National Conference in Mechanical Engineering Research and Postgraduate Studies (2nd NCMER 2010)
3-4 DECEMBER 2010, Faculty of Mechanical Engineering, UMP Pekan, Kuantan, Pahang, Malaysia; pp. 12-22
ISBN: 978-967-0120-04-1; Editors: M.M. Rahman, M.Y. Taib, A.R. Ismail, A.R. Yusoff, and M.A.M. Romlay

ONE-DIMENSIONAL HEAT TRANSFER IN A CONVECTION-TIP RECTANGULAR THIN FIN: A COMPARATIVE STUDY

M.M. Fakir[1], S. Basri[2], Rosli A. Bakar[1] and M.M. Rahman[1]

[1]Faculty of Mechanical Engineering, Universiti Malaysia Pahang
26600 UMP, Pekan, Pahang, Malaysia
Phone: +609-4242337, Fax: +609-4242202
E-mail: moslem@ump.edu.my
[2]Department of Aerospace Engineering
Faculty of Engineering, Universiti Putra Malaysia
43400 UPM, Serdang, Selangor, Malaysia.

ABSTRACT

Among various numerical solution techniques, finite element method (FEM) and differential quadrature method (DQM) are two important of those. Usually elements are sub-divided uniformly in FEM (conventional FEM, CFEM) to obtain temperature distribution behavior in a fin. Hence, extra computational complexity is needed to obtain a fair solution with required accuracy. In this paper, non-uniform sub-elements are considered for FEM (efficient FEM, EFEM) solution to reduce the computational complexity. Then EFEM is applied for the solution of one-dimensional heat transfer problem in an convection-tip thin rectangular fin. The obtained results are compared with CFEM and efficient DQM (EDQM, with non-uniform mesh generation). It is found that the EFEM exhibits approximately 100% and 99% accuracy compared to CFEM and EDQM respectively showing its potentiality**.**

***Keywords**:* Efficient finite element method, efficient differential quadrature method, Heat transfer problem, convection-tip fin.

INTRODUCTION

Presently there are many numerical solution techniques known to the computational mechanics community. FEM is one of those numerical solution techniques to solve structural, mechanical, heat transfer, and fluid dynamics which arise in problems of engineering and physical sciences (Strang and Fix, 1973; Tirupathi and Ashok, 1997; Li, 2004; Fairag, 2001). Here, conventional FEM (CFEM) means the used elements are of same size and uniformly distributed. In its application to the solution of engineering problems, the finite element discretization has been implemented almost to the spatial problems. For dynamic or time dependent problems whose solutions as functions of time are of interest, a step by step procedure of finite difference is usually employed with computational complexity.

For heat transfer problems, rapid changes of heat/temperature distributions take place near the element boundary (and at the boundary). It is very important to know these temperature change behavior of an element prior to its use. Hence, to get an actual picture using FEM, the element is usually subdivided into very small sub-elements uniformly (conventional FEM, CFEM), which leads to huge amount of complexity, memory consumption and computational time (Park, 1996). Otherwise, error flow occurs with unreliable results (Strang and Fix, 1973; Tirupathi and Chandrupatla, 1997; Park, 1996).

On the other hand, to get a clear picture about the temperature changes near (and at) the element boundary, better to subdivide the elements into very small sub-elements at the boundary only, followed by relatively bigger elements gradually towards the mid-point of the element non-uniformly (efficient FEM, EFEM). This may serve the intended purpose without any additional burden and this is highlighted in this paper with improved accuracy (approximately 65%) compared to CFEM. Hence, here, focus is given to develop and apply efficient (non-uniform mesh density) nodal points distribution algorithm for automatic mesh (elements) generation to optimize CFEM solution.

DQM is another numerical solution technique to solve above mentioned problems efficiently (Bellman and Casti, 1971; Bellman and Casti, 1972; Bert et al., 1989; Bert and Malik, 1996; Shu et al., 2001; Fakir et al., 2002; Fakir et al., 2003). The essence of the DQM is that the partial derivative of a function is approximated by a weighted linear sum of the function values at given discrete points. Bellman and Casti (Bellman and Casti, 1971; Bellman and Casti, 1972) developed this numerical solution technique in the early 1970s and since then, the technique has been successfully employed in a variety of problems in engineering and physical sciences. To make the DQM more efficient with less computational complexity, efficient DQM (EDQM) was proposed in (Shu et al., 2001; Fakir et al., 2002; Fakir et al., 2003) with non-uniformly distributed mesh points.

