Compatible Pair of Nontrivial Actions for Cyclic Groups of 3-Power Order

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Abstract — A compatible action played an important role before computing the nonabelian tensor product. Number of compatible actions will give the different nonabelian tensor product. Thus, this paper present some exact number of compatible pairs of actions for cyclic groups focusing on groups of 3-power order. Some necessary and sufficient number theoretical conditions for a pair of cyclic groups of *p*-power order with nontrivial actions act compatibly on each other are applied to find the exact number of compatible pairs of actions. Group, Algorithm and Programming (GAP) software is used to find some examples on a different case. Results on the compatible pair of nontrivial actions of order three and nine are given.

Keywords— *cyclic groups; compatible actions; nonabelian tensor product;*

1. INTRODUCTION

Let G and H be groups which act on each other and each of which acts on itself by conjugation, then the actions are compatible if ${}^{(g_h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g'))$ and ${}^{(h_g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$ for all $g, g' \in G$ and $h, h' \in H$. This compatible action played an important role before computing the nonabelian tensor product. Brown and Loday [1] were the first who introduced the concept of the nonabelian tensor product of groups with the compatible actions. The construction of this concept has its origins in the algebraic K-theory and in homotopy theory. Several authors had investigated the concept of the nonabelian tensor product of groups, starting with Brown et. al [2] had studied group theoretical aspects of the nonabelian tensor product extensively. An overview on some known results on the nonabelian tensor product and literature was illustrated by Kappe [3]. There are many researchers had investigated the compatibly conditions. Visscher [4] studied the nonabelian tensor product of *p*-power order and he focused on the cyclic groups. He provided a necessary and sufficient conditions for a pair of cyclic groups to act compatibility. Visscher [4] also gave a complete conditions when one of the actions is trivial or both actions are trivial in cyclic groups of p-power order. Mohamad et. al [5] investigated the nonabelian tensor product and the compatible actions for the pair of cyclic groups of order p^2 , and he gave a necessary and sufficient condition for a pair of cyclic groups of p-power order to act compatibility on each other by considering the order of the actions is p, where p is an odd prime. Two years later, Sulaiman et. al [6] computed the compatible pair of nontrivial actions for some cyclic groups of 2-power order involving at least one of the actions having order greater than two. In this research, all the compatible actions involving both actions having order greater than two will be determined. Groups, Algorithm and Programming (GAP) software [7] is used to determine the conditions in which the actions satisfying the compatibility conditions in the finite cyclic groups of 3-power order.

2. SOME OF THE PRELIMINARY RESULTS ON THE COMPATIBILITY CONDITIONS

The nonabelian tensor product for a pair of a groups *G* and *H* is denoted by $G \otimes H$ which is defined only when *g* and *h* act on each other for all $g \in G$ and $h \in H$ such that the actions satisfying the compatibility conditions. Starting with the definitions are given as follows:

Definition 2.1 [5]

Let G and H be groups. An action of G on H is a mapping, $\Phi: G \to Aut(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$

for all $g, g' \in G$ and $h \in H$.

Definition 2.2 [5]

Let G and H be groups which act on each other and each of which acts on itself by conjugation. Then the actions are compatible if

$${}^{(^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{(^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

Since our study focusing on the cyclic groups, the compatibility conditions are hold for the abelain groups will be decrease to ${}^{(^{g}h)}g' = {}^{h}g'$ and ${}^{(^{h}g)}h' = {}^{g}h'$ for all $g, g' \in G$ and $h, h' \in H$. For the compatibility of actions of G and H on each other, its sufficient for the compatibility conditions to hold on the generators of G and H. Visscher [4] had done that and he provided some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibly on each other.

3. COMPATIBLE CONDITIONS OF NONTRIVIAL ACTION

From the past section, we have seen that Visscher [4] provided some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibility on each other and he had completed the classification of compatible mutual actions for a pair of cyclic groups of p-power order where p is an odd prime. Our aim is to determine the compatible conditions when both order of the actions are greater than two. The computations of the compatible conditions are doing by using GAP software. New GAP coding has been built to compute the compatible actions.

