Stability analysis of explicit and implicit stochastic Runge-Kutta methods for stochastic differential equations

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Abstract. This paper concerns to the stability analysis of explicit and implicit stochastic Runge-Kutta methods in approximating the solution of stochastic models. The stability analysis of the schemes in mean-square norm is investigated. Linear stochastic differential equations are used as test equations to demonstrate the efficiency of the proposed schemes.

1. Introduction

Modelling of the physical and biological systems via stochastic differential equations (SDEs) had been intensively researched over the last few decades. General form of SDEs is

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad y \in \mathbb{R}^m \tag{1}$$

where the deterministic term $f(y(t))$ is the drift coefficient, the stochastic terms $g(y(t))$ is diffusion coefficients and the $W(t)$ is independent Wiener processes, which increment $\Delta W(t) = W(t + \Delta t) - W(t)$ is a Gaussian random variable $N(0, \Delta t)$. In SDEs, the state variable is governed by the uncontrolled fluctuations or environmental noise, hence contribute to the complexity of finding the analytical solution of the systems. Recently, there has been much interest in designing a reliable and efficient numerical integrator for SDEs. Effort has been made to develop the explicit and implicit numerical methods for solving SDEs. The method is called explicit if it is explicit in both deterministic and stochastic components. Meanwhile, if both deterministic and stochastic components are implicit, a numerical integrator is then called as implicit method. Not much work has been done to introduce an implicit numerical integrator of SDEs. Implicit methods have better convergence and stability compare than explicit and semi-implicit methods [1]. Among of the recent works is Milstein et al. [2] who proposed the balanced implicit method for the numerical solutions of SDEs. For balanced implicit methods, the type and degree of implicitness and its stability properties can be chosen by appropriate weights. Then the implicit 2-stage stochastic Runge-Kutta (SRK2) method of SDEs was introduced by Burrage and Tian [3]. Later is an extension work of explicit SRK2 method that was developed in [4]. Rosli et al. [5] investigated the stability analysis of explicit stochastic Runge-Kutta methods with 1.0 and 1.5 order of convergence. In this paper, the stability analysis of 2-stage explicit and implicit stochastic Runge-Kutta methods to simulate the solution of SDEs is presented. The arrangement of this paper is as follows; Section 2 considers the numerical schemes of

explicit and implicit SRK2 of SDEs. Mean square stability functions and regions for both methods are presented in Section 3. The numerical experiments of the schemes and the numerical results are reported in Section 4.

2. Stochastic Runge-Kutta for SDEs

A general class of s-stage SRK methods was derived by Rümelin [6] for solving SDEs in equation (1). The general formulation is given by

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + J_1 \sum_{j=1}^{s} b_{ij} g(Y_j), \quad i = 1,...,s$$

$$y_{n+1} = y_n + h\sum_{j=1}^{s} \alpha_i f(Y_j) + J_1 \sum_{j=1}^{s} \gamma_j g(Y_j). \tag{2}$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are $s \times s$ matrices of real elements, $\alpha^T = (\alpha_1,...,\alpha_s)$ and $\gamma^T = (\gamma_1,...,\gamma_s)$ are row vectors $\in \mathbb{R}^s$. Burrage and Burrage [4] developed a numerical scheme of two stage explicit SRK method based as in equation (2). The method is 1.0 order of convergence and the matrix coefficients obtained are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ \dfrac{2}{3} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \dfrac{2}{3} & 0 \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{3}{4} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{3}{4} \end{bmatrix}$$

It can be written as

$$Y_1 = y_n, \ Y_2 = y_n + \frac{2}{3}hf(Y_1) + \frac{2}{3}J_1 g(Y_1)$$

$$y_{n+1} = y_n + h\left[\frac{1}{4}f(Y_1) + \frac{3}{4}f(Y_2)\right] + J_1\left[\frac{1}{4}g(Y_1) + \frac{3}{4}g(Y_2)\right] \tag{3}$$

Burrage and Tian [3] proposed a stiffly accurate diagonal implicit SRK (SADISRK2) method with matrix coefficients

$$\mathbf{A} = \begin{bmatrix} 1-\dfrac{\sqrt{2}}{2} & 0 \\ \dfrac{\sqrt{2}}{2} & 1-\dfrac{\sqrt{2}}{2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1-\dfrac{\sqrt{2}}{2} & 0 \\ \dfrac{\sqrt{2}}{2} & 1-\dfrac{\sqrt{2}}{2} \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \dfrac{\sqrt{2}}{2} & 1-\dfrac{\sqrt{2}}{2} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \dfrac{\sqrt{2}}{2} & 1-\dfrac{\sqrt{2}}{2} \end{bmatrix}$$