Hence, in this paper, one-dimensional (1-D) heat conduction problems in a thin convection-tip rectangular fin is solved using EFEM by means of the accurate discretization and solver (code) and then the results are compared with CFEM and EDQM to verify EFEM efficiency. The paper is organized as follows. Section II presents the governing equation with EFEM rules, followed by simulation set-up and assumptions, results and discussions, and finally conclusion of the paper.

ONE-DIMENSIONAL EFFICIENT FINITE ELEMENT METHOD

One dimensional (1-D) heat conduction equation is shown in Eq. (1) (Tirupathi and Ashok, 1997; Hinton and Owen, 1985; Tiwari et al., 2003; Wang and Tian, 2005; Lo and Wang, 2005; Ozisik, 1985).

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+Q=0 \tag{1}$$

with the boundary conditions $T\big|_{x=0}=T_0$ and $q\big|_{x=L}=h(T_L-T_\infty)$ as shown in Figure 1. Here, the heat flux $q=-k\dfrac{dt}{dx}$.

Figure 1 shows the 1-D element discretization in the x-direction. The temperature T at various nodal points are the unknowns except at node 1, where, $T_1=T_0$ with initial temperature T_0. Within a typical element '*ei or e*' the local node numbers are i and $i+1$ with coordinates x_i and x_{i+1} and element length, $l_{ei}=x_{i+1}-x_i$. For example, $e1$ whose local node numbers are 1 and 2 with coordinates x_1 and x_2, and element length $l_{e1}=x_2-x_1$ respectively.

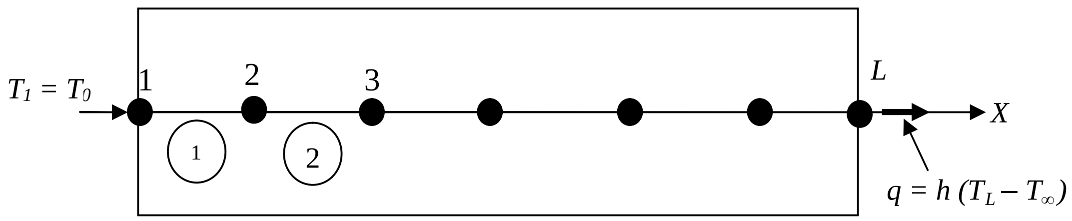

Figure 1: Boundary conditions for 1-D heat conduction

One-dimensional thin rectangular fin is shown in Figure 2. Heat is transmitted along its length by conduction and dissipated from its lateral surfaces to the surroundings by convection. The governing equation for the temperature in the fin is given in Eq. (1).

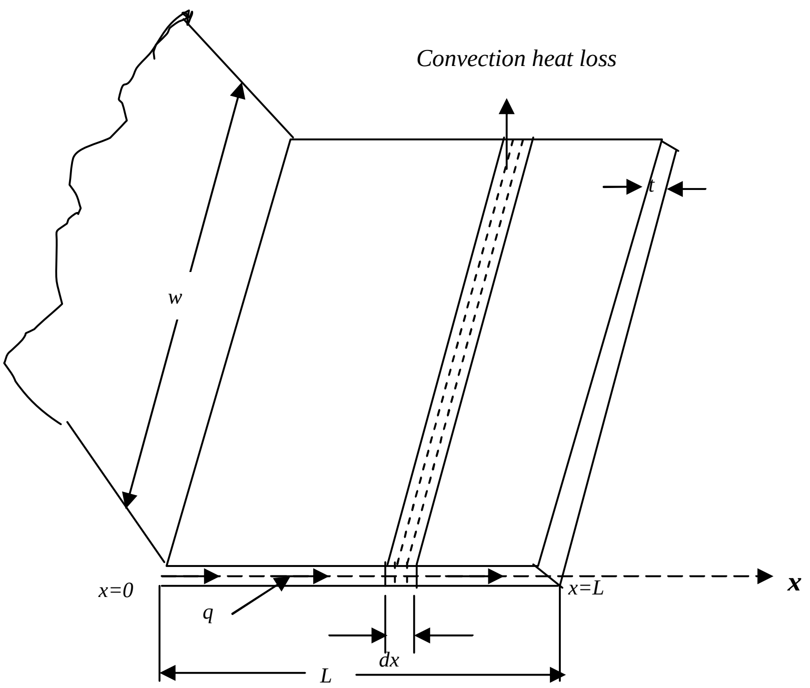

Figure 2: Thin rectangular fin

The parameter, M is given by $M^2 = \dfrac{hp}{kA_C}$,

where, p is the fin perimeter (m) and A_c is the cross sectional area of the fin [m^2]. Fin length, width and thickness are L, w and t respectively.