Let $G = \langle x \rangle \cong C_{3^{\alpha}}$ and $H = \langle y \rangle \cong C_{3^{\beta}}$ be finite cyclic groups of 3-power order. Let the actions of *G* and *H* on each other be such that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$. Now, we consider that both actions have order three. The compatible conditions which are satisfied by *k* and *l* are presented in **Table 1** when G = H and in **Table 2** when $G \neq H$.

Table 1: The values of k and l in which the action of order three satisfying the compatible conditions when G = H.

Groups	k	l	Groups	k	l
G = H = C	4	4	G = H = C	82	82
$0 - 11 - C_{3^2}$	7	7	$0 = 11 = C_{3^5}$	163	163
$G = H = C_{3^3}$	10	10	C = H = C	244	244
	19	19	$G = H = C_{3^6}$	487	487
$G = H = C_{3^4}$	28	28	G = H = C	730	730
	55	55	$0 = 11 = C_{3^7}$	1459	1459

Groups	k	l		Groups	k	l	Groups	k	l
$G = C_{2}$	4	10		$G = C_{-3}$	10	4	$G = C_{-4}$	28	4
and	4	19		and	10	7	and	28	7
and	7	10		and	19	4	and	55	4
$H = C_{3^3}$	7	19		$H = C_{3^2}$	19	7	$H = C_{3^2}$	55	7
$G = C_{2^2}$	4	28		$G = C_{2^3}$	10	28	$G = C_{24}$	28	10
and	4	55		and	10	55	and	28	19
	7	28			19	28		55	10
$H = C_{3^4}$	7	55		$H = C_{3^4}$	19	55	$H = C_{3^3}$	55	19
$G = C_{3^2}$	4	82		$G = C_{3^3}$	10	82	$G = C_{3^4}$	28	82
and	4	163		and	10	163	and	28	163
	7	82			19	82		55	82
$H = C_{3^5}$	7	163		$H = C_{3^5}$	19	163	$H = C_{3^5}$	55	163
$G = C_{3^2}$	4	244		$G = C_{3^3}$	10	244	$G = C_{3^4}$	28	244
and	4	487		and	10	487	and	28	487
and	7	244			19	244	and	55	244
$H = C_{3^{6}}$	7	487		$H = C_{3^6}$	19	487	$H = C_{3^{6}}$	55	487
$G = C_{12}$	4	730		$G = C_{2^3}$	10	730	$G = C_{24}$	28	730
3-	4	1459		and	10	1459	and	28	1459
and	7	730		and	19	730	and	55	730
$H = C_{3^7}$	7	1459		$H = C_{3^7}$	19	1459	$H = C_{3^7}$	55	1459
Groups	k	l		Groups	k	l	Groups	k	l
$Groups$ $G = C_{2^5}$	k 82	<i>l</i> 4	-	$Groups$ $G = C_{2^6}$	k 244	<i>l</i> 4	Groups $G = C_{2^7}$	k 730	<i>l</i> 4
$Groups$ $G = C_{3^{5}}$ and	k 82 82	<i>l</i> 4 7	-	$G = C_{3^6}$	k 244 244	<i>l</i> 4 7	$Groups$ $G = C_{3^7}$ and	k 730 730	<i>l</i> 4 7
$Groups$ $G = C_{3^5}$ and	k 82 82 163	l 4 7 4		$Groups$ $G = C_{3^6}$ and	k 244 244 487	<i>l</i> 4 7 4	$Groups$ $G = C_{3^7}$ and	k 730 730 1459	<i>l</i> 4 7 4
$Groups$ $G = C_{3^5}$ and $H = C_{3^2}$	k 82 82 163	<i>l</i> 4 7 4 7		$Groups$ $G = C_{3^6}$ and $H = C_{3^2}$	k 244 244 487 487	<i>l</i> 4 7 4 7	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$	k 730 730 1459 1459	<i>l</i> 4 7 4 7
$Groups$ $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$	k 82 82 163 163 82	<i>l</i> 4 7 4 7 10		$Groups$ $G = C_{3^6}$ and $H = C_{3^2}$ $G = C_{7^6}$	k 244 244 487 487 244	<i>l</i> 4 7 4 7 10	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$ $G = C_{3^{7}}$	k 730 730 1459 1459 730	<i>l</i> 4 7 4 7 10
