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NUMBER OF COMPATIBLE PAIR FOR NONTRIVIAL ACTIONS OF FINITE CYCLIC 2-GROUPS

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Abstract. The nonabelian tensor product of groups has its origins in the algebraic K-theory and homotopy theory. The nonabelian tensor product for a pair of groups is defined when the action act compatibly on each other. The compatible pair for nontrivial actions for finite cyclic 2-groups can be found by using the necessary and sufficient conditions of two finite cyclic 2-groups act compatibly on each other. Hence, the exact number of compatible pair of nontrivial actions for finite cyclic 2-groups are computed and given as a main result in this paper.

Keywords Compatible Actions, Cyclic Groups, Nonabelian Tensor Product

1.0 INTRODUCTION

The nonabelian tensor product for groups denoted by $G \otimes H$ where G and H are groups, started in connection with a generalized Van Kampen Theorems and the structure has its origins in the algebraic K-theory and also in homotopy theory. Brown and Loday [1] introduced the concept of the nonabelian tensor product of groups with compatible actions which extending the ideas of Whitehead [2]. The nonabelian tensor product can be defined in [1] as a group generated by the symbols $g \otimes h$ with relations $gg' \otimes h = \left({}^{g}g' \otimes {}^{g}h\right)\left(g \otimes h\right)$, $g \otimes hh' = \left(g \otimes h\right)\left({}^{h}g \otimes {}^{h}h'\right)$ and satisfy the compatibility conditions, ${}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g'))$ and ${}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$ for all $g, g' \in G$ and $h, h' \in H$.

From the idea of Brown *et al.* in [3], studied on the group theoretic properties and the explicit computation of nonabelian tensor square, there are several researchers that started work in group theoretical aspects of nonabelian tensor product extensively. In [3], they also provided eight open problems regarding the nonabelian tensor square and nonabelian tensor product.

In this research, we are interested with the concept of the nonabelian tensor product where the actions act compatibly on each other. Ellis and McDermott [4] have checked the compatible actions for two difference groups, D_4 and Q_8 when calculating the nonabelian tensor products. They found that only 292 compatible pair of actions between D_4 and Q_8. Visscher in [5] determined the characterization on the compatible conditions and provided some necessary and sufficient conditions for a pair of cyclic groups of *p*-power order when *p* is prime.

As a continuation of Visscher's work, Mohamad [6] focused on determination of the characterization and represented new necessary and sufficient conditions for a pair of cyclic groups of *p*-power order where the order of the actions is included as one of the condition in the characterizations. A paper by Mohamad *et al.* [7] gave the compatible pair of actions and the nonabelian tensor product of cyclic groups of order p^2 with the actions of order *p*. While, Sulaiman *et al.* [8] studied on compatible pair of nontrivial actions of order two and four for cyclic groups of 2-power order. In this paper, our aim to determine the exact number of compatible pair of actions for finite cyclic 2-groups but focused only when one of the actions has order greater than two.

2.0 THE PREPARATORY RESULTS

In this section, all related definitions and previous results on compatible conditions that have been done before are stated. Firstly, the definition of an action of a group *G* on a group *H* is given in the following.

Definition 2.1 Action [5]
Let *G* and *H* be groups. An action of *G* on H is a mapping, $\Phi:\ G \to \text{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)\big(\Phi(g')(h)\big)$$

for all $g, g' \in G$ and $h \in H$.

In this paper, our consideration case is on cyclic groups. If *G* and *H* are cyclic groups, then the action of group *G* act on group *H* be required to have the property that the identity in *G* acts as the identity mapping on *H*. Thus, all elements in *G* act as automorphism *H* on *H*.

Next, the definition of compatible action is given follows:

Definition 2.2 Compatible Action [1]
Let *G* and *H* be groups which act on each other. These mutual action are said to be compatible with each other and with the actions of *G* and *H* on themselves by conjugation if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } ^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

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In the next theorem, Dummit and Foote in [9] gave the group which isomorphic to the automorphism groups of finite cyclic groups of 2-power order.