In this case, The heat flux is $q = h(T - T_\infty) = -k\dfrac{dT}{dx}$, perimeter, $p = 2(w+t)$, cross-section area, $A_C = w \times t$ and $\dfrac{p}{A_C} = \dfrac{2(w+t)}{w \times t} \approx \dfrac{2}{t}$.

The convection heat loss in the fin is equivalent to negative heat source and can be expressed as follows: $Q = -\frac{(p\,dx)h(T - T_\infty)}{A_C dx} = -\frac{ph}{A_C}(T - T_\infty)$.

After manipulating, Eq. (1) can be expressed in Eq. (2).

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{ph}{A_C}(T - T_\infty) = 0 \tag{2}$$

To calculate the approximate solution *T(x)*, the mathematical formulation using Galerkin's approach (Tirupathi and Ashok, 1997) is write in Eq. (3).

$$\int_0^L \phi\left[\frac{d}{dx}\left(k\frac{dT}{dx} - \frac{ph}{A_C}(T - T_\infty)\right)\right]dx = 0 \tag{3}$$

where ϕ is a test function constructed from the same basis functions as those of *T*, with $\phi(0) = 0$. Integrating by parts Eq. (3) becomes,

$$\phi\, k\frac{dT}{dx}\Big|_0^L - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx - \int_0^L \phi\frac{ph}{A_C}(T - T_\infty)dx = 0 \tag{4}$$

Since $\phi(0) = 0$ and $q = h(T_L - T_\infty)$, Eq. (4) is expressed as in Eq. (5),

$$-\phi(L)h(T_L - T_\infty) - \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}dx - \int_0^L \phi\frac{ph}{A_C}(T - T_\infty)dx = 0 \tag{5}$$

A global virtual temperature vector is defined as $\psi = [\psi_1, \psi_2, \psi_3, ..., \psi_L]^T$ then within each element, the test function becomes, $\phi(i) = N_i\psi_i$. Here, N is the element shape function and $N_L = 1$ at the element boundary (Figure 1). Therefore we can write as Eq. (6).

$$\phi(L) = [N\psi]_L = \psi_L \tag{6}$$

As, $\frac{dT}{dx} = \mathbf{B}_T\mathbf{T}^e$, from Eq. (6), $\frac{d\phi}{dx} = \mathbf{B}_T\psi$, then, $\left(\frac{d\phi}{dx}\right)^T \times \left(\frac{dT}{dx}\right) = (\mathbf{B}_T^T\boldsymbol{\psi}^T)(\mathbf{B_T}\mathbf{T}^{ei})$ and $\mathbf{B}_T^{\mathbf{T}}\mathbf{B}_T = \frac{1}{(l_{ei})^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

The element conductivity matrix is expressed in Eq. (7).

$$k_T = \frac{k_{ei}\,l_{ei}}{2}\int_{-1}^{1}\mathbf{B}_T^{\mathbf{T}}\mathbf{B}_T d\xi = \frac{k_{ei}}{l_{ei}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{7}$$

where, ξ varies from -1 to $+1$ and $\xi = \frac{2}{x_{i+1} - x_i}(x - x_i) - 1 \quad with\ d\xi = \frac{2}{x_{i+1} - x_i} dx$.

The element heat rate vector due to the heat source is written by Eq. (8).

$$\mathbf{R} = \mathbf{r}_Q = \frac{Q_{ei}\, l_{ei}}{2} \int_{-1}^{1} \mathbf{N}^{\mathbf{T}} d\xi = \frac{Q_{ei} l_{ei}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{8}$$

With the help of Eqs. (3-8), Eq. (2) can be transformed into either Eq. (9) or Eq. (10)

$$-\psi_l h(T_L - T_\infty) - \sum_{ei} \boldsymbol{\psi}^{\mathbf{T}} \left(\frac{k_{ei}\, l_{ei}}{2} \int_{-1}^{1} \mathbf{B}_T^{\mathbf{T}} \mathbf{B}_T d\xi \right) \mathbf{T}^e + \sum_{ei} \boldsymbol{\psi}^{\mathbf{T}} \frac{Q_{ei}\, l_{ei}}{2} \int_{-1}^{1} \mathbf{N}^{\mathbf{T}} d\xi = 0 \tag{9}$$