$Groups$ $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$	k 82 82 163 82 82 82	<i>l</i> 4 7 4 7 10 19	-	$Groups$ $G = C_{3^6}$ and $H = C_{3^2}$ $G = C_{3^6}$ ond	k 244 244 487 487 244 244	<i>l</i> 4 7 4 7 10 19	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$ $G = C_{3^{7}}$ ond	k 730 730 1459 1459 730 730	<i>l</i> 4 7 4 7 10 19
$Groups$ $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$ and $H = C_{3^5}$	k 82 82 163 82 82 82 163	<i>l</i> 4 7 4 7 10 19 10		$Groups$ $G = C_{3^{6}}$ and $H = C_{3^{2}}$ $G = C_{3^{6}}$ and $H = C_{3^{6}}$	k 244 244 487 487 244 244 487	<i>l</i> 4 7 4 7 10 19 10	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$ $G = C_{3^{7}}$ and $H = C_{3^{7}}$	k 730 730 1459 1459 730 730 730 1459	<i>l</i> 4 7 4 7 10 19 10
$Groups$ $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$ and $H = C_{3^3}$	k 82 82 163 163 82 82 163 163	<i>l</i> 4 7 4 7 10 19 10 19		$Groups$ $G = C_{3^6}$ and $H = C_{3^2}$ $G = C_{3^6}$ and $H = C_{3^3}$	k 244 244 487 487 244 244 244 487 487 487	<i>l</i> 4 7 4 7 10 19 10 19	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$ $G = C_{3^{7}}$ and $H = C_{3^{3}}$	k 730 730 1459 1459 730 730 730 1459 1459	<i>l</i> 4 7 4 7 10 19 10 19
$Groups$ $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$ and $H = C_{3^3}$ $G = C_{3^5}$	k 82 82 163 82 82 163 82 163 82 163 82 82 163 82 163 82	<i>l</i> 4 7 4 7 10 19 10 19 28		$Groups$ $G = C_{3^{6}}$ and $H = C_{3^{2}}$ $G = C_{3^{6}}$ and $H = C_{3^{3}}$ $G = C_{3^{6}}$	k 244 244 487 487 244 244 244 244 247 248 244 244 244 244 244 244 487 244	<i>l</i> 4 7 4 7 10 19 10 19 28	$Groups$ $G = C_{3^{2}}$ and $H = C_{3^{2}}$ $G = C_{3^{2}}$ and $H = C_{3^{3}}$ $G = C_{3^{2}}$	k 730 730 1459 1459 730 730 730 1459 730 730 730 730 730 730 730 730	<i>I</i> 4 7 4 7 10 19 10 19 28
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Groups $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$ and $H = C_{3^3}$ $G = C_{3^5}$ and $H = C_{3^4}$ $G = C_{3^5}$ and $H = C_{3^6}$ $G = C_{3^5}$	k 82 82 163 82 82 163 82 163 163 82 163 82 163 82 163 82 163 82 163 82 163 82 163 82 163 82	l 4 7 10 19 10 19 28 55 28 55 24 487 244 487 730		Groups $G = C_{3^6}$ and $H = C_{3^2}$ $G = C_{3^6}$ and $H = C_{3^3}$ $G = C_{3^6}$ and $H = C_{3^4}$ $G = C_{3^6}$ and $H = C_{3^5}$ $G = C_{3^6}$	k 244 244 487 487 244 244 487 487 244 487 244 487 244 244 244 244 487 244 487 244 487 244 244 244 244 244 244 244 244 487 244	I 4 7 10 19 10 19 28 55 28 55 82 163 82 163 730	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$ $G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$ $G = C_{3^{7}}$	k 730 730 1459 1459 730 730 1459 730 1459 1459 730 1459 730 730 1459 730 1459 730 1459 730 730 730 730 730 730 730 730 730 730 730 730 730 730 730 730 730	I 4 7 10 19 10 19 28 55 28 55 344 487 244 487 2188
