Compatible Pair of Nontrivial Actions for Cyclic Groups of 3-Power Order

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Abstract — A compatible action played an important role before computing the nonabelian tensor product. Number of compatible actions will give the different nonabelian tensor product. Thus, this paper present some exact number of compatible pairs of actions for cyclic groups focusing on groups of 3-power order. Some necessary and sufficient number theoretical conditions for a pair of cyclic groups of *p*-power order with nontrivial actions act compatibly on each other are applied to find the exact number of compatible pairs of actions. Group, Algorithm and Programming (GAP) software is used to find some examples on a different case. Results on the compatible pair of nontrivial actions of order three and nine are given.

Keywords— cyclic groups; compatible actions; nonabelian tensor product;

1. Introduction

Let G and H be groups which act on each other and each of which acts on itself by conjugation, then the actions are compatible if ${}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g'))$ and ${}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$ for all $g, g' \in G$ and $h, h' \in H$. This compatible action played an important role before computing the nonabelian tensor product. Brown and Loday [1] were the first who introduced the concept of the nonabelian tensor product of groups with the compatible actions. The construction of this concept has its origins in the algebraic K-theory and in homotopy theory. Several authors had investigated the concept of the nonabelian tensor product of groups, starting with Brown et. al [2] had studied group theoretical aspects of the nonabelian tensor product extensively. An overview on some known results on the nonabelian tensor product and literature was illustrated by Kappe [3]. There are many researchers had investigated the compatibly conditions. Visscher [4] studied the nonabelian tensor product of *p*-power order and he focused on the cyclic groups. He provided a necessary and sufficient conditions for a pair of cyclic groups to act compatibility. Visscher [4] also gave a complete conditions when one of the actions is trivial or both actions are trivial in cyclic groups of *p*-power order. Mohamad et. al [5] investigated the nonabelian tensor product and the compatible actions for the pair of cyclic groups of order p^2, and he gave a necessary and sufficient condition for a pair of cyclic groups of *p*-power order to act compatibility on each other by considering the order of the actions is *p*, where *p* is an odd prime. Two years later, Sulaiman et. al [6] computed the compatible pair of nontrivial actions for some cyclic groups of 2-power order involving at least one of the actions having order greater than two. In this research, all the compatible actions involving both actions having order greater than two will be determined. Groups, Algorithm and Programming (GAP) software [7] is used to determine the conditions in which the actions satisfying the compatibility conditions in the finite cyclic groups of 3-power order.

2. Some of The Preliminary Results on The Compatibility Conditions

The nonabelian tensor product for a pair of a groups G and H is denoted by $G \otimes H$ which is defined only when g and h act on each other for all $g \in G$ and $h \in H$ such that the actions satisfying the compatibility conditions. Starting with the definitions are given as follows:

Definition 2.1 [5]

Let G and H be groups. An action of G on H is a mapping, $\Phi : G \to \mathrm{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$

for all $g, g' \in G$ and $h \in H$.

Definition 2.2 [5]

Let G and H be groups which act on each other and each of which acts on itself by conjugation. Then the actions are compatible if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

Since our study focusing on the cyclic groups, the compatibility conditions are hold for the abelain groups will be decrease to ${}^{({}^{g}h)}g' = {}^{h}g'$ and ${}^{({}^{h}g)}h' = {}^{g}h'$ for all $g, g' \in G$ and $h, h' \in H$. For the compatibility of actions of G and H on each other, its sufficient for the compatibility conditions to hold on the generators of G and H. Visscher [4] had done that and he provided some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibly on each other.

3. COMPATIBLE CONDITIONS OF NONTRIVIAL ACTION

From the past section, we have seen that Visscher [4] provided some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibility on each other and he had completed the classification of compatible mutual actions for a pair of cyclic groups of p-power order where p is an odd prime. Our aim is to determine the compatible conditions when both order of the actions are greater than two. The computations of the compatible conditions are doing by using GAP software. New GAP coding has been built to compute the compatible actions.

Let $G = \langle x \rangle \cong C_{3^{\alpha}}$ and $H = \langle y \rangle \cong C_{3^{\beta}}$ be finite cyclic groups of 3-power order. Let the actions of G and H on each other be such that ${}^{y}x = x^{k}$ and ${}^{x}y = y^{l}$ for $k, l \in \mathbb{N}$. Now, we consider that both actions have order three. The compatible conditions which are satisfied by k and l are presented in **Table 1** when $G = H$ and in **Table 2** when $G \neq H$.

Table 1: The values of k and l in which the action of order three satisfying the compatible conditions when $G = H$.

Groups	***k***	***l***		**Groups**	***k***	***l***
$G = H = C_{3^2}$	4	4		$G = H = C_{3^5}$	82	82
	7	7			163	163
$G = H = C_{3^3}$	10	10		$G = H = C_{3^6}$	244	244
	19	19			487	487
$G = H = C_{3^4}$	28	28		$G = H = C_{3^7}$	730	730
	55	55			1459	1459

Table 2: The values of *k* and *l* in which the action of order three satisfying the compatible conditions when $G \neq H$.

