Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6571–6582

http://www.ripublication.com/gjpam.htm

Numerical solution of second order delay type differential equation by collocation method via first Boubeker polynomials

Muhammad Bilal[1] **and Norhayati Rosli**
Faculty of Industrial Science and Technology,
University Malaysia Pahang, Malaysia.
E-mail: norhayati@ump.edu.my

Iftikhar Ahmad
Department of Mathematics,
University Of Gujrat, Jalal Pur Jatan, Pakistan.
E-mail: dr.iftikhar@uog.edu.pk

Safi Ullah
Department of Information Technology,
University Of Gujrat, Jalal Pur Jatan, Pakistan.
E-mail: safi.ullah@uog.edu.pk

Abstract

In this paper a numerical method has been applied to solve second order pantograph differential equations via Boubeker polynomial. To examine the validity and convergence of the proposed method, we presented some test problems of second order Pantograph differential equations. Moreover, results obtained through the proposed technique are in comparison with the reported results and it was found good in agreement.

AMS subject classification:
Keywords: First Boubeker polynomials, Collocation method, Second order Pantograph equation.

[1]Corresponding author.

1. Introduction

This research work is focus with the generalization of Delay differential equation famous as Pantograph equation (with variable as well as constant coefficients) which also contains a linear functional opine (with advanced cases as well as retarded cases or in some proportional delays). Usually delay differential equation with some time lags are known as Pantograph differential equation. The name of Pantograph equation comes from the work of Ockendon and Tayler work [1]. Due to its wide range of application many authors studied Pantograph equation analytically as well as numerical aspects [[6]-[7]]. Pantograph equations play a major role in describing many different real life phenomena. They emerge in industrial applications [8] and in studies which depend on nonlinear dynamic systems, biology,control theory, astro-physics,cell growth, electro dynamic, economy, and many others [[1],[8],[9]]. In addition,properties of the analytical and numerical approximated solutions of pantograph equations have been investigated by several authors [[1]-[4],[10]-[14]]. A Neuro heuristic computational intelligence method used for the solution of nonlinear pantograph systems by Raja [16]. Osman applied polynomial interpolation for solving pantograph equation [19]. Neural network method used for solving multi-pantograph equation by iftikhar [17].

Our basic purpose of present study is to create a numerical method which is called collocation method which depend on Boubeker polynomial and collocation points for approximated solution of Pantograph differential equation [[2]-[5]]. A chebyshev collocation method was applied for solving singular integral equation with cosecant kernel [18]. Some more work by different authors [20]-[21]. In present study, we will examine the generalized Pantograph differential equation.

$$U^m(x) = \sum_{j=0}^{J} \sum_{k=0}^{m-1} G_{jk}(x) U^k(\mu_j x + \gamma_j) + h(x) \tag{1}$$

$$\sum_{k=0}^{m-1} c_{ik} U^k(0) = \lambda_i \tag{2}$$

Where $G_{jk}(x)$ and $h(x)$ are the analytic functions; c_{ik}, λ_i, μ_j and γ_j are real or complex constants. Our focus is to get approximated solution of problem (1) and (2) as truncated First Boubeker expansion defined as

$$U(x) = \sum_{n=0}^{M} \eta B_n(x), 0 \leq x \leq b < \infty \tag{3}$$

Where $B_n\ (x), n = 0,\ 1,\ 2, \ldots$ denote the First Boubeker series; $\eta_n, 0 \leq n \leq M$ are unknown First Boubeker coefficients, and M will be chosen any positive integer such that $M \geq m$. Here is universal standard First Boubeker polynomials are described by

$$B_0(x) = 1$$

$$B_1(x) = x$$

$$B_2 = x^2 + 2$$

$$B_m(x) = XB_{m-1}(x) - B_{m-2}(x) for$$

and a monomial definition of these polynomials was established by Labiadh et al. [5].

$$B_n = \sum_{p=0}^{\frac{n}{2}} [\frac{n-4p}{n-o} C_{n-p}^{p}].(-1)^p.x^n - 2p \tag{4}$$