Groups $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$ and $H = C_{3^3}$ $G = C_{3^5}$ and $H = C_{3^4}$ $G = C_{3^5}$ and $H = C_{3^6}$ $G = C_{3^5}$ and $H = C_{3^6}$ $G = C_{3^5}$ and	k 82 82 163 82 82 163 82 163 163 82 163 82 163 82 163 82 163 82 163 82 163 82 163 82 163 82 82 82 82 82 82 82 82 82 82 82 82 82	l 4 7 4 7 10 19 10 19 28 55 28 55 244 487 244 487 730 1459		Groups $G = C_{3^6}$ and $H = C_{3^2}$ $G = C_{3^6}$ and $H = C_{3^3}$ $G = C_{3^6}$ and $H = C_{3^4}$ $G = C_{3^6}$ and $H = C_{3^5}$ $G = C_{3^6}$ and $H = C_{3^5}$ $G = C_{3^6}$	k 244 244 487 487 244 244 487 487 487 244 244 487 244 244 244 487 244 487 244 487 244 244 244 244 244 244 244 244 244 244 244 244 244 244 244 244 244 244	I 4 7 10 19 10 19 28 55 28 55 82 163 82 163 730 1459	$Groups$ $G = C_{3^{7}}$ and $H = C_{3^{2}}$ $G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$ $G = C_{3^{7}}$ and $G = C_{3^{7}}$	k 730 730 1459 1459 730 730 730 1459 730 1459 730 1459 730 730 730 1459 730 1459 730 730 730 730 730 730 730 730 730 730 730 730 730 730 730	I 4 7 10 19 10 19 28 55 28 55 344 487 244 487 2188 4375
Groups $G = C_{3^5}$ and $H = C_{3^2}$ $G = C_{3^5}$ and $H = C_{3^3}$ $G = C_{3^5}$ and $H = C_{3^4}$ $G = C_{3^5}$ and $H = C_{3^6}$ $G = C_{3^5}$ and $H = C_{3^6}$ $G = C_{3^5}$ and $H = C_{3^6}$	k 82 82 163 82 82 163 82 163 163 82 163 82 163 82 163 163 82 163 82 163 82 163 82 163 82 163 82 163 82 163 82 163	l 4 7 4 7 10 19 10 19 28 55 28 55 244 487 244 487 730 1459 730		Groups $G = C_{3^6}$ and $H = C_{3^2}$ $G = C_{3^6}$ and $H = C_{3^5}$ $G = C_{3^6}$ and $H = C_{3^4}$ $G = C_{3^6}$ and $H = C_{3^6}$ $G = C_{3^6}$ $G = C_{3^6}$	k 244 244 487 487 244 244 487 487 244 487 244 244 244 244 244 487 244 487 244 244 244 244 244 487 244 487 244 487 244 487 244 487 244 487	I 4 7 10 19 10 19 28 55 28 55 82 163 82 163 730 1459 730	Groups $G = C_{3^7}$ and $H = C_{3^2}$ $G = C_{3^7}$ and $H = C_{3^3}$ $G = C_{3^7}$ and $H = C_{3^4}$ $G = C_{3^7}$ and $H = C_{3^6}$ $G = C_{3^7}$ and $H = C_{3^6}$ $G = C_{3^7}$ and $H = C_{3^6}$ $G = C_{3^7}$ and $G = C_{3^7}$ $G = C_{3^7}$	k 730 730 1459 1459 730 730 1459 730 1459 1459 730 1459 730 730 1459 730 1459 730 730 1459 730 730 1459 730 1459 730 1459	I 4 7 10 19 10 19 28 55 28 55 344 487 244 487 2188 4375 2188