Or,

$$\boldsymbol{\psi}^{\mathbf{T}} \mathbf{K}_T \mathbf{T} + \psi_L h T_L = \boldsymbol{\psi}^{\mathbf{T}} \mathbf{R} + \psi_L h T_\infty \tag{10}$$

For convection-tip fin, the base of the fin is held at a constant temperature, T_0 and the tip of the fin is a convection surface, and the final global matrix shown in Eq. (10) can be written as Eq. (11).

$$\begin{pmatrix} A_{22} & A_{23} & \dots\ A_{2L} \\ A_{32} & A_{33} & \dots\ A_{3L} \\ \dots & \dots & \dots\ \dots \\ A_{L2} & A_{L3} & \dots\ A_{LL} + h \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ \dots \\ T_L \end{pmatrix} = \begin{pmatrix} R_{2\infty} \\ R_{3\infty} \\ \dots \\ R_{L\infty} + hT_\infty \end{pmatrix} - \begin{pmatrix} A_{21} T_0 \\ A_{31} T_0 \\ \dots \\ A_{L1} T_0 \end{pmatrix} \tag{11}$$

Using Eq. (11) and the efficient FEM (EFEM) algorithm, the approximate solution *T(x)* has been obtained. The 1-D EFEM algorithm (rule) is depicted in terms of self-explanatory flow chart in Figure 3. Example of non-uniform and uniform mesh distributions and element lengths are depicted in Figures 4 and 5 respectively.

Simulation Set-Up and Assumptions

Table I shows the considered parameters and their corresponding values used to obtain simulation results using FORTRAN 90 software. We used these values to obtain the temperature distribution for EFEM, CFEM, EDQM and exact methods. We considered, $M^2 = hP/kA = 1$ and the associated assumptions (in Table I) to compare the obtained FEM results with DQM (Fakir et al., 2002) and exact solution (Ozisik, 1985). Here to mention that, to obtain 1-D DQM solutions, element material properties, fin-width and fin-thickness are not required (which is the shortcoming of the method). The errors in FEM and DQM solutions are computed compared to exact solution.