It can be written as

$$Y_1 = y_n + \left(1-\frac{\sqrt{2}}{2}\right)hf(Y_1) + \left(1-\frac{\sqrt{2}}{2}\right)J_1 g(Y_1)$$

$$Y_2 = y_n + h\left[\left(\frac{\sqrt{2}}{2}\right)f(Y_1) + \left(1-\frac{\sqrt{2}}{2}\right)f(Y_2)\right] + J_1\left[\left(\frac{\sqrt{2}}{2}\right)g(Y_1) + \left(1-\frac{\sqrt{2}}{2}\right)g(Y_2)\right] \tag{4}$$

$$y_{n+1} = y_n + h\left[\left(\frac{\sqrt{2}}{2}\right)f(Y_1) + \left(1-\frac{\sqrt{2}}{2}\right)f(Y_2)\right] + J_1\left[\left(\frac{\sqrt{2}}{2}\right)g(Y_1) + \left(1-\frac{\sqrt{2}}{2}\right)g(Y_2)\right]$$

SADISRK2 in equation (4) is 1.0 order of convergence. The next section dealt with the stability analysis of explicit SRK2 and implicit SADISRK2. The stability functions are derived and the regions of the stability functions are plotted.

3. Mean-Square (MS) Stability Analysis

Consider, a linear SDE of the Stratonovich type

$$dy = \left(1 - \frac{3}{2}\delta\right)\lambda\, y\, dt + \sqrt{-\delta\lambda}\, y\, dW(t) \tag{5}$$

for $\lambda h \in (-\infty, 0)$, $\delta \in [0,1)$, $t \geq 0$, $y_0 > 0$, $\lambda < 0$.

3.1. MS-Stability of Explicit SRK2 Approximations Scheme

The MS-stability functions and MS-stability region for the discrete time approximation derivative-free method of SRK2 of explicit and implicit schemes are presented in this section. By applying the SRK2 of Burrage's scheme to the linear test in equation (5), the following approximation solution for the process, y at time, t_{n+1} is

$$y_{n+1} = \left(1 + h\lambda - \frac{3}{2}h\lambda\delta + \frac{1}{2}h^2\lambda^2 - \frac{3}{2}h^2\lambda^2\delta + h\lambda J_1\sqrt{-\delta\lambda} + \frac{9}{8}h^2\lambda^2\delta^2\right.$$
$$\left. - \frac{3}{2}h\lambda\delta J_1\sqrt{-\delta\lambda} + J_1\sqrt{-\delta\lambda} - \frac{1}{2}\delta\lambda J_1^2\right) y_n \tag{6}$$

with the intermediate stages of $Y_1 = y_n$ and $Y_2 = y_n + \frac{2}{3}hf(Y_1) + \frac{2}{3}J_1 g(Y_1)$, where

$$f(Y_1) = \left(1 - \frac{3}{2}\delta\right)\lambda y_n, \quad g(Y_1) = \sqrt{-\delta\lambda}\, y_n,$$

$$f(Y_2) = \left(1 - \frac{3}{2}\delta\right)\lambda\left(y_n + \frac{2}{3}h\left(1 - \frac{3}{2}\delta\right)\lambda y_n + \frac{2}{3}J_1\sqrt{-\delta\lambda}\, y_n\right),$$

$$g(Y_2) = \sqrt{-\delta\lambda}\left(y_n + \frac{2}{3}h\left(1 - \frac{3}{2}\delta\right)\lambda y_n + \frac{2}{3}J_1\sqrt{-\delta\lambda}\, y_n\right).$$

The transfer function, $G_{n+1}(\lambda h, \delta)$ at time, t_{n+1} can be computed by taking the ratio $\left|\dfrac{y_{n+1}}{y_n}\right|$ to equation (6). This yield

$$G_{n+1_{SRK2}} = 1 + h\lambda - \frac{3}{2}h\lambda\delta + \frac{1}{2}h^2\lambda^2 - \frac{3}{2}h^2\lambda^2\delta + h\lambda J_1\sqrt{-\delta\lambda} + \frac{9}{8}h^2\lambda^2\delta^2 - \frac{3}{2}h\lambda\delta J_1\sqrt{-\delta\lambda}$$
$$+ J_1\sqrt{-\delta\lambda} - \frac{1}{2}\delta\lambda J_1^2 \tag{7}$$