Table 2: The values of *k* and *l* in which the action of order three satisfying the compatible conditions when $G \neq H$.

This lead us to the following results:

Theorem 3.1

Let $C_{3^{\alpha}} = \langle x \rangle$ and $C_{3^{\beta}} = \langle y \rangle$ be cyclic groups of 3-power order. Furthermore, let $C_{3^{\alpha}}$ and $C_{3^{\beta}}$ act on each other so that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$ with (k, 3) = (l, 3) = 1. Then $C_{3^{\alpha}}$ and $C_{3^{\beta}}$ act compatibly on each other when $\alpha = \beta$ with ${}^{y^{3}}x = x$ and ${}^{x^{3}}y = y$, if and only if k and l are congruent to one of the followings:

(i)
$$k = l \equiv 1 + 3^{\alpha - 1} \mod 3^{\alpha}$$
,

(ii) $k = l \equiv 1 - 3^{\alpha - 1} \mod 3^{\alpha}$.

Theorem 3.2

Let $C_{3^{\alpha}} = \langle x \rangle$ and $C_{3^{\beta}} = \langle y \rangle$ be cyclic groups of 3-power order. Furthermore, let $C_{3^{\alpha}}$ and $C_{3^{\beta}}$ act on each other so that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$ with (k,3) = (l,3) = 1. Then $C_{3^{\alpha}}$ and $C_{3^{\beta}}$ act compatibly on each other when $\alpha \neq \beta$ with ${}^{y^{3}}x = x$ and ${}^{x^{3}}y = y$, if and only if k and l are congruent to one of the followings:

- (i) $k \equiv 1 + 3^{\alpha 1} \mod 3^{\alpha} \mod l \equiv 1 + 3^{\beta 1} \mod 3^{\beta}$,
- (ii) $k \equiv 1 + 3^{\alpha 1} \mod 3^{\alpha} \mod l \equiv 1 3^{\beta 1} \mod 3^{\beta}$,
- (iii) $k \equiv 1 3^{\alpha 1} \mod 3^{\alpha} \mod l \equiv 1 + 3^{\beta 1} \mod 3^{\beta}$,
- (iv) $k \equiv 1 3^{\alpha 1} \mod 3^{\alpha} \mod l \equiv 1 3^{\beta 1} \mod 3^{\beta}$.

Next, let the actions of *G* and *H* be on each other such that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$. Now, consider that the actions has order nine and the other one has order three. The compatible conditions are satisfied by *k* and *l* are given in **Table 3** and **Table 4** when G = H and when $G \neq H$ respectively.

Table 3: The values of k for the action has order nine and l for the action has order three satisfying the compatible conditions when the groups G and H are same.

Groups	k	l	Groups	k	l
G = H = C	4	19	G = H = C	82	487
$0 - 11 - 0_{3^3}$	25	19	$0 - 11 - 0_{3^6}$	649	487
G = H = C	10	55	G = H = C	244	1459
$0 - 11 - C_{3^4}$	73	55	$0 - H - C_{3^7}$	487	1459
G = H = C	28	163	G = H = C	730	4375
$0 - 11 - C_{3^5}$	217	163	$0 - 11 - C_{3^8}$	5833	4375

