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A general numerical approximation of construction of axisymmetric ideal plastic plane deformation of a granular material

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Abstract. A general numerical approximation of the stress equilibrium equations and constructing axisymmetric ideal plastic plane deformation of a granular material is considered. The stress components are assumed to satisfy the Coulomb yield criterion and the self-weight of the material is neglected. The standard method of numerical approximation leads to the construction of the small segments of the stress characteristic field. Using the Matlab program, the method is applied to a problem of granular indentation by a smooth flat surface.

1. Introduction

It may be noted that many established concepts of theories governing the deformation of granular materials such as dry grain or sand have been extensively studied, such theories being demonstrated by the work of, amongst others, Spencer [1], de Jong [2], Goodman and Cowin [3] and Mehrabadi and Cowin [4]. The stress equilibrium equations and Coulomb's failure criterion provide a complete mathematical theory for the deformation of granular materials. This mathematical formulation enables to solve boundary value problem in geophysics.

Using these theories, the plasticity fields are solved analytically, however, the simplest way in solving the plasticity fields problems will be by numerically constructing the deformation field in the case of axially symmetry. Papers [5] and [6] used numerical method for the construction of the granular deformation field, however, the work was either tedious or could be solved only to the solution of that particular problem. Now that such developments are in place, it is highly desirable that there should be a general and simple method in constructing any geometrical stress characteristic that may arise. This is the motivation for the work presented here.

In this paper we attempt to develop a numerical method of constructing axisymmetric deformation fields, which, is general and simple to use. In order to make this paper self-contained, brief review of the relevant parts of the theory is given in Section 2. Section 3 discusses the basis in constructing of an axisymmetric stress characteristics field using the numerical method. This section aims to locate a new point (say *C*) which is the intersection point of α and β characteristics curvilinear curves that passes through two neighbouring points (say *A* and *B* respectively). For the construction we consider a rigid-perfectly plastic material which obeys Coulomb yield criterion. In section 4, an example of

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axisymmetric plastic plane deformation field solved using the numerical technique described are shown.

2. Axisymmetric ideal plastic plane deformation: Governing equations

Suppose that the planar deformation to take place in a Cartesian coordinate system O*xy* then, the stress components σ_{11}, σ_{22} and σ_{12} are define in terms of pressure *p* along the α / β characteristics as

$$\begin{aligned} \sigma_{11} &= -p + q\cos 2\psi \\ \sigma_{22} &= -p - q\cos 2\psi \\ \sigma_{12} &= q\sin 2\psi \end{aligned} \tag{1}$$

where ψ is the angle of $\alpha-$ and $\beta-$ characteristics inclined with respect to the *x*-axis at $\pi/4+\phi/2$ and $-\pi/4-\phi/2$ respectively. The stress equilibrium equations for planar deformation are

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} &= -\rho X \\ \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} &= -\rho Y \end{aligned} \tag{2}$$

where *X* and *Y* are the body force acting in the *x* and *y* directions respectively and ρ is the bulk density of the granular material. With pressure $p = (\sigma_{11} + \sigma_{22})/2$ and $q = p\sin\phi + c\cos\phi$, equations (1) are then substituted into the equilibrium equations (2) and by eliminating differentiated *q*, the equilibrium equations become

$$\begin{aligned} \left(-1+\sin\phi\cos 2\psi\right)\frac{\partial p}{\partial x} + \sin\phi\sin 2\psi\frac{\partial p}{\partial y} + 2q\left(\cos 2\psi\frac{\partial \psi}{\partial y} - \sin 2\psi\frac{\partial \psi}{\partial x}\right) &= \rho X \\ \sin\phi\cos 2\psi\frac{\partial p}{\partial x} - \left(1+\sin\phi\cos 2\psi\right)\frac{\partial p}{\partial y} + 2q\left(\cos 2\psi\frac{\partial \psi}{\partial x} + \sin 2\psi\frac{\partial \psi}{\partial y}\right) &= \rho Y. \end{aligned} \tag{3}$$

This gives a hyperbolic systems of equations in *p* and ψ along the stress characteristic curves in *x* and *y* direction. The inverse relationship of equations (3) are

$$\begin{aligned} \cos\phi\frac{\partial p}{\partial s_\alpha} + 2q\frac{\partial \psi}{\partial s_\alpha} &= \rho\left[X\sin(\psi+\varepsilon) - Y\cos(\psi+\varepsilon)\right] \text{ along the } \alpha-\text{characteristic,} \\ \cos\phi\frac{\partial p}{\partial s_\beta} - 2q\frac{\partial \psi}{\partial s_\beta} &= \rho\left[-X\sin(\psi-\varepsilon) - Y\cos(\psi-\varepsilon)\right] \text{ along the } \beta-\text{characteristic.} \end{aligned} \tag{4}$$

3. Technique for the numerical solution

Finite-difference technique is used to solve the stress equations (4) numerically,. The stress boundary value problem is the determination of the stress characteristics field when two intersecting characteristics are known. Figure 1 shows three points *A*, *B* and *C* where the stress variables and (*x*, *y*)-coordinates are assumed known at the points *A* and *B*. Two characteristics curves α and β that pass through points A and B are constructed and intersect each other at point C.