Square both sides of equation (7) and then take the expectation of the Stratonovich integrals where $E(J_1^4) = \dfrac{h^2}{2}$, $E(J_1^3) = E(J_1) = 0$, $E(J_1^2) = h$, we have

$$G_{1_{SRK2}} = 1 + \left(-\frac{27}{16}\lambda^3\delta^3 + \frac{9}{4}\lambda^3\delta^2 - \frac{3}{4}\lambda^3\delta - \frac{3}{2}\lambda^4\delta - \frac{27}{8}\lambda^4\delta^3 + \frac{27}{8}\lambda^4\delta^2 + \frac{81}{64}\lambda^4\delta^4 + \frac{1}{4}\lambda^4\right)h^4$$
$$+ \left(\frac{9}{4}\lambda^2\delta^2 - \frac{3}{2}\lambda^2\delta - \frac{9}{2}\lambda^3\delta + \frac{27}{4}\lambda^3\delta^2 - \frac{27}{8}\lambda^3\delta^3 + \lambda^3\right)h^3 \tag{8}$$
$$+ \left(2\lambda^2 - 6\lambda^2\delta + \frac{9}{2}\lambda^2\delta^2 + \frac{1}{16}\lambda\delta^2 - \lambda\delta\right)h^2 + \left(-3\lambda\delta + 2\lambda\right)h$$

Let $\overline{\lambda} = \lambda h$ yield

$$G_{1,SRK2} = 1 + \left(\frac{73}{8}\delta^2 + 2 - 9\delta\right)\overline{\lambda^2} + \left(1 - 6\delta - \frac{27}{4}\delta^3 + \frac{45}{4}\delta^2\right)\overline{\lambda^3}$$

$$+ \left(\frac{27}{8}\delta^2 + \frac{1}{4} + \frac{81}{64}\delta^4 - \frac{27}{8}\delta^3 - \frac{3}{2}\delta\right)\overline{\lambda^4} + (2 - 5\delta)\overline{\lambda} \tag{9}$$

The stability region of a stability of equation (9) is plotted in Maple 16 and the region is illustrated in Figure 1.
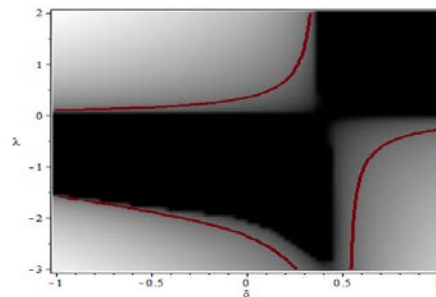

Figure 1. The stability region of a stability of equation (9).

3.2 MS-Stability of Implicit SRK2 Method

The MS-stability function of implicit SRK2 that was proposed by Burrage and Tian [3] is derived. We apply SRK2 of implicit to the linear test of equation (2). The approximate solution for the process, *y* at time, t_{n+1} is

$$y_{n+1} = \left(1 - \frac{3}{2}h\delta\lambda^2 J^2 + \frac{9}{4}h\delta^2\lambda^2 J^2 - \frac{3}{2}h\lambda\delta J\sqrt{-\delta\lambda} + \frac{3}{4}h\sqrt{2}\delta\lambda^2 J^2 - \frac{9}{8}h\sqrt{2}\delta^2\lambda^2 J^2 - \frac{9}{2}h^2\lambda^2\delta J\sqrt{-\delta\lambda} + \right.$$

$$\frac{27}{8}h^2\lambda^2\delta^2 J - \frac{3}{4}\sqrt{2}h^2\lambda^2 J\sqrt{-\delta\lambda} + \frac{1}{4}\sqrt{-\delta\lambda}\sqrt{2}\delta\lambda J^3 + h\lambda + \frac{1}{2}\lambda^2 h^2 + \frac{1}{2}\lambda^3 h^3 - \frac{1}{4}\sqrt{2}\lambda^3 h^3 + J\sqrt{-\delta\lambda}$$