Groups	*k*	*l*	Groups	*k*	*l*	Groups	*k*	*l*
$G = C_{3^2}$ and $H = C_{3^3}$	4	10	$G = C_{3^3}$ and $H = C_{3^2}$	10	4	$G = C_{3^4}$ and $H = C_{3^2}$	28	4
	4	19		10	7		28	7
	7	10		19	4		55	4
	7	19		19	7		55	7
$G = C_{3^2}$ and $H = C_{3^4}$	4	28	$G = C_{3^3}$ and $H = C_{3^4}$	10	28	$G = C_{3^4}$ and $H = C_{3^3}$	28	10
	4	55		10	55		28	19
	7	28		19	28		55	10
	7	55		19	55		55	19
$G = C_{3^2}$ and $H = C_{3^5}$	4	82	$G = C_{3^3}$ and $H = C_{3^5}$	10	82	$G = C_{3^4}$ and $H = C_{3^5}$	28	82
	4	163		10	163		28	163
	7	82		19	82		55	82
	7	163		19	163		55	163
$G = C_{3^2}$ and $H = C_{3^6}$	4	244	$G = C_{3^3}$ and $H = C_{3^6}$	10	244	$G = C_{3^4}$ and $H = C_{3^6}$	28	244
	4	487		10	487		28	487
	7	244		19	244		55	244
	7	487		19	487		55	487
$G = C_{3^2}$ and $H = C_{3^7}$	4	730	$G = C_{3^3}$ and $H = C_{3^7}$	10	730	$G = C_{3^4}$ and $H = C_{3^7}$	28	730
	4	1459		10	1459		28	1459
	7	730		19	730		55	730
	7	1459		19	1459		55	1459
Groups	***k***	***l***	**Groups**	***k***	***l***	**Groups**	***k***	***l***
$G = C_{3^5}$ and $H = C_{3^2}$	82	4	$G = C_{3^6}$ and $H = C_{3^2}$	244	4	$G = C_{3^7}$ and $H = C_{3^2}$	730	4
	82	7		244	7		730	7
	163	4		487	4		1459	4
	163	7		487	7		1459	7
$G = C_{3^5}$ and $H = C_{3^3}$	82	10	$G = C_{3^6}$ and $H = C_{3^3}$	244	10	$G = C_{3^7}$ and $H = C_{3^3}$	730	10
	82	19		244	19		730	19
	163	10		487	10		1459	10
	163	19		487	19		1459	19
$G = C_{3^5}$ and $H = C_{3^4}$	82	28	$G = C_{3^6}$ and $H = C_{3^4}$	244	28	$G = C_{3^7}$ and $H = C_{3^4}$	730	28
	82	55		244	55		730	55
	163	28		487	28		1459	28
	163	55		487	55		1459	55
$G = C_{3^5}$ and $H = C_{3^6}$	82	244	$G = C_{3^6}$ and $H = C_{3^5}$	244	82	$G = C_{3^7}$ and $H = C_{3^6}$	730	344
	82	487		244	163		730	487
	163	244		487	82		1459	244
	163	487		487	163		1459	487
$G = C_{3^5}$ and $H = C_{3^7}$	82	730	$G = C_{3^6}$ and $H = C_{3^7}$	244	730	$G = C_{3^7}$ and $H = C_{3^8}$	730	2188
	82	1459		244	1459		730	4375
	163	730		487	730		1459	2188
	163	1459		487	1459		1459	4375

This lead us to the following results:

Theorem 3.1

Let $C_{3^\alpha} = \langle x \rangle$ and $C_{3^\beta} = \langle y \rangle$ be cyclic groups of 3-power order. Furthermore, let C_{3^α} and C_{3^β} act on each other so that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{N}$ with $(k,3) = (l,3) = 1$. Then C_{3^α} and C_{3^β} act compatibly on each other when $\alpha = \beta$ with ${}^{y^3}x = x$ and ${}^{x^3}y = y$, if and only if k and l are congruent to one of the followings:

(i) $k = l \equiv 1 + 3^{\alpha-1} \bmod 3^\alpha$,

(ii) $k = l \equiv 1 - 3^{\alpha-1} \bmod 3^\alpha$.

Theorem 3.2

Let $C_{3^\alpha} = \langle x \rangle$ and $C_{3^\beta} = \langle y \rangle$ be cyclic groups of 3-power order. Furthermore, let C_{3^α} and C_{3^β} act on each other so that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{N}$ with $(k,3) = (l,3) = 1$. Then C_{3^α} and C_{3^β} act compatibly on each other when $\alpha \neq \beta$ with ${}^{y^3}x = x$ and ${}^{x^3}y = y$, if and only if k and l are congruent to one of the followings:

(i) $k \equiv 1 + 3^{\alpha-1} \bmod 3^\alpha$ and $l \equiv 1 + 3^{\beta-1} \bmod 3^\beta$,

(ii) $k \equiv 1 + 3^{\alpha-1} \bmod 3^\alpha$ and $l \equiv 1 - 3^{\beta-1} \bmod 3^\beta$,

(iii) $k \equiv 1 - 3^{\alpha-1} \bmod 3^\alpha$ and $l \equiv 1 + 3^{\beta-1} \bmod 3^\beta$,

(iv) $k \equiv 1 - 3^{\alpha-1} \bmod 3^\alpha$ and $l \equiv 1 - 3^{\beta-1} \bmod 3^\beta$.