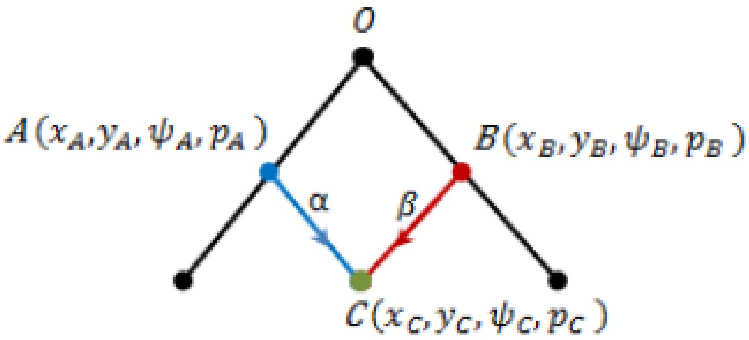

Figure 1. Numerical construction of stress characteristics.

Consider that at the points *A* and *B*, their coordinates (x_A, y_A) and (x_B, y_B), pressure p_A, p_B and gradient ψ_A, ψ_B are known, then the coordinates (x_C, y_C), pressure p_C and gradient ψ_C at point *C* are calculated as follows

$$\begin{aligned} y_A - y_C &= \tan\left(\psi_\alpha - \varepsilon\right)\left(x_A - x_C\right) \text{along the } \alpha-\text{characteristic } AC \\ y_B - y_C &= \tan\left(\psi_\beta + \varepsilon\right)\left(x_B - x_C\right) \text{ along the } \beta-\text{characteristic } BC \end{aligned} \tag{5}$$

where $\psi_\alpha = \left(\psi_A + \psi_C\right)/2$ and $\psi_\beta = \left(\psi_B + \psi_C\right)/2$ along the α and $\beta-$characteristics respectively.

By using the equations (5) along the α and $\beta-$characteristics, the following approximations are made

$$\begin{aligned} \cos\phi\left(p_A - p_C\right) + 2q_\alpha\left(\psi_A - \psi_C\right) &= F_\alpha s_A^\alpha \\ \cos\phi\left(p_B - p_C\right) + 2q_\beta\left(\psi_B - \psi_C\right) &= F_\beta s_B^\beta \end{aligned} \tag{6}$$

where

$$\begin{aligned} q_\alpha &= \sin\phi\left(p_A + p_C\right)/2 + c\cos\phi \\ q_\beta &= \sin\phi\left(p_B + p_C\right)/2 + c\cos\phi \\ F_\alpha &= \rho\left(X\sin\left(\psi_\alpha + \varepsilon\right) - Y\cos\left(\psi_\alpha + \varepsilon\right)\right) \\ F_\beta &= \rho\left(-X\sin\left(\psi_\beta - \varepsilon\right) + Y\cos\left(\psi_\beta - \varepsilon\right)\right). \end{aligned} \tag{7}$$

The four equations in equations (5) and equations (6) contain four unknowns x_C, y_C, ψ_C and p_C. A zeroth approximation for ψ_C and p_C denoted by $\psi_C{}^0$ and $p_C{}^0$ respectively is taken as

$$\begin{aligned} \psi_C{}^0 &= \left(\psi_A + \psi_B\right)/2 \\ p_C{}^0 &= \left(p_A + p_B\right)/2 \end{aligned} \tag{8}$$

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Let

$$\begin{aligned}&{\psi_\alpha}^1 = \left(\psi_A + {\psi_C}^0\right)/2 \text{ along the } \alpha-\text{characteristic } AC\\ &{\psi_\beta}^1 = \left(\psi_B + {\psi_C}^0\right)/2 \text{ along the } \beta-\text{characteristic } BC.\end{aligned} \tag{9}$$

Substituting equations (9) into equations (5) and solved the two linear equations in x_C and y_C simultaneously gives the first approximation

$$x^1_C = \frac{x_B \tan\left(\psi^1_\beta + \varepsilon\right) - x_A \tan\left(\psi^1_\alpha - \varepsilon\right) + y_A - y_B}{\tan\left(\psi^1_\beta + \varepsilon\right) - \tan\left(\psi^1_\alpha - \varepsilon\right)}$$

$$y^1_C = \frac{\left(x_B - x_A\right)\left(\tan\left(\psi^1_\alpha - \varepsilon\right)\tan\left(\psi^1_\beta + \varepsilon\right)\right) + y_A \tan\left(\psi^1_\beta + \varepsilon\right) - y_B \tan\left(\psi^1_\alpha - \varepsilon\right)}{\tan\left(\psi^1_\beta + \varepsilon\right) - \tan\left(\psi^1_\alpha - \varepsilon\right)}.$$