$$- \frac{27}{16}\sqrt{2}\delta^2 h^2\lambda^2 J\sqrt{-\delta\lambda} + \frac{9}{4}\sqrt{2}h^2\lambda^2\delta J\sqrt{-\delta\lambda} - \frac{3}{2}h\lambda\delta - \frac{3}{2}h^2\lambda^2\delta + \frac{9}{8}h^2\lambda^2\delta^2 - \frac{1}{2}\delta\lambda J^2 - \frac{9}{4}\lambda^3 h^3\delta \tag{10}$$

$$+ \frac{27}{8}\lambda^3 h^3\delta - \frac{27}{16}\lambda^3 h^3\delta^3 + h\lambda J\sqrt{-\delta\lambda} + \frac{3}{2}h^2\lambda^2 J\sqrt{-\delta\lambda} + \frac{9}{8}\sqrt{2}\lambda^3 h^3\delta - \frac{27}{6}\sqrt{2}\lambda^3 h^3\delta^2 + \frac{27}{32}\sqrt{2}\lambda^3 h^3\delta^3$$

$$\left. - \frac{1}{2}\sqrt{-\delta\lambda}\delta\lambda J^3\right)y_n$$

By using the similar way as SRK2 method, the MS-stability function of SRK2 in equation (10) is

$$
\begin{aligned}
G_{1_{\text{Implicit,RK2}}} = 1 + & \left(\begin{array}{l} -\dfrac{27}{8}\delta - \dfrac{729}{256}\sqrt{2}\delta^6 - \dfrac{1}{4}\sqrt{2} + \dfrac{3645}{128}\delta^4 - \dfrac{135}{16}\sqrt{2}\delta^2 + \dfrac{3}{8} - \dfrac{1215}{64}\sqrt{2}\delta^4 \\[2mm] -\dfrac{405}{16}\delta^3 + \dfrac{135}{8}\sqrt{2}\delta^3 + \dfrac{9}{4}\sqrt{2}\delta - \dfrac{2187}{128}\delta^5 + \dfrac{729}{64}\sqrt{2}\delta^5 + \dfrac{2187}{512}\delta^6 + \dfrac{405}{32}\delta^2 \end{array} \right) \overline{\lambda^6} \\[3mm]
+ & \left(\begin{array}{l} \dfrac{1}{2} - \dfrac{3645}{64}\sqrt{2}\delta^4 + \dfrac{45}{8}\sqrt{2}\delta + \dfrac{2835}{32}\delta^4 + 45\delta^2 - \dfrac{75}{8}\delta + \dfrac{2673}{128}\sqrt{2}\delta^5 \\[2mm] +\dfrac{945}{16}\sqrt{2}\delta^3 - \dfrac{225}{8}\sqrt{2}\delta^2 - \dfrac{1485}{16}\delta^3 - \dfrac{4131}{128}\delta^5 - \dfrac{1}{4}\sqrt{2} \end{array} \right) \overline{\lambda^5} \\[3mm]
+ & \left(\begin{array}{l} -\dfrac{\sqrt{2}}{2} + \dfrac{945}{32}\delta^4 - \dfrac{945}{16}\delta^3 + \dfrac{5}{4} - \dfrac{159}{8}\sqrt{2}\delta^2 + \dfrac{11}{2}\sqrt{2}\delta \\[2mm] -\dfrac{243}{16}\sqrt{2}\delta^4 + \dfrac{117}{4}\sqrt{2}\delta^3 + \dfrac{675}{16}\delta^2 - \dfrac{25}{2}\delta \end{array} \right) \overline{\lambda^4} \\[3mm]
+ & \left(-13\sqrt{2}\delta^2 + \dfrac{151}{16}\sqrt{2}\delta^3 - \dfrac{33}{2}\delta + \dfrac{149}{4}\delta^2 + \dfrac{21}{4}\sqrt{2}\delta - \dfrac{819}{32}\delta^3 + 2 - \dfrac{\sqrt{2}}{2} \right) \overline{\lambda^3} \\[3mm]
+ & \left(\dfrac{3\sqrt{2}\delta}{2} - 12\delta + \dfrac{113}{8}\delta^2 - \dfrac{5}{2}\sqrt{2}\delta^2 + 2 \right) \overline{\lambda^2} + \left(2 - 5\delta \right) \overline{\lambda}
\end{aligned}
\tag{11}
$$

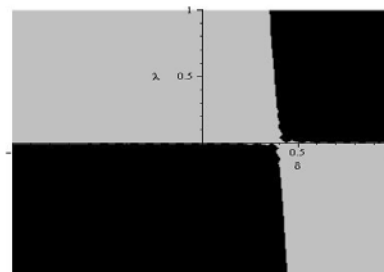The stability region of the stability of equation (11) is illustrated in Figure 2.