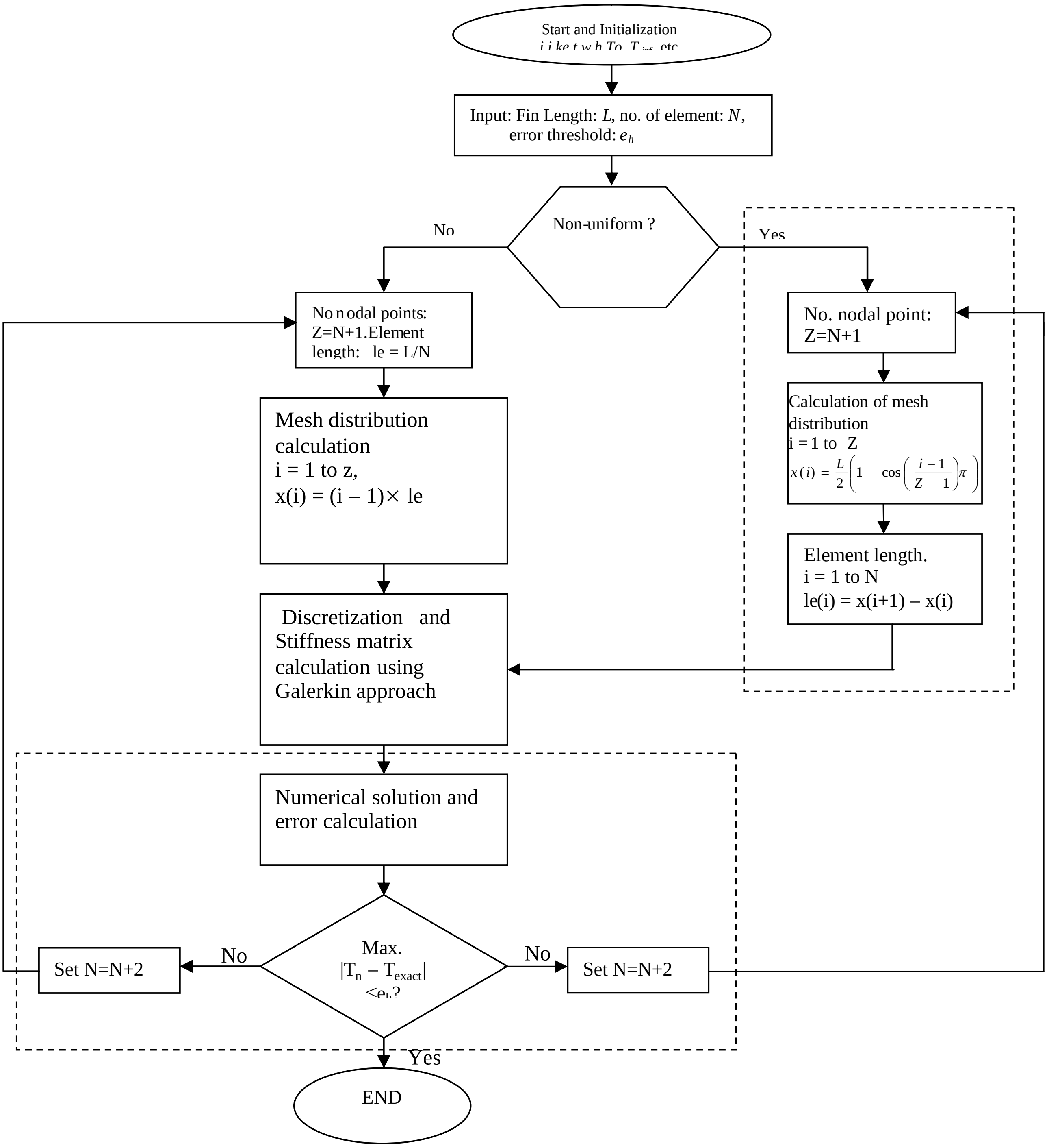

Figure 3: Efficient discretization and solution rule for 1-D FEM

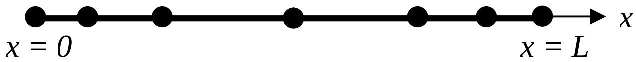

Figure 4: Example 1-D efficient mesh distribution and element lengths

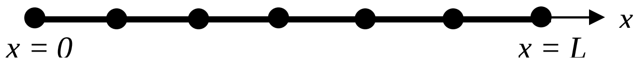

Figure 5: Example 1-D conventional mesh distribution and element lengths

Table 1: Input Parameters and Assumptions for 1-D Rectangular Fin

Input Parameters	**Assumed value for Convection-Tip Fin**
Boundary and other values: Initial temperature (T0) Ambient temperature (T∞) Heat flux (q) % Error threshold (eh)	1 OC 0 OC Variable 0 - 0.1
Element Type (NNODE): Linear (for 1-D)	2
Element material properties: Thermal conductivity ($k_e = k$) Convective heat transfer coefficient *(h)* Heat source *(Q)*	7.03125 $W/(m\ ^0C)$ 9 $W/m^2\ ^0C$ 0 $W/m^3\ ^0C$
Element (Fin) dimension: length (*L*) along x-axis width (*w*) thickness (*t*) Number of elements (*N*)	1 m Variable to make *M* = 1 Variable to make *M* = 1 11 - 104

RESULTS AND DISCUSSION

The comparison of convergence results of convection-tip fin as shown in Figure 6, contain the maximum absolute percentage errors in the FEM and DQM solutions obtained with uniformly (conventional) and non-uniformly (efficient) distributed nodal (mesh) points. It is essential to know, how many mesh points (elements) are required to obtain a convergent FEM solution in the solution domain.In Figure 6, it is apparent that for all cases, the solutions converge smoothly for all Z within the solution domain. Figure 7 shows the convergent numerical and exact solutions (fin temperature) and the corresponding percentage errors for *N = 100* elements (FEM case) which is equivalent to $Z = 101$ mesh points (both FEM and DQM cases). These results are obtained at an interval of $\Delta x = 0.1$ along the fin length, $0 \le x \le 1$, using cubic spline interpolation. It is seen that, all numerical solutions are very close to exact solutions throughout the length of the fin with temperature variations $T_0 = 1^0 C$ at base of the fin (*x* = 0) to $T_L = 0.328^0 C$ at the tip of the fin (*x* = 1). The comparison shows similar results as in Figure 8 except EFEM yields result with higher accuracy, of one order of magnitude or more with increasing Z (for $Z > 20$) compared to that with CFEM. Here, EFEM results converges from $Z = 80$ showing best result at $Z = 90$ to 101, EDQM (Fakir et al., 2002) shows similar results with some oscillations, whereas CFEM does not exhibit any best convergence.

