$ , which thickening the thermal boundary layer thickness.

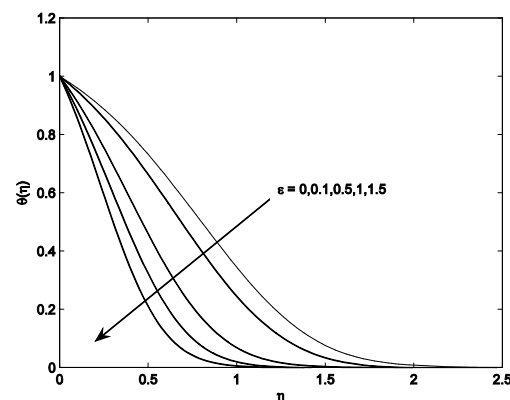
Figure 4 shows the temperature profiles for various values of  $v$ . It is found that the present of plate velocity parameter ( $v > 0$ ) reduces the thermal boundary layer thickness. Physically, the increase of  $v$  results to the increase of ratio velocity differences between the plate and the fluid which enhance the fluid to move away from the plate region rapidly. This situation reduces the thermal diffusivity and thinning the thermal boundary layer thicknesses.



**Fig. 2. Temperature profiles  $\theta(\eta)$  for various values of  $N_b$  and  $N_t$  when  $Pr = 7, Ec = 0.1, Le = 10$  and  $v = 0.5$ .**



**Fig. 3. Temperature profiles  $\theta(\eta)$  for various values of  $Le$  and  $Ec$  when  $N_b = N_t = 0.1, Pr = 7$  and  $v = 0.5$ .**

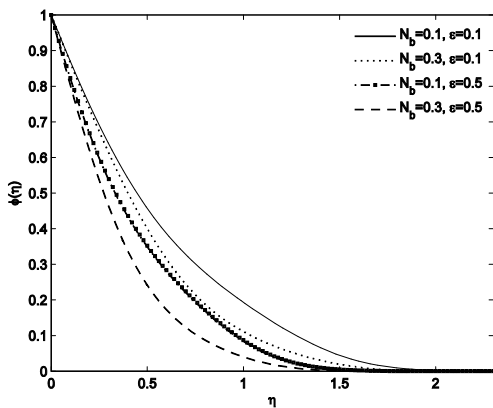


**Fig. 4. Temperature profiles  $\theta(\eta)$  for various values of  $v$  when  $N_b = N_t = 0.1, Pr = 7, Le = 10$  and  $Ec = 0.1$ .**

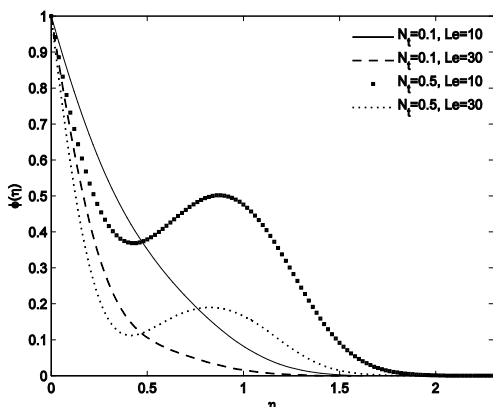
Figures 5 to 7 show the variation of nanoparticle volume fraction  $w(y)$  for various values of parameters. The increase of  $N_b$  in Fig. 5 results to

the decrease of  $w(y)$ . Also, the effects of  $V$  is similar as in Fig. 4. The moving effects reduces the nanoparticle volume fraction  $w(y)$ .

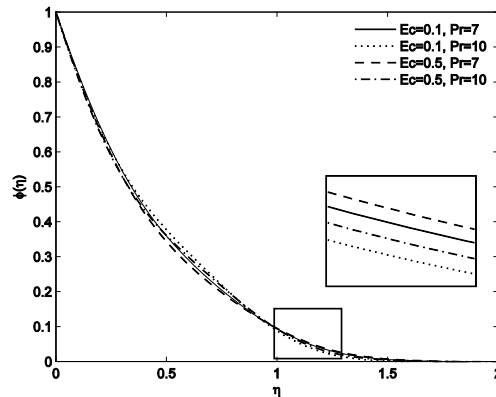
The effects of  $N_t$  and  $Le$  are shown in Fig. 6. Both parameters consume the contra trend where the increase of  $Le$  is to the decrease the nanoparticle volume fraction  $w(y)$  while  $w(y)$  increases as  $N_t$  increases. Further, the nanoparticle volume fraction  $w(y)$  increases monotonically as  $Pr$  decreases. Meanwhile, the Eckert number  $Ec$  shows the contradict trends. Large values of  $Ec$  means large viscous dissipation effect and this enhance the nanoparticle volume fraction  $w(y)$ . The variations of the nanoparticle volume fraction for both parameters  $Pr$  and  $Ec$  are plotted in Fig. 7.