By using these coordinates, we can then solve for the approximation to pressure p and gradient ψ at point C

$$\psi^1_C = \frac{\left(p_A - p_B\right)\cos\phi + 2q^1_\alpha \psi_A + 2q^1_\beta \psi_\beta - F^1_\alpha \delta s^1_\alpha + F^1_\beta \delta s^1_\beta}{2q^1_\alpha + 2q^1_\beta}$$

$$p^1_C = \frac{p_A \cos\phi + 2q^1_\alpha\left(\psi_A - \psi^1_C\right) - F^1_\alpha \delta s^1_\alpha}{\cos\phi}.$$

where

$$\begin{aligned}
\delta s^1_\alpha &= y_A - y^1_C / \sin\left(\psi^1_\alpha - \varepsilon\right),\\
\delta s^1_\beta &= y_B - y^1_C / \sin\left(\psi^1_\beta + \varepsilon\right),\\
q^1_\alpha &= \left(p_A + p^1_C\right)\sin\phi + c\cos\phi,\\
q^1_\beta &= \left(p_B + p^1_C\right)\sin\phi + c\cos\phi,\\
F^1_\alpha &= \rho\left(X\sin\left(\psi^1_\alpha + \varepsilon\right) - Y\cos\left(\psi^1_\alpha + \varepsilon\right)\right),\\
F^1_\beta &= \rho\left(-X\sin\left(\psi^1_\beta - \varepsilon\right) + Y\cos\left(\psi^1_\beta - \varepsilon\right)\right).
\end{aligned}$$

4. Numerical construction of axisymmetric ideal plastic plane deformation

As an application of the method, we consider the normal plane strain indentation of a semi-infinite granular material by a block with smooth flat surface. Consider figure 2, the surface *AG* is the surface under the punch and all the values for *x*, *y*, *p* and ψ are known. The axisymmetric stress field in the region *AGH* may be constructed numerically as follows. The surface *AG* is divided into (*n*-1) equal segments where all the coordinates are known. The α and β characteristics meet all the points at the punch surface *AG* at $\pi/4 + \phi/2$. The point *C*(1,2) is first to be constructed using the numerical procedure outlined in Section 3. Then the procedure is repeated to point *E*(2,2) until the point *K*(*n*-1,2) is reached. Following the above method of axisymmetric stress field construction, the calculation is repeated and continued until the point *H*(1,*n*) in figure 2 is obtained.

Thus by employing a simple computer programming using Matlab, the whole stress characteristics in the region *AGH* can be easily constructed starting from the punch surface *AG*.

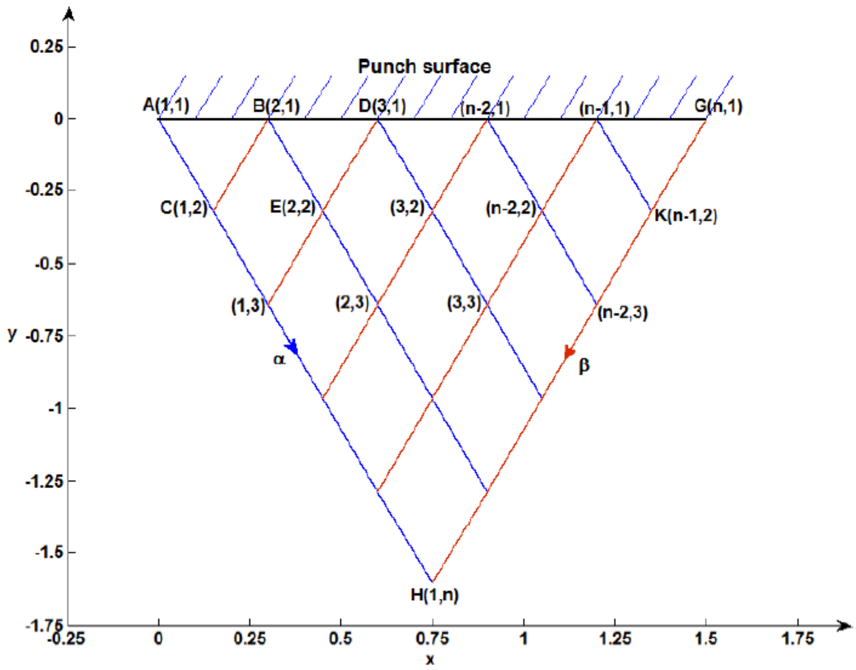

Figure 2. Geometrical construction of axisymmetric stress characteristics fields from the punch surface.

5. Discussion

When the granular material is indented by the punch, the granular particles move vertically downwards directly under the punch surface. Since the pressure p and gradient ψ along the punch surface are constant, this yields the triangular region *AGH* under the punch in which the stresses are all constant. This numerical result obeys the analytical work by Spencer [1] and finite element method by Murthy [7].

6. Conclusion

The numerical method for constructing axisymmetric stress characteristics fields from the punch surface in the triangular region described in this paper is fairly general and simple. This method can be constructed to other types of boundary value problems in plastic deformation. This will be a future work.

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