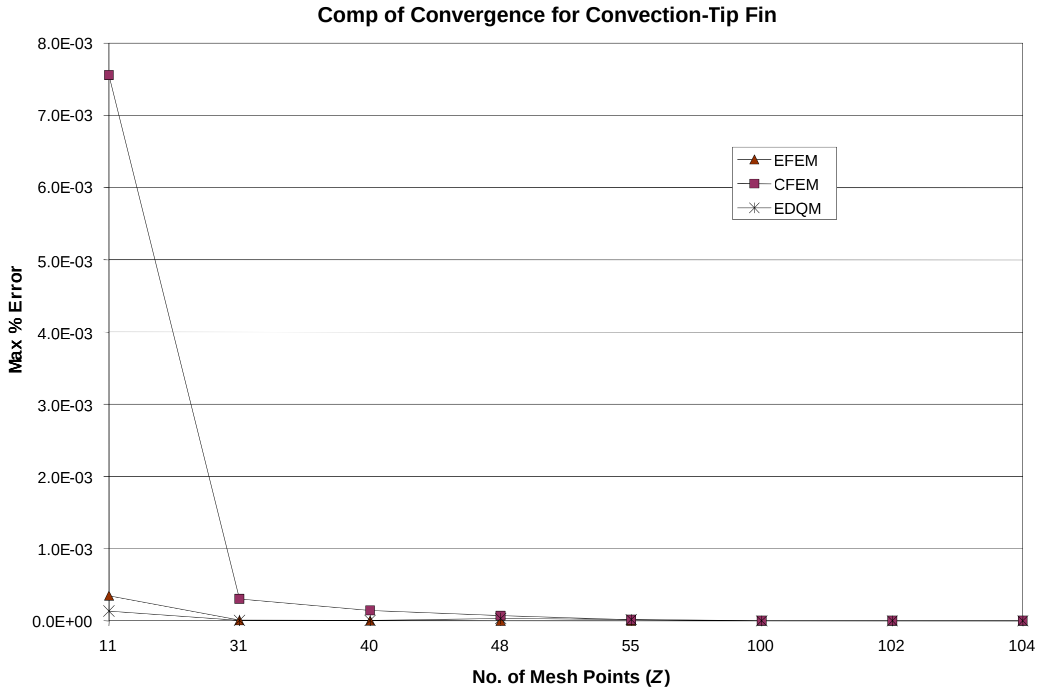

Figure 6: Comparison of convergence of Convection-tip fin-temperature in terms of maximum % error for CFEM, EFEM and EDQM solutions (Z = 11 to 104)