Figure 2. The stability region of stability in equation (11).

Based on Figures 1 and 2, it is clear that the implicit SRK2 method shows better stability result compare than the explicit Burrage and Platen's scheme. It can be confirmed by performing numerical experiments that is presented in the next section.

4. Numerical Experiment

We carried out the numerical experiment to examine the stability properties of the explicit and implicit SRK2 methods. The following numerical experiments show that the step size, h influences the mean-square stability of the corresponding methods. We used linear SDE in equation (5) as a test equation by choosing a set of parameters $\lambda = -2$ and at the critical point of $\delta = 0.5$ with a step size of 1.0, 0.5, 0.25 and 0.125. Therefore, we have

$$
dy = -0.4 y dt + 0.6325 y \circ dW(t)
\tag{12}
$$

The second moment of y_T for $T \in [0,10]$ are estimated and the expectation of $|y_n|^2$ for $N = 10$ sample paths with 5 batches are computed as

$$E\left|y_n\right|^2 = \frac{1}{5\times10}\sum_{i=1}^{50}\left|y_n(\varpi_i)\right|^2 .$$

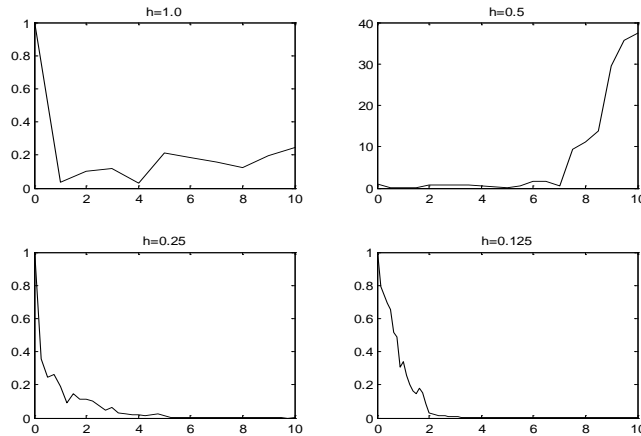The results are illustrated in Figures 3a and 3b.

Figure 3a. Numerical solution of SDE equation (12) via SRK2 of Burrage's scheme.
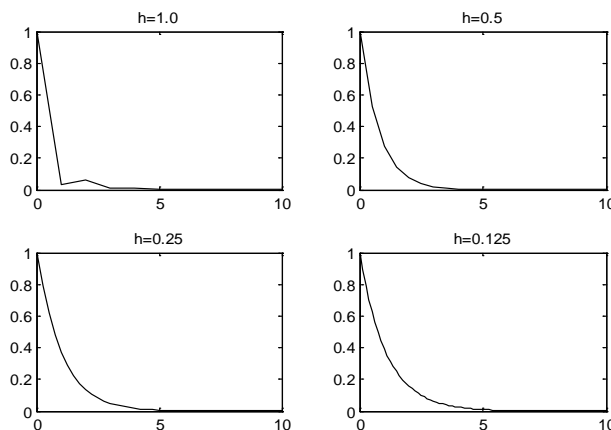

Figure 3b. Numerical solution of SDE in equation (12) via implicit of SRK2 method.

Based on Figure 3a, as the values of step size increases ($h = 1.0, 0.5$), the results are numerically unstable. However, for $h = 0.25, 0.125$ the numerical solution of SDE in equation (12) is show the stability of the solution. When implicit method of SRK2 is applied to SDE in equation (12), the solutions tend to zero for all values of $h = 1.0, 0.5, 0.25$ and 0.125 as shown in Figure 3b. This indicates that the implicit method is numerically stable compare than the explicit method.

5. Conclusion

We have presented the stability function and stability region for explicit and implicit derivative-free SRK2 methods for a linear test in equation (1). It can be seen that, the implicit SRK2 method shows better stability region compare than explicit schemes. The theoretical finding is confirmed by the numerical experiment. For various values of step size, implicit method that was performed to a linear test equation indicates numerical stability. Whereas, the explicit SRK2 methods of Burrage scheme show numerical instability for certain values of step size.

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