Compatible Pair of Nontrivial Actions for Cyclic Groups of 3-Power Order

Mohammed Khalid Shahoodh, Mohd Sham Mohamad, Yuhani Yusof, Sahimel Azwal Sulaiman

Abstract — A compatible action played an important role before computing the nonabelian tensor product. Number of compatible actions will give the different nonabelian tensor product. Thus, this paper present some exact number of compatible pairs of actions for cyclic groups focusing on groups of 3-power order. Some necessary and sufficient number theoretical conditions for a pair of cyclic groups of $p$-power order with nontrivial actions act compatibly on each other are applied to find the exact number of compatible pairs of actions. Group, Algorithm and Programming (GAP) software is used to find some examples on a different case. Results on the compatible pair of nontrivial actions of order three and nine are given.

Keywords — cyclic groups; compatible actions; nonabelian tensor product;

1. INTRODUCTION

Let $G$ and $H$ be groups which act on each other and each of which acts on itself by conjugation, then the actions are compatible if $v(g) g' = (v(g')) g$ and $v(h) h' = (v(h')) h$ for all $g, g' \in G$ and $h, h' \in H$. This compatible action played an important role before computing the nonabelian tensor product. Brown and Loday [1] were the first who introduced the concept of the nonabelian tensor product of groups with the compatible actions. The construction of this concept has its origins in the algebraic $K$-theory and in homotopy theory. Several authors had investigated the concept of the nonabelian tensor product of groups, starting with Brown et. al [2] had studied group theoretical aspects of the nonabelian tensor product extensively. An overview on some known results on the nonabelian tensor product and literature was illustrated by Kappe [3].

There are many researchers had investigated the compatibility conditions. Visscher [4] studied the nonabelian tensor product of $p$-power order and he focused on the cyclic groups. He provided a necessary and sufficient conditions for a pair of cyclic groups to act compatibility. Visscher [4] also gave a complete conditions when one of the actions is trivial or both actions are trivial in cyclic groups of $p$-power order. Mohamad et. al [5] investigated the nonabelian tensor product and the compatible actions for the pair of cyclic groups of order $p^3$, and he gave a necessary and sufficient condition for a pair of cyclic groups of $p$-power order to act compatibility on each other by considering the order of the actions is $p$, where $p$ is an odd prime. Two years later, Sulaiman et. al [6] computed the compatible pair of nontrivial actions for some cyclic groups of 2-power order involving at least one of the actions having order greater than two. In this research, all the compatible actions involving both actions having order greater than two will be determined. Groups, Algorithm and Programming (GAP) software [7] is used to determine the conditions in which the actions satisfying the compatibility conditions in the finite cyclic groups of 3-power order.

2. SOME OF THE PRELIMINARY RESULTS ON THE COMPATIBILITY CONDITIONS

The nonabelian tensor product for a pair of a groups $G$ and $H$ is denoted by $G \otimes H$ which is defined only when $g$ and $h$ act on each other for all $g \in G$ and $h \in H$ such that the actions satisfying the compatibility conditions. Starting with the definitions are given as follows:
Definition 2.1 [5]

Let $G$ and $H$ be groups. An action of $G$ on $H$ is a mapping, $\Phi : G \rightarrow \text{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$

for all $g, g' \in G$ and $h \in H$.

Definition 2.2 [5]

Let $G$ and $H$ be groups which act on each other and each of which acts on itself by conjugation. Then the actions are compatible if

$$g'g = \Phi(g)(g'g) \quad \text{and} \quad h'h = \Phi(h)(h'h)$$

for all $g, g' \in G$ and $h, h' \in H$.

Since our study focusing on the cyclic groups, the compatibility conditions are hold for the abelian groups will be decrease to

$$g'g = \Phi(g)(g'g) \quad \text{and} \quad h'h = \Phi(h)(h'h)$$

for all $g, g' \in G$ and $h, h' \in H$.