**Fig. 5. Variation of nanoparticle volume fraction  $w(y)$ : for various values of  $N_b$  and  $v$  when  $Pr = 7, N_t = Ec = 0.1$  and  $Le = 10$ .**



**Fig. 6. Variation of nanoparticle volume fraction  $w(y)$ : for different values of  $N_t$  and  $Le$  when  $Pr = 7, N_b = Ec = 0.1$  and  $v = 0.5$ .**



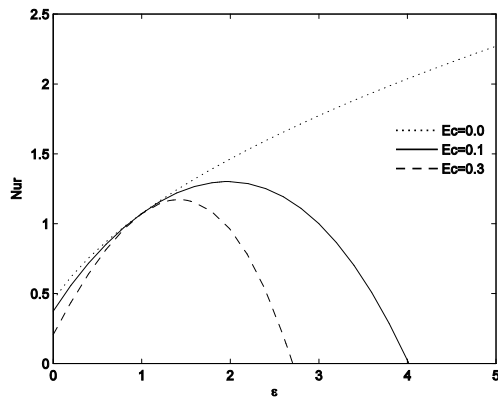
**Fig. 7. Variation of nanoparticle volume fraction  $w(y)$ : for different values of  $Pr$  and  $Ec$  when  $N_b = N_t = 0.1, Le = 10$  and  $v = 0.5$ .**

Figures 8 and 9 illustrate the variation of the reduced Nusselt number  $Nur$  and the reduced Sherwood number  $Shr$  with  $V$  for various values of  $Ec$ . From Fig. 8, it is found that the presence of viscous dissipation ( $Ec \neq 0$ ) changes the variation of  $Nur$  curve to a quadratic curve. Also, the presence of  $Ec$  reduces the range of  $v$  for which the solution exists. It is clearly shown when  $Ec = 0.1$ , the physical acceptable solution occurs until  $v = 4.0189$  while when the value of  $Ec$  increases ( $Ec = 0.3$ ),  $Nur$  stops at  $v = 2.7153$ . Since  $Nu \cong 0$ , this means that there is no convection occurs beyond this value, which leads to pure conduction heat transfer. Furthermore, it is noticed that when  $v = 1$ , which results from the equivalent of stretching and external velocity at the plate surface,  $Ec$  does not affect the values of  $Nur$ .

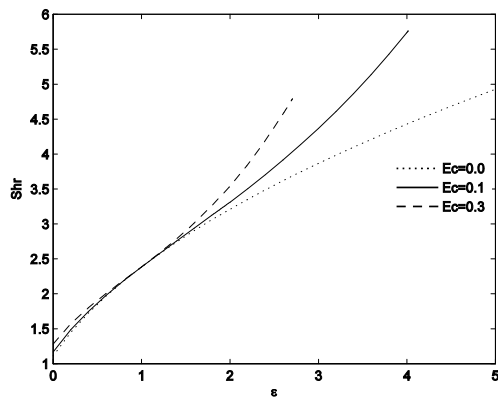
In Fig. 9, it is noticed that the physical acceptable solution for  $v$  has a similar trend as in Fig. 8. Further, it is found that, at large values of  $V$ , larger values of  $Ec$  produce large values of  $Shr$  which physically means large convective mass transfer. This is due to the increase of the ratio of kinetic energy over enthalpy in increasing  $Ec$ . Also, it is noticed that the variation of  $Shr$  is unique for all  $Ec$  when  $v = 1$  as in Fig. 8.

Figures 10 and 11 show the variation of  $Nur$  and  $Shr$  with  $Pr$  for various values of  $Le$ . From Fig. 10, it is found that small values of  $Pr$  such as liquid metal ( $Pr \ll 1$ ), produce very small values of  $Nur$  which imply no convection occurs or the heat transfer is in a pure conduction situation. It is realistic since liquid metals have high thermal conductivity but low viscosity. For fixed values of  $Pr$ , increasing  $Le$  results in the decrease of  $Nur$  but an increase of  $Shr$ .