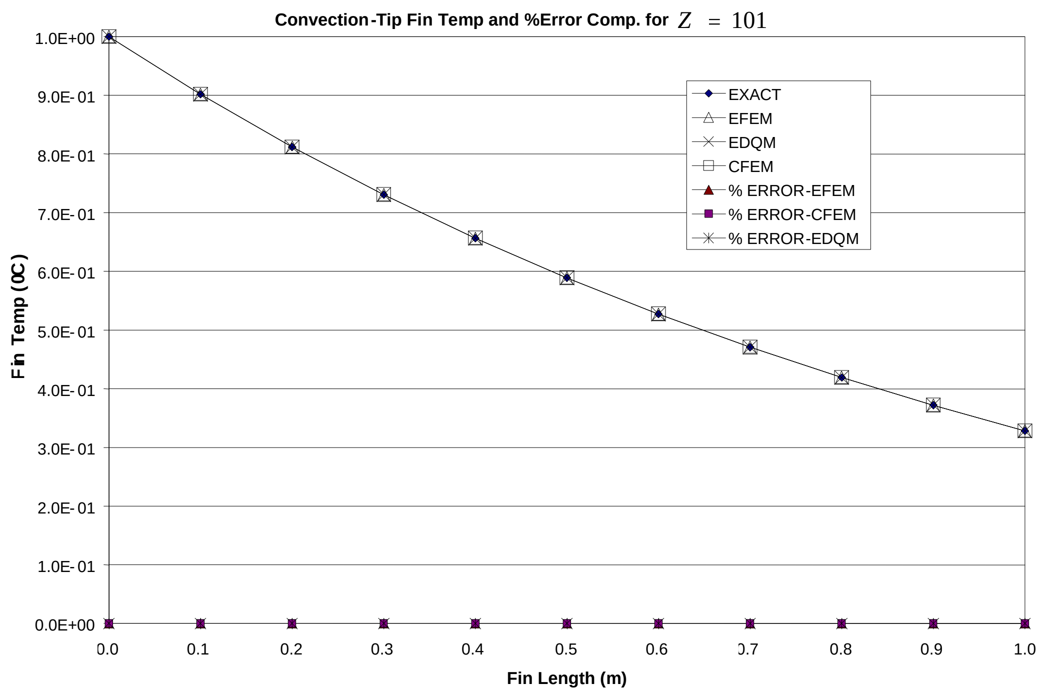

Figure 7: Convection-tip fin-temperature distribution for exact, EFEM, CFEM and EDQM along with its respective % errors (Z =100)

The comparison of CFEM, EFEM and EDQM maximum percentage errors is shown in Figure 8. There is no error at the base of the fin due to initial temperature $T_0 = 1^0 C$ (Figures 5). The errors almost remain the same with EFEM and EDQM (Fakir et al., 2002) except negligible increase at the middle of the fin due to nodal point distribution with maximum spacing there. Whereas, with CFEM, it increases gradually along the length of the fin with the maximum percentage error 3.31×10^{-6} at the tip (x = 1). The EDQM converges with oscillations (instability) throughout the solution domain. The average % error in CFEM, EDQM (Fakir et al., 2002) and EFEM are 1.69×10^{-6}, 3.08×10^{-9} and 2.24×10^{-11} respectively, which shows nearly 100% and 99% improvements in EFEM results compared to CFEM and EDQM demonstrating its superiority.

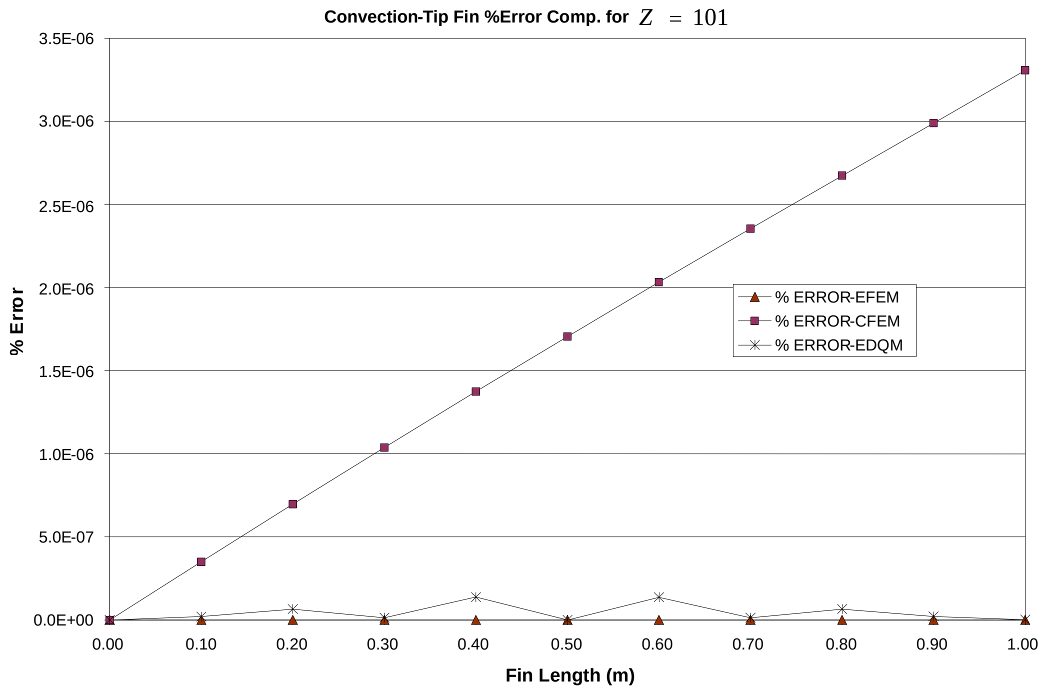

Figure 8: Comparison of % error among different methods along the fin-length for Z =101.