3. Compatible Conditions of Nontrivial Action

From the past section, we have seen that Visscher [4] provided some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibly on each other and he had completed the classification of compatible mutual actions for a pair of cyclic groups of $p$-power order where $p$ is an odd prime. Our aim is to determine the compatible conditions when both order of the actions are greater than two. The computations of the compatible conditions are doing by using GAP software. New GAP coding has been built to compute the compatible actions.

Let $G = \{x \} \cong C_3$ and $H = \{y \} \cong C_3$ be finite cyclic groups of 3-power order. Let the actions of $G$ and $H$ on each other be such that $x^k = x^k$ and $y^l = y^l$ for $k, l \in \mathbb{N}$. Now, we consider that both actions have order three. The compatible conditions which are satisfied by $k$ and $l$ are presented in Table 1 when $G = H$ and in Table 2 when $G \neq H$.

<table>
<thead>
<tr>
<th>Groups</th>
<th>$k$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = H \cong C_3$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$G = H \cong C_3$</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>$G = H \cong C_3$</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>55</td>
<td>55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Groups</th>
<th>$k$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = H \cong C_3$</td>
<td>82</td>
<td>82</td>
</tr>
<tr>
<td></td>
<td>163</td>
<td>163</td>
</tr>
<tr>
<td>$G = H \cong C_3$</td>
<td>244</td>
<td>244</td>
</tr>
<tr>
<td></td>
<td>487</td>
<td>487</td>
</tr>
<tr>
<td>$G = H \cong C_3$</td>
<td>730</td>
<td>730</td>
</tr>
<tr>
<td></td>
<td>1459</td>
<td>1459</td>
</tr>
</tbody>
</table>
Table 2: The values of $k$ and $l$ in which the action of order three satisfying the compatible conditions when $G \neq H$.

<table>
<thead>
<tr>
<th>Groups</th>
<th>$k$</th>
<th>$l$</th>
<th>Groups</th>
<th>$k$</th>
<th>$l$</th>
<th>Groups</th>
<th>$k$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>10</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>10</td>
<td>4</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>28</td>
<td>4</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>19</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>10</td>
<td>7</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>28</td>
<td>7</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>28</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>4</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>4</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>25</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>7</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>7</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>55</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>28</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>10</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>163</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>163</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>19</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>7</td>
<td>82</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>82</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>82</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>7</td>
<td>163</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>163</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>163</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>244</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>244</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>244</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>4</td>
<td>247</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>247</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>247</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>7</td>
<td>244</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>244</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>244</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>7</td>
<td>1459</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>1459</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>1459</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>7</td>
<td>730</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>730</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>730</td>
</tr>
<tr>
<td>$G = C_p$ and $H = C_p$</td>
<td>7</td>
<td>1459</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>19</td>
<td>1459</td>
<td>$G = C_p$ and $H = C_p$</td>
<td>55</td>
<td>1459</td>
</tr>
</tbody>
</table>

This lead us to the following results:

**Theorem 3.1**

Let $C_p = \langle x \rangle$ and $C_p = \langle y \rangle$ be cyclic groups of 3-power order. Furthermore, let $C_p$ and $C_p$ act on each other so that $x^k = x^l$ and $y^k = y^l$ for $k, l \in \mathbb{N}$ with $(k, 3) = (l, 3) = 1$. Then $C_p$ and $C_p$ act compatibly on each other when $\alpha = \beta$ with $x^k = x^l$ and $y^k = y^l$, if and only if $k$ and $l$ are congruent to one of the followings:

(i) $k \equiv l \equiv 1 + 3^{i} \mod 3^e$.

(ii) $k \equiv l \equiv 1 - 3^{i} \mod 3^e$. 