2. Fundamental Relations

Consider the Eq. (1) and to find the matrix equation, in the start we need to convert $U(x)$ which is the solution and its derivative $U^k(x)$ described by the truncated boubeker series (3) to matrix form.

$$U(x) = B(x)\ \eta \text{ and } U^{(k)}(x) = B^{(k)}(x)\ \eta,\ k = 0, 1, 2, \ldots \tag{5}$$

$$B(x) = [B_0(x)\ B_1(x)\ B_2(x)\ , \ldots,\ B_M(x)]$$

$$\eta = [\eta_0\ \eta_1\ , \ldots,\ \eta_M\ ]^T$$

After using expression (4) and $n = 0,\ 1\ , \ldots,\ M$, we get the analogous matrix which defined by

$$B(x)\ =\ X(x)\ Z^{'} \tag{6}$$

'=transpose of matrix
where

$$X\ (x) = [1\ x\ .....\ x^M]$$

if N is odd, then

$$Z = \begin{bmatrix} \psi_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \psi_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ \psi_{2,1} & 0 & \psi_{2,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_{M-1,\frac{M-1}{2}} & 0 & \psi_{M-1,\frac{M-3}{2}} & 0 & \cdots & \psi_{M-1,0} & 0 \\ 0 & \psi_{M,\frac{M-1}{2}} & 0 & \psi_{M,\frac{M-3}{2}} & \cdots & 0 & \psi_{M,0} \end{bmatrix};$$

if N is even,then

$$Z = \begin{bmatrix} \psi_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \psi_{1,\,0} & 0 & 0 & \cdots & 0 & 0 \\ \psi_{2,\,1} & 0 & \psi_{2,\,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \psi_{M-1,\frac{M-2}{2}} & 0 & \psi_{M-1,\,\frac{M-4}{2}} & \cdots & \psi_{M-1,0} & 0 \\ \psi_{M,\,\frac{M}{2}} & 0 & \psi_{M,\frac{M-2}{2}} & 0 & \cdots & 0 & \psi_{M,0} \end{bmatrix}$$

where

$$B_n(x) = \sum_{p=0}^{\frac{n}{2}} \psi_{n,p} T^{n-2p}, n = 0, 1, \ldots, M$$

$$\psi_{n,p} = [\frac{n-4p}{n-p} C_{n-p}^{p}](-1)^p$$

From Eq. (6)

$$B^{(k)}(x) = X^{(k)}(x) Z' \tag{7}$$

To access the matrix $X^{(k)}(x)$ in form of the matrix $X(x)$, generally this procedure is followed:

$$\begin{aligned} X^{(1)}(x) &= X(x)\, P^1 \\ X^{(2)}(x) &= X^{(1)}(x) P = X(x) P^2 \\ X^{(3)}(x) &= X^{(2)}(x) P = X(x) P^3 \\ &\vdots \\ X^{(k)}(x) &= X^{(k-1)}(x)\, P = X(x)\, P^k \end{aligned} \tag{8}$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

likewise we have the relations between $X^{(k)}(\mu_j x + \gamma_j)$ and $X(x)$ which is as follows:

$$X(\mu_j x + \gamma_j) = [1\; (\mu_j x + \gamma_j)\; (\mu_j x + \gamma_j)^2 \cdots (\mu_j x + \gamma_j)^n] = X(x)\, A(\mu_j,\, \gamma_j) \tag{9}$$

$$X^{(k)}(\mu_j x + \gamma_j) = X(\mu_j + \gamma_j) P^k \tag{10}$$

and from (9) and (10)