Theorem 2.3 [9]
Let G be a cyclic group of order 2^m, $m \geq 3$. Then $\text{Aut}(G) \cong C_2 \times C_{2^{m-2}}$ and $|\text{Aut}(G)| = \varphi(2^m) = 2^{m-1}$.

There are three automorphism of order two in each finite cyclic groups of 2-power order as stated in Corollary 2.4 below.
Corollary 2.4 [6]
Let $G = \langle g \rangle \cong C_{2^m}, m \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2$. If t is an integer such that $\sigma(g) = g^t$, then

$$t \equiv 2^{m-1} + 1 \bmod 2^m, t \equiv 2^{m-1} - 1 \bmod 2^m \text{ or } t \equiv -1 \bmod 2^m$$

Next, Mohamad in [6] represent the necessary and sufficient for the compatible pair of actions for any two automorphisms with specific order are determined in the following theorem.

Theorem 2.5 [6]
Let $G = \langle x \rangle \cong C_{2^m}$ and $H = \langle y \rangle \cong C_{2^n}$.Let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2^s$, $s \geq 2$ and $\sigma' \in$ $\text{Aut}(H)$ $m \geq 4, n \geq 1$.

i If $\sigma(x) = x^t$ with $t \equiv (-1)^i 5^j \bmod 2^m$ and $i = 1$, then (σ, σ') is a compatible pair if and
only if $\sigma'(y) = y^{t'}$ with $t' \equiv 1 \bmod 2^n$ and $t' \equiv 2^{n-1} + 1 \bmod 2^n$.

ii If $\sigma(x) = x^t$ with $t \equiv (-1)^i 5^j \bmod 2^m$ and $i = 0$, then (σ, σ') is a compatible pair if and
only if $|\sigma'| \leq 2^{m-s}$ provided $n \leq m - s + 2$.

For next section, the exact number of compatible pair of actions for finite cyclic 2-groups are calculated but focused only when one of the actions has order greater than two.

3.0 THE EXACT NUMBER OF COMPATIBLE PAIR OF ACTIONS

As an extension from Mohamad [6] work, by using the characterization of all compatible pairs of two nontrivial actions, we are determine the exact number of compatible pair of actions for finite cyclic 2-groups but focusing when one of the actions has order greater than two. The following proposition shows the number of compatible pair of actions that have order greater than two.

Proposition 3.1 Let $G = \langle x \rangle \cong C_{2^m}$ and $\sigma \in \text{Aut}(G)$. If $|\sigma| > 2$, then there are $2^{m-1} - 4$ number of automorphisms.

Proof Let $G = \langle x \rangle \cong C_{2^m}$ and $\sigma \in \text{Aut}(G)$. By Theorem 2.4, $\text{Aut(G)} \cong C_2 \times C_{2^{m-2}}$ and $|\text{Aut}(G)| = \varphi(2^m) = 2^{m-1}$. Furthermore, its only one automorphism of order one and three automorphisms of order 2 by Corollary 2.3. Thus, there are $2^{m-1}-1-3=2^{m-1}-4$. □

Now, let $\sigma(x)=x^t$ be an automorphism of order 2^s where $s\geq 2$. Then, the values of *t* can be describe as $t\equiv(-1)^i5^j \bmod 2^m$. We divided into two cases which $i=0$ and $i=1$. Lemma 3.2 gives the exact number of compatible pair of actions where $\sigma(x)=x^t$ with $t\equiv(-1)^i5^j \bmod 2^m$ and $i=1$.

Lemma 3.2 Let $G=\langle x\rangle\cong C_{2^m}$ and $H=\langle y\rangle\cong C_{2^n}, m\geq 4, n\geq 1$. Furthermore, let $\sigma\in$ Aut(*G*) with $|\sigma|=2^s$, $s\geq 2$ and $\sigma'\in$Aut(*H*). If $\sigma(x)=x^t$ with $t\equiv(-1)^i5^j \bmod 2^m$ and $i=1$, then there are 2^s compatible pair of actions.