794
Theorem 3.2

Let \( C_3^\alpha = \langle x \rangle \) and \( C_3^\beta = \langle y \rangle \) be cyclic groups of 3-power order. Furthermore, let \( C_3^\alpha \) and \( C_3^\beta \) act on each other so that \( x^k = x^l \) and \( y^k = y^l \) for \( k, l \in \mathbb{N} \) with \( (k, 3) = (l, 3) = 1 \). Then \( C_3^\alpha \) and \( C_3^\beta \) act compatibly on each other when \( \alpha \neq \beta \) with \( x^k = x^l \) and \( y^k = y^l \), if and only if \( k \) and \( l \) are congruent to one of the followings:

(i) \( k \equiv 1 + 3^\alpha - 1 \mod 3^\alpha \) and \( l \equiv 1 + 3^\beta - 1 \mod 3^\beta \),
(ii) \( k \equiv 1 + 3^\alpha - 1 \mod 3^\alpha \) and \( l \equiv 1 - 3^\beta - 1 \mod 3^\beta \),
(iii) \( k \equiv 1 - 3^\alpha - 1 \mod 3^\alpha \) and \( l \equiv 1 + 3^\beta - 1 \mod 3^\beta \),
(iv) \( k \equiv 1 - 3^\alpha - 1 \mod 3^\alpha \) and \( l \equiv 1 - 3^\beta - 1 \mod 3^\beta \).

Next, let the actions of \( G \) and \( H \) be on each other such that \( x^k = x^l \) and \( y^k = y^l \) for \( k, l \in \mathbb{N} \). Now, consider that the actions has order nine and the other one has order three. The compatible conditions are satisfied by \( k \) and \( l \) are given in Table 3 and Table 4 when \( G = H \) and when \( G \neq H \) respectively.

Table 3: The values of \( k \) for the action has order nine and \( l \) for the action has order three satisfying the compatible conditions when the groups \( G \) and \( H \) are same.

<table>
<thead>
<tr>
<th>Groups</th>
<th>( k )</th>
<th>( l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G = H = C_3^\alpha )</td>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>19</td>
</tr>
<tr>
<td>( G = H = C_3^\beta )</td>
<td>10</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>73</td>
<td>55</td>
</tr>
<tr>
<td>( G = H = C_3^\gamma )</td>
<td>28</td>
<td>163</td>
</tr>
<tr>
<td></td>
<td>217</td>
<td>163</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Groups</th>
<th>( k )</th>
<th>( l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G = H = C_3^\alpha )</td>
<td>82</td>
<td>487</td>
</tr>
<tr>
<td></td>
<td>649</td>
<td>487</td>
</tr>
<tr>
<td>( G = H = C_3^\beta )</td>
<td>244</td>
<td>1459</td>
</tr>
<tr>
<td></td>
<td>487</td>
<td>1459</td>
</tr>
<tr>
<td>( G = H = C_3^\gamma )</td>
<td>730</td>
<td>4375</td>
</tr>
<tr>
<td></td>
<td>5833</td>
<td>4375</td>
</tr>
</tbody>
</table>
The result is given in the following theorem:

**Theorem 3.3**

Let \( G = \langle x \rangle \cong C_p \) and \( H = \langle y \rangle \cong C_p \) be cyclic groups of 3-power order. Furthermore, let \( G \) and \( H \) act on each other so that 
\[ \alpha x \beta \]  
\[ \gamma y \delta \]  
for \( k, l \in \mathbb{N} \) with \( (k,3) = (l,3) = 1 \). Then \( G \) and \( H \) act compatibly on each other with the order of the action of \( H \) on \( G \) is nine, and the order of the action of \( G \) on \( H \) is three, if and only if \( k \) and \( l \) are congruent to one of the followings:
The National Conference for Postgraduate Research 2016, Universiti Malaysia Pahang

4. CONCLUSION

In this paper, three theorems have been studied according to the order of the actions that satisfying the compatible actions, which wrapped when both actions have order three, and when one of the actions has order nine and the other one has order three.

REFERENCES