Groups	k	l	Groups	k	l	Groups	k	l
$G = C_{3^3}$	4	55	$G = C_{3^4}$	10	19	$G = C_{3^5}$	28	19
$H = C_{3^4}$	25	55	$H = C_{3^3}$	73	19	$H = C_{3^3}$	217	19
$G = C_{3^3}$	4	163	$G = C_{3^4}$	10	163	$G = C_{3^5}$	28	55
$H = C_{3^5}$	25	163	$H = C_{3^5}$	73	163	$H = C_{3^4}$	217	55
$G = C_{3^3}$	4	487	$G = C_{3^4}$	10	487	$G = C_{3^5}$	28	487
$H = C_{3^6}$	25	487	$H = C_{3^6}$	73	487	$H = C_{3^6}$	217	487
$G = C_{3^3}$	4	1459	$G = C_{3^4}$	10	1459	$G = C_{3^5}$	28	1459
$H = C_{3^7}$	25	1459	$H = C_{3^7}$	73	1459	$H = C_{3^7}$	217	1459
$G = C_{3^3}$	4	4375	$G = C_{3^4}$	10	4375	$G = C_{3^5}$	28	4375
$H = C_{3^8}$	25	4375	$H = C_{3^8}$	73	4375	$H = C_{3^8}$	217	4375
Groups	k	l	Groups	k	1	Groups	k	1
		-			i			•
$G = C_{3^6}$ and	82	19	$G = C_{3^7}$ and	244	19	$G = C_{3^8}$ and	730	19
$G = C_{3^{6}}$ and $H = C_{3^{3}}$	82	19	$G = C_{3^{7}}$ and $H = C_{3^{3}}$	244	19	$G = C_{3^8}$ and $H = C_{3^3}$	730	19
$G = C_{3^6}$ and $H = C_{3^3}$	82 649	19 19	$G = C_{3^7}$ and $H = C_{3^3}$	244 1459	19 19	$G = C_{3^8}$ and $H = C_{3^3}$	730 5833	19 19
$G = C_{3^6}$ and $H = C_{3^3}$ $G = C_{3^6}$ and	82 649 82	19 19 55	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and	244 1459 244	19 19 19 55	$G = C_{3^8}$ and $H = C_{3^3}$ $G = C_{3^8}$ and	730 5833 730	19 19 55
$G = C_{3^{6}}$ and $H = C_{3^{3}}$ $G = C_{3^{6}}$ and $H = C_{3^{4}}$	82 649 82 649	19 19 55 55	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$	244 1459 244 1459	19 19 55 55	$G = C_{3^8}$ and $H = C_{3^3}$ $G = C_{3^8}$ and $H = C_{3^4}$	730 5833 730 5833	19 19 55 55
$G = C_{3^6}$ and $H = C_{3^1}$ $G = C_{3^6}$ and $H = C_{3^4}$ $G = C_{3^6}$ and	82 649 82 649 82	19 19 55 55 163	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{7}}$ and	244 1459 244 1459 244	19 19 55 55 163	$G = C_{3^{8}}$ and $H = C_{3^{3}}$ $G = C_{3^{8}}$ and $H = C_{3^{4}}$ $G = C_{3^{8}}$ and	730 5833 730 5833 730	19 19 55 55 163
$G = C_{3^{6}}$ and $H = C_{3^{7}}$ $G = C_{3^{6}}$ and $H = C_{3^{4}}$ $G = C_{3^{6}}$ and $H = C_{3^{6}}$	82 649 82 649 82 649	19 19 55 55 163 163	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{5}}$	244 1459 244 1459 244 1459	19 19 55 55 163 163	$G = C_{3^{8}}$ and $H = C_{3^{3}}$ $G = C_{3^{8}}$ and $H = C_{3^{4}}$ $G = C_{3^{8}}$ and $H = C_{3^{5}}$	730 5833 730 5833 730 5833 5833	19 19 55 55 163 163
$G = C_{3^{6}}$ and $H = C_{3^{3}}$ $G = C_{3^{6}}$ and $H = C_{3^{4}}$ $G = C_{3^{6}}$ and $H = C_{3^{5}}$ $G = C_{3^{6}}$ and $H = C_{3^{6}}$ and	82 649 82 649 82 649 82 82	19 19 55 55 163 163 1459	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{5}}$ $G = C_{3^{7}}$ and $H = C_{3^{7}}$ and	244 1459 244 1459 244 1459 244	19 19 55 55 163 163 487	$G = C_{3^{8}}$ and $H = C_{3^{3}}$ $G = C_{3^{8}}$ and $H = C_{3^{4}}$ $G = C_{3^{8}}$ and $H = C_{3^{5}}$ $G = C_{3^{8}}$ and $H = C_{3^{5}}$ $G = C_{3^{8}}$ and	730 5833 730 5833 730 5833 730 5833 730	19 19 55 55 163 163 487
$G = C_{3^{6}}$ and $H = C_{3^{7}}$ $G = C_{3^{6}}$ and $H = C_{3^{4}}$ $G = C_{3^{6}}$ and $H = C_{3^{5}}$ $G = C_{3^{6}}$ and $H = C_{3^{7}}$	82 649 82 649 82 649 82 649 82 649	19 19 55 55 163 163 1459 1459	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$	244 1459 244 1459 244 1459 244 1459	19 19 55 55 163 163 487 487	$G = C_{3^8}$ and $H = C_{3^3}$ $G = C_{3^8}$ and $H = C_{3^4}$ $G = C_{3^8}$ and $H = C_{3^5}$ $G = C_{3^8}$ and $H = C_{3^6}$	730 5833 730 5833 730 5833 730 5833 730 5833	19 19 55 55 163 163 487 487
$G = C_{3^6}$ and $H = C_{3^1}$ $G = C_{3^6}$ and $H = C_{3^4}$ $G = C_{3^6}$ and $H = C_{3^5}$ $G = C_{3^6}$ and $H = C_{3^7}$ $G = C_{3^6}$ and $H = C_{3^7}$	82 649 82 649 82 649 82 649 82 649 82	19 19 55 55 163 1459 1459 4375	$G = C_{3^{7}}$ and $H = C_{3^{3}}$ $G = C_{3^{7}}$ and $H = C_{3^{4}}$ $G = C_{3^{7}}$ and $H = C_{3^{5}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$ $G = C_{3^{7}}$ and $H = C_{3^{6}}$	244 1459 244 1459 244 1459 244 1459 244	19 19 55 55 163 163 487 487 4375	$G = C_{3^{8}}$ and $H = C_{3^{3}}$ $G = C_{3^{8}}$ and $H = C_{3^{4}}$ $G = C_{3^{8}}$ and $H = C_{3^{5}}$ $G = C_{3^{8}}$ and $H = C_{3^{6}}$	730 5833 730 5833 730 5833 730 5833 730 5833 730	19 19 55 55 163 163 487 487 487 1459