Next, let the actions of G and H be on each other such that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{N}$. Now, consider that the actions has order nine and the other one has order three. The compatible conditions are satisfied by k and l are given in **Table 3** and **Table 4** when $G = H$ and when $G \neq H$ respectively.

Table 3: The values of k for the action has order nine and l for the action has order three satisfying the compatible conditions when the groups G and H are same.

Groups	*k*	*l*		Groups	*k*	*l*
$G = H = C_{3^3}$	4	19		$G = H = C_{3^6}$	82	487
	25	19			649	487
$G = H = C_{3^4}$	10	55		$G = H = C_{3^7}$	244	1459
	73	55			487	1459
$G = H = C_{3^5}$	28	163		$G = H = C_{3^8}$	730	4375
	217	163			5833	4375

Table 4: The values of *k* for the action has order nine and *l* for the action has order three satisfying the compatible conditions when the groups *G* and *H* are different.

Groups	***k***	***l***		**Groups**	***k***	***l***		**Groups**	***k***	***l***
$G=C_{3^3}$ and $H=C_{3^4}$	4	55		$G=C_{3^4}$ and $H=C_{3^3}$	10	19		$G=C_{3^5}$ and $H=C_{3^3}$	28	19
	25	55			73	19			217	19
$G=C_{3^3}$ and $H=C_{3^5}$	4	163		$G=C_{3^4}$ and $H=C_{3^5}$	10	163		$G=C_{3^5}$ and $H=C_{3^4}$	28	55
	25	163			73	163			217	55
$G=C_{3^3}$ and $H=C_{3^6}$	4	487		$G=C_{3^4}$ and $H=C_{3^6}$	10	487		$G=C_{3^5}$ and $H=C_{3^6}$	28	487
	25	487			73	487			217	487
$G=C_{3^3}$ and $H=C_{3^7}$	4	1459		$G=C_{3^4}$ and $H=C_{3^7}$	10	1459		$G=C_{3^5}$ and $H=C_{3^7}$	28	1459
	25	1459			73	1459			217	1459
$G=C_{3^3}$ and $H=C_{3^8}$	4	4375		$G=C_{3^4}$ and $H=C_{3^8}$	10	4375		$G=C_{3^5}$ and $H=C_{3^8}$	28	4375
	25	4375			73	4375			217	4375
Groups	***k***	***l***		**Groups**	***k***	***l***		**Groups**	***k***	***l***
$G=C_{3^6}$ and $H=C_{3^3}$	82	19		$G=C_{3^7}$ and $H=C_{3^3}$	244	19		$G=C_{3^8}$ and $H=C_{3^3}$	730	19
	649	19			1459	19			5833	19
$G=C_{3^6}$ and $H=C_{3^4}$	82	55		$G=C_{3^7}$ and $H=C_{3^4}$	244	55		$G=C_{3^8}$ and $H=C_{3^4}$	730	55
	649	55			1459	55			5833	55
$G=C_{3^6}$ and $H=C_{3^5}$	82	163		$G=C_{3^7}$ and $H=C_{3^5}$	244	163		$G=C_{3^8}$ and $H=C_{3^5}$	730	163
	649	163			1459	163			5833	163
$G=C_{3^6}$ and $H=C_{3^7}$	82	1459		$G=C_{3^7}$ and $H=C_{3^6}$	244	487		$G=C_{3^8}$ and $H=C_{3^6}$	730	487
	649	1459			1459	487			5833	487
$G=C_{3^6}$ and $H=C_{3^8}$	82	4375		$G=C_{3^7}$ and $H=C_{3^8}$	244	4375		$G=C_{3^8}$ and $H=C_{3^7}$	730	1459
	649	4375			1459	4375			5833	1459

The result is given in the following theorem:

Theorem 3.3

Let $G=\langle x\rangle \cong C_{3^{\alpha}}$ and $H=\langle y\rangle \cong C_{3^{\beta}}$ be cyclic groups of 3-power order. Furthermore, let *G* and *H* act on each other so that ${}^{y}x=x^{k}$ and ${}^{x}y=y^{l}$ for $k,l\in\mathbb{N}$ with $(k,3)=(l,3)=1$. Then *G* and *H* act compatibly on each other with the order of the action of *H* on *G* is nine, and the order of the action of *G* on *H* is three, if and only if *k* and *l* are congruent to one of the followings:

(i) $k \equiv 1+3^{\alpha-2} \bmod 3^{\alpha}$ and $l \equiv 1-3^{\beta-1} \bmod 3^{\beta}$.

(ii) $k \equiv 1-3^{\alpha-2} \bmod 3^{\alpha}$ and $l \equiv 1-3^{\beta-1} \bmod 3^{\beta}$.

4. CONCLUSION

In this paper, three theorems have been studied according to the order of the actions that satisfying the compatible actions, which wrapped when both actions have order three, and when one of the actions has order nine and the other one has order three.

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