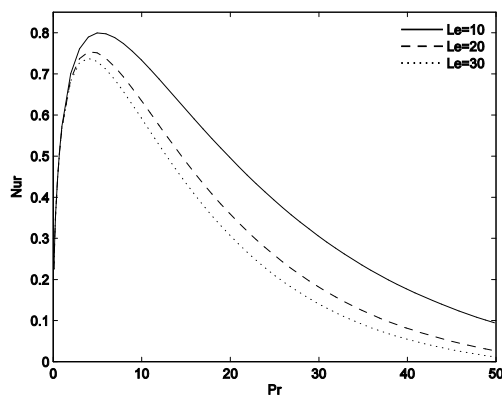




**Fig. 8.** Variation of reduced Nusselt number with  $V$  for various values of  $Ec$  when  $Pr = 7$ ,  $Le = 10$  and  $N_b = N_t = 0.1$ .



**Fig. 9.** Variation of reduced Sherwood number with  $V$  for various values of  $Ec$  when  $Pr = 7$ ,  $Le = 10$  and  $N_b = N_t = 0.1$ .

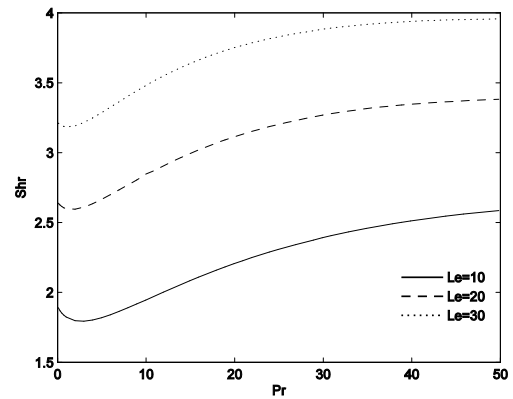


**Fig. 10.** Variation of reduced Nusselt number with  $Pr$  for various values of  $Le$  when  $N_b = N_t = Ec = 0.1$  and  $v = 0.5$ .

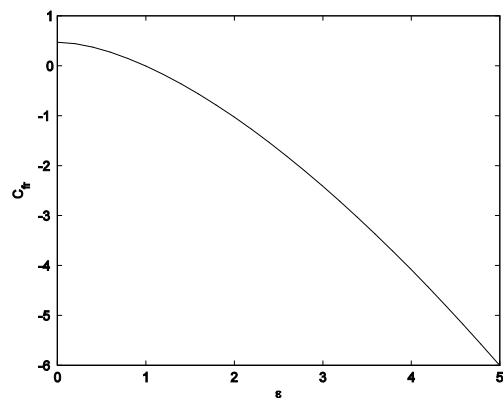
Lastly, Fig. 12 presents the variation of the reduced skin friction coefficient  $C_{fr}$  for various values of  $v$  which produce  $f'(0) = v$  and  $f'(y) = 1$  as  $y \rightarrow \infty$ . From this figure, it is found that the value of  $C_{fr}$  is

positive for  $v < 1$ , and  $C_{fr}$  decreases to 0 as  $v$  approach 1. The value  $C_{fr} = 0$  at  $v = 1$  is due to the plate moves in the same velocity with the fluid. Further, from this figure, we can conclude that  $C_{fr}$  decreases as  $v$  increases.

From this study it is worth mentioning that only moving parameter affects the velocity profiles and the value of the skin friction coefficient. It is clear from the ordinary differential Eqs. (8) to (10) and boundary conditions (11).



**Fig. 11.** Variation of reduced Sherwood number with  $Pr$  for various values of  $Le$  when  $N_b = N_t = Ec = 0.1$  and  $v = 0.5$ .



**Fig. 12.** Reduced skin friction coefficient for various values of  $v$ .

## 5. CONCLUSION

In this paper, the boundary layer flow over a moving plate in a nanofluid in the presence of viscous dissipation was numerically studied. It was shown how the Prandtl number  $Pr$ , plate velocity parameter  $v$ , Brownian motion parameter  $N_b$ , thermophoresis parameter  $N_t$ , Eckert number  $Ec$  and Lewis number  $Le$  affect the values of the reduced Nusselt and Sherwood numbers and the skin friction coefficient as well as the concentration and temperature profiles. As a conclusion, the thermal boundary layer thickness depends strongly on these parameters. It was found that in the

presence of viscous dissipation, the range of the plate velocity parameter for which the solution exists reduces, which physically leads to pure conduction to occur. Further, the same situation occurs to the nanofluid with low Prandtl number. This may be explained as the low Prandtl value means the fluid is low in viscosity but high in thermal conductivity such as liquid metal. The increase of  $N_b$ ,  $N_t$  and  $Ec$  results to the increase in thermal boundary layer thickness. Meanwhile, the increase of  $Pr$ ,  $\nu$  and  $Le$  reduced the thermal boundary layer thickness.

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