CONCLUSION

The solutions of the temperature distribution in convection-tip 1-D rectangular fin are computed numerically using FEM. The results are found to be good agreement with the exact solution. It is found that the unequally spaced element distribution yield more accurate results than equally spaced for FEM solution. The solution converges smoothly as the number of elements reach to the optimum value. The results of EFEM shows remarkable enhancement compared to CFEM and agree very well with EDQM with very small difference showing its potentiality. Hence EFEM is suitable to test the temperature distribution scenario in any thin metal fin prior to its design and practical implementation.

REFERENCES

Bellman, R.E. and Casti, J. 1971. Differential quadrature and long-term integration. J Math and Appl., 34: 235-238.

Bellman, R.E., Kashef, B.G. and Casti, J. 1972. Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations. J Comput Phys, 10: 40-52.

Bert, C.W. and Malik, M. 1996. Differential quadrature method in computational mechanics: a review. Appl. Mech. Rev, 49(1): 1-27.

Bert, C.W., Striz, A.G. and Jang, S.K.1989. Nonlinear bending analysis of orthotropic rectangular plates by the method of differential quadrature. *J. Comput. Mech.*, 5: 217-226.

Fairag, F. 2001. Numerical computations of viscous, incompressible flow problems using a two-level finite element method. Technical Report of ArXiv: Math, NA/0109109.

Fakir, M.M., Mawlood, M.K., Asrar, W., Basri, S. and Omar, A.A. 2002. Rectangular fin temperature distribution by the method of differential quadrature. Journal of the Institute of Engineers, Malaysia (IEM), 63(4): 41-47.

Fakir, M.M., Mawlood, M.K., Asrar, W., Basri, S. and Omar, A.A. 2003. Triangular fin temperature distribution by the method of differential quadrature. Journal Mekanikal, Malaysia, 15: 20-31.

Hinton, E. and Owen, D.R.J. 1985. An introduction to finite element computations, UK: Pineridge Press Limited.

Li, B. 2004. Numerical method for a parabolic stochastic partial differential equation. Master's thesis, Chalmers University of Technology.

Lo, S.H. and Wang, W.X. 2005. Finite element mesh generation over intersecting curved surfaces by tracing of neighbours. Journal of Finite Elements in Analysis and Design, 41: 351-370.

Ozisik, M.N. 1985. Heat transfer: a basic approach. New York: McGraw-Hill.

Park, S. 1996. Development and applications of finite elements in time domain. PhD Thesis, Virginia Polytechnic Institute and State University, USA.

Shu, C., Chen, W., Xue, H. and Du, H. 2001. Numerical analysis of grid distribution effect on the accuracy of differential quadrature analysis of beams and plates by error estimation of derivative approximation. Int. J. Numer. Methods Eng., 51(2): 159-179.

Strang, G. and Fix, G.J. 1973. An analysis of the finite element method. New Jersey: Prentice-Hall, Inc.

Tirupathi, R.C. and Ashok, D.B. 1997. Introduction to finite elements in engineering. New Jersey: Prentice-Hall International, Inc.

Tiwari, S., Biswas, G., Prasad, P.L.N. and Basu, S. 2003. Numerical prediction of flow and heat transfer in a rectangular channel with a built-in circular tube. ASME Journal of Heat Transfer, 125(3): 413-421.

Wang, B.L. and Tian, Z.H. 2005. Application of finite element-finite difference method to the determination of transient temperature field in functionally graded materials. Journal of Finite Elements in Analysis and Design, 41: 335-349.

NOMENCLATURE

$T = T(x)$	the surface temperature at any point of the fin
k $[(W/m)^0 C]$	thermal conductivity in x directions
Q $[W/m^3]$	internal heat source
q $[W/m^3]$	convection heat flux
h $[(w/m^2)^0 C]$	convection heat transfer coefficient
q_0	the initial heat flux
T_0	the initial fin temperatures
T_∞	the surrounding (or ambient) temperature
A_c	the cross sectional area of the fin
L	fin length along x-direction
w	fin width along y - direction
t	fin thickness
N	the number of the elements along x - direction
Z	the number of nodal points along x - direction
l_e	the element length
l_{ei}	length of ith element
k_{ei}	thermal conductivity of ith element
Q_{ei}	internal heat of ith element
T_n	numerical temperature solution
T_{exact}	exact temperature solution
e_n	calculated absolute error
e_h	error threshold