Table 4: The values of k for the action has order nine and l for the action has order three satisfying the compatible conditions when the groups G and H are different.

The result is given in the following theorem:

Theorem 3.3

Let $G = \langle x \rangle \cong C_{3^{q}}$ and $H = \langle y \rangle \cong C_{3^{\beta}}$ be cyclic groups of 3-power order. Furthermore, let *G* and *H* act on each other so that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$ with (k, 3) = (l, 3) = 1. Then *G* and *H* act compatibly on each other with the order of the action of *H* on *G* is nine, and the order of the action of *G* on *H* is three, if and only if *k* and *l* are congruent to one of the followings:

- (i) $k \equiv 1 + 3^{\alpha 2} \mod 3^{\alpha} \mod l \equiv 1 3^{\beta 1} \mod 3^{\beta}$.
- (ii) $k \equiv 1 3^{\alpha 2} \mod 3^{\alpha} \mod l \equiv 1 3^{\beta 1} \mod 3^{\beta}$.

4. CONCLUSION

In this paper, three theorems have been studied according to the order of the actions that satisfying the compatible actions, which wrapped when both actions have order three, and when one of the actions has order nine and the other one has order three.

REFERENCES

- [1] R. Brown, D. L. Johnson, and E. F. Robertson, "Excision Homotopique En Basse Dimension", C.R. Acad. Sci. Ser. I Math. Paris. 298, pp.353-356, 1984.
- [2] R. Brown, D. L. Johnson, and E. F. Robertson, "Some Computations of Non-Abelian Tensor Products of Groups", J. of Algebra.111, pp.177-202, 1987.
- [3] L. C. Kappe, "Nonabelian tensor products of groups": the commutator connection. Bath: Proc Groups St. Andrews. LMS Lecture Notes, 1997.
- [4] M. P. Visscher, "On the Nonabelian Tensor Products of Groups". PhD Dissertation, State University of New York, Binghamton, NY, 1998.
- [5] M. S. Mohamad, N. H. Sarmin, N. M. M. Ali and L. C. Kappe, "The Computation of The Nonabelian Tensor Product of Cyclic groups of Order p^2 " journal Technologi (Science & Engineering) Suppl 1 Penerbit UTM Press. Universiti Teknologi Malaysia, vol. 57, pp. 35-44, March 2012.
- [6] S. A.Sulaiman, M. S. Mohamad, Y. Yusof, N. H. Sarmin, M. M. N. Ali, L. T. Ken, and T. Ahmad, "Compatible pair of nontrivial actions for some cyclic groups of 2-power order". In The 2nd Ism International Statistical Conference 2014 (Ism-Ii): Empowering the Applications of Statistical and Mathematical Sciences, Vol.1643, pp. 700-705. AIP Publishing, February 2015.
- [7] <u>http://www.gap-system.org</u>. The GAP Group, GAP-Groups, Algorithm, and programming, Version 4.7, 2015.