$$X^{(k)}(\mu_j x + \gamma_j) = X(x) A(\mu_j + \gamma_j) P^k \tag{11}$$

where

$A(\mu_j,\ \gamma_j)$

$$= \begin{bmatrix} \binom{0}{0}(\mu_j)^0(\gamma_j)^0 & \binom{1}{0}(\mu_j)^0(\gamma_j)^1 & \binom{2}{0}(\mu_j)^0(\gamma_j)^2 & \cdots & \binom{M}{0}(\mu_j)^0(\gamma_j)^M \\ 0 & \binom{1}{1}(\mu_j)^1(\gamma_j)^0 & \binom{2}{1}(\mu_j)^1(\gamma_j)^1 & \cdots & \binom{M}{1}(\mu_j)^1(\gamma_j)^{M-1} \\ 0 & 0 & \binom{2}{2}(\mu_j)^2(\gamma_j)^0 & \cdots & \binom{N}{2}(\mu_j)^2(\gamma_j)^{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{M}{M}(\mu_j)^M(\gamma_j)^0 \end{bmatrix}$$

and for $\mu_j \neq 0, \gamma_j = 0$

$$A(\mu_j, 0) = \begin{bmatrix} (\mu_j)^0 & 0 & \cdots & 0 & \\ 0 & (\mu_j)^1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & (\mu_j)^M \end{bmatrix}$$

Therefore, by putting the matrix forms in Eq. (7) and Eq. (8) into Eq. (5), the resultant matrix is

$$U^{(k)}(x) = X(x)P^k Z' \eta \tag{12}$$

and by using the (5), (7) and (11)

$$U^{(k)}(\mu_j x + \gamma_j) = B^{(k)}(\mu_j x + \gamma_j)\eta = X^{(k)} A(\mu_j x, \gamma_j) P^k Z' \eta, k = 0, 1, 2, \ldots \tag{13}$$

3. Methodology

In the present section we will form the basic form of matrix equation analogous to Eq. (1). By putting equations Eq. (12) and (13) in Eq. (1), and after solving we get the matrix equation

$$X(x)P^m Z' \eta = \sum_{j=0}^{J} \sum_{k=0}^{m-1} G_{jk}(x) X(x) A(\mu_j, \gamma_j) P^k Z' \eta + g(x) \tag{14}$$

By using in Eq. (14) the collocation points X_i defined by$X_i = \dfrac{b}{N}$, $i = 0, 1, \ldots, M$, we obtain the system of equations in the form of matrix.

$$X\ (x_i)P^m Z' \eta = \sum_{j=0}^{J} \sum_{k=0}^{m-1} G_{jk}(x_i) X(x_i) A(\mu_j, \gamma_j) P^k Z' \eta + h(x), i = 0, 1, \ldots, M$$

or detailed fundamental matrix equation

$$XP^mZ' - \sum_{j=0}^{J}\sum_{k=0}^{m-1} G_{jk}XA(\mu_j,\gamma_j)P^kZ'\eta = G \tag{15}$$

where

$$G_{jk} = \begin{bmatrix} G_{jk}\,(x_0) & 0 & 0 & \cdots & 0 \\ 0 & G_{jk}\,(x_1) & 0 & \cdots & 0 \\ 0 & 0 & G_{jk}\,(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{jk}(x_M) \end{bmatrix}, H = \begin{bmatrix} h(x_0) \\ h(x_1) \\ h(x_2) \\ h(x_3) \\ \vdots \\ h(x_M) \end{bmatrix}$$

$$X = \begin{bmatrix} X(x_0) \\ X(x_1) \\ X(x_2) \\ X(x_3) \\ \vdots \\ X(x_M) \end{bmatrix} = G_{jk} = \begin{bmatrix} 1 & x_0 & {x_0}^2 & \cdots & {x_0}^M \\ 1 & x_1 & {x_1}^2 & \cdots & {x_1}^M \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^M \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_M & {x_M}^2 & \cdots & {x_M}^M \end{bmatrix}$$

Hence, Eq. (15) can be written in the form

$$S\eta = H \text{ or } [S;\, H],\, S = [s_{pq}],\, p, q = 0, 1, \ldots, n \tag{16}$$

where

$$S = [s_{pq}] = XP^mZ' - \sum_{j=0}^{J}\sum_{k=0}^{m-1} G_{jk}TA(\mu_j,\gamma_j)P^kZ'$$