Proof Let $G=\langle x\rangle\cong C_{2^m}$ and $H=\langle y\rangle\cong C_{2^n}$. Furthermore, let $\sigma\in$Aut(*G*) with $|\sigma|=2^s$, $s\geq 2$ and $\sigma'\in$Aut(*H*). By Theorem 2.5, if $\sigma(x)=x^t$ with $t\equiv(-1)^i5^j \bmod 2^m$ and $i=1$, then (σ,σ') is compatible pair if and only if $\sigma'(y)=y^{t'}$ with $t'\equiv 1 \bmod 2^n$ and $t'\equiv 2^{n-1}+1 \bmod 2^n$. Since $i=1$, we are only considered 2^{s-1} number of automorphism σ . If $t'\equiv 1 \bmod 2^n$, then 2^{s-1} number of compatible pair of actions. If $t'\equiv 2^{n-1}+1 \bmod 2^n$, then the respective automorphism is order two. Thus, another 2^{s-1} number of compatible pairs are exists. Therefore, the total number of compatible pair of actions are $2^{s-1}+2^{s-1}=2^s$. □

Next, the following lemma gives the exact number of compatible pair of actions where $\sigma(x)=x^t$ with $t\equiv(-1)^i5^j \bmod 2^m$ and $i=0$.

Lemma 3.3 Let $G=\langle x\rangle\cong C_{2^m}$ and $H=\langle y\rangle\cong C_{2^n}$, $m\geq 4$, $n\geq 1$. Furthermore, let $\sigma\in$Aut(*G*) with $|\sigma|=2^s$, $s\geq 2$ and $\sigma'\in$Aut(*H*) with $|\sigma'|=2^{s'}$. If $\sigma(x)=x^t$ with $t\equiv(-1)^i5^j \bmod 2^m$ and $i=0$, then there are $2^{r'-1}$ compatible pair of actions with $r'=\min\{m,n\}$.

Proof Let $G=\langle x\rangle\cong C_{2^m}$ and $H=\langle y\rangle\cong C_{2^n}$. Furthermore, let $\sigma\in$Aut(*G*) with $|\sigma|=2^s$, $s\geq 2$ and $\sigma'\in$Aut(*H*) with $|\sigma'|=2^{s'}$. Let $\sigma(x)=x^t$ with $t\equiv(-1)^i5^j \bmod 2^m$ and $i=0$. By Theorem 2.5 in [6], σ is compatible with all σ' provided $s'\leq m'-s$ and $s'\leq m-s$ or $s'\leq\min\{m'-s,m-s\}$. By letting $r'=\min\{n,m\}$, we have that all σ' where $|\sigma'|\leq 2^{r'-s}$ are compatible with σ when $|\sigma'|=2^s$. Furthermore, $2^2+2^3+\ldots+2^{r'-s}=\dfrac{2(2^{r'-s}-1)}{2-1}-2=2^{r'-s+1}-4$ number of automorphisms which we are considered when $s\geq 2$. Again by using Theorem 2.5 on $\sigma'\in$Aut(C_{2^n}), only half automorphisms are considered for each s' which compatible with σ. Therefore, we have $\dfrac{2^{r'-s+1}-4}{2}=2^{r'-s}-2$. Note that, σ is also compatible with trivial automorphism and one element with $|\sigma'| = 2$. Now, we have $2^{r'-s} - 2 + 2 = 2^{r'-s}$ number of compatible pair of actions with specific σ that have order 2^s. Since, there are 2^{s-1} elements when $i = 0$ have order 2^s, then the total number of compatible pair of actions when $|\sigma'| = 2^s$ are $2^{s-1}\left(2^{r'-s}\right) = 2^{r'-1}$ when $r' = \min\{m, n\}$. □

4.0 CONCLUSION

The exact number of compatible pair of actions for finite cyclic groups of 2-power order but focused only when one of the actions has order greater than two was determined. We can conclude that the exact number of compatible pair of nontrivial actions where $\sigma(x) = x^t$ with $t \equiv (-1)^i 5^j \bmod 2^m$ and $i = 0$ are $2^{r'-1}$ when $r' = \min\{m, n\}$. While, the exact number of compatible pair of actions where $\sigma(x) = x^t$ with $t \equiv (-1)^i 5^j \bmod 2^m$ and $i = 1$ are 2^s.

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