Eq. (16) corresponds to the system of (N+1) algebraic equation along the unknown Boubeker coefficients $\eta_0,\ \eta_1,\ \eta_2,\ \cdots,\ \eta_N$. On the other hand, the matrix in the forms of conditions (2), as follows:

$$\sum_{k=0}^{m-1} c_{ik}X(0)P^kZ'\eta = [\lambda_i]$$

briefly

$$Y_i\eta = [\lambda_i] \text{ or } [Y_i;\, \lambda_i],\ i = 0,\ 1,\ \ldots,\ m-1 \tag{17}$$

where

$$Y_i = [y_{i0}y_{i1}, \ldots, y_{iM}] = \sum_{k=0}^{m-1} c_{ik}X(0)P^kZ'$$

Finally, to access the approximated result of Eq. (1) with Eq. (2), by substitute the row matrices (17) by the last m rows of the (16), we obtain the new matrix.

$$[\hat{S};\,\hat{H}] = \left[\begin{array}{cccccccc} s_{00} & s_{01} & s_{02} & \cdots & s_{0M} & ; & h(x_0) \\ s_{10} & s_{11} & s_{12} & \cdots & s_{1M} & ; & h(x_1) \\ s_{20} & s_{21} & s_{22} & \cdots & s_{2M} & ; & h(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ s_{(M-m)0} & s_{(M-m)1} & s_{(M-m)2} & \cdots & s_{(M-m)M} & ; & h(x_{(M-m)}) \\ y_{00} & y_{01} & y_{02} & \cdots & z_{0M} & ; & \lambda_0 \\ z_{10} & z_{11} & u_{12} & \cdots & z_{1M} & ; & \lambda_1 \\ y_{20} & y_{21} & y_{22} & \cdots & z_{2N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ y_{(m-1)0} & y_{(m-1)\,1} & y_{(m-1)\,2} & \cdots & y_{(m-1)\,M} & ; & \lambda_{m-1} \end{array}\right) \tag{18}$$

If rank $\hat{S}$ = rank $[\hat{S};\,\hat{H}] = M+1$ then we can write $\eta = (\hat{S})^{-1}\hat{H}$ where coefficient $\eta(\,\eta_0\;\eta_1\;\eta_2,\;\ldots,\;\eta_M)$ is individually determined. Also the Eq. (1) under the Eq. (2) has a unique approximated solution. By using First Boubeker expansion Eq. (3) we obtain approximated solution of Eq. (1).

4. Illustrative Examples

Present section represents the numerical examples which given to show the convergence,effectiveness properties and accuracy of the present method with Boubeker polynomials, these performance will be performed on the computer using a computer code in MATLAB R2016a.

4.1. Example 1

Let us take second order functional type pantograph differential equation.

$$u''\left(\frac{x}{2}\right) + \left[u'\left(\frac{x}{2}\right)\right]^2 - \frac{1}{4}u\left(\frac{t}{2}\right) - \frac{1}{4}u(t) = 0, 0 \le x \ \le 1 \tag{19}$$

with initial condition

$$u(0) = u'(0) = 1n \tag{20}$$

We assume that the solution of first Boubeker polynomials is

$$u(x) = \sum_{n=0}^{M} \eta_n B_n(x)$$

The basic matrix form of example 1 is

$$\left(XP^2ZA\left(\frac{1}{2},0\right) + \left[XPA\left(\frac{1}{2},0\right)\right]^2 + G_{10}XZA\left(\frac{1}{2},0\right) + G_{10}XZ\right)\eta = H$$

The collocation points are computed as follows

$$x_1 = 0, x_2 = \frac{1}{5}, x_3 = \frac{2}{5}, x_4 = \frac{3}{5}, x_5 = \frac{4}{5}, x_6 = 1$$

Table 1: Results from the present method and the absolute errors (AE) of present method of Example 1 for N= 5, 19 and 29.

i	x	Collocation Method						
		Exact	N_5		N_{19}		N_{29}	
		$u(x_i) = e^{x_i}$	$u(x_i)$	$Er(x_i)$	$u(x_i)$	$Er(x_i)$	$u(x_i)$	$Er(x_i)$
1	0.0	1	1	0	1	0	1	0
2	0.20	1.2214027568	1.2237930	0.00239025	1.22152170	0.00074942	1.221402750	5.0000E-08
3	0.40	1.4918246973	1.5070550	0.15230310	1.49128393	0.00054076	1.491824699	1.7000E-09
4	0.60	1.8221188046	1.8654930	0.03374200	1.81814800	0.003970800	1.822117762	1.0426E-06
5	0.80	2.2255409247	2.3061530	0.08061208	2.21580000	0.00974092	2.225540814	1.1070E-07
6	1.0	2.7182818228	2.8200980	0.10181618	2.70530500	0.01297682	2.718281827	4.2000E-09

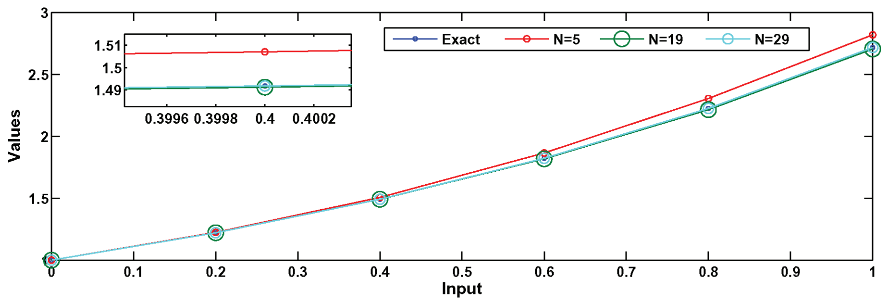

Figure 1: Figure shows the comparison between approximated result for different values of N with exact result of Example 1.

Table 1 shows the Absolute error between the approximated result and different values of N. Tables shows as we increase the size of matrix, the approximated result was more near to exact result. We monitor that the absolute error decreases fast as we increase the number of iterations. Thus a few iteration gives high degree of accuracy. Fig. 3 clearly shows the results taken when N=29 is gives best approximated result. Error goes to e^{09} while applying the Present methodology with $N = 29$.

4.2. Example 2

Let us take second order functional type pantograph DE.

$$u''(x) - \frac{8}{3}u'\left(\frac{x}{2}\right)u(x) - 8x^2u\left(\frac{x}{2}\right) = -\frac{4}{3} - \frac{22x}{3} - 7x^2 - \frac{5}{3}x^3, 0 \leq x \ \leq 1 \quad (21)$$

with initial condition

$$u(0) = u(1) = 1 \quad (22)$$

We assume that the solution of first Boubeker polynomials is

$$u(x) = \sum_{n=0}^{M} \eta_n B_n(x)$$

The fundamental matrix form of Example 2 is

$$\left(XP^2Z + G1XPZA\left(\frac{1}{2}, 0\right) XZ + G2XZA\left(\frac{1}{2}, 0\right)\right) \eta = H$$

Table 2: Absolute errors of Example 2 present methodology for N= 3, 11 and 15.

i	x	Collocation Method						
		Exact	N_3		N_{11}		N_{15}	
		$u(x_i) = e^{x_i}$	$u(x_i)$	$Er(x_i)$	$u(x_i)$	$Er(x_i)$	$u(x_i)$	$Er(x_i)$
1	0.0	1	1	0	1	0	1	0
2	0.20	1.192000	1.134612	0.057388	1.105110	0.08689	1.192675	0.000675
3	0.40	1.336000	1.247982	0.088018	1.377144	0.041144	1.336453	0.000453
4	0.60	1.384000	1.294045	0.089955	1.382809	0.001191	1.384832	0.000832
5	0.80	1.288000	1.226739	0.061261	1.287967	3.3E-05	1.288004	4E-06
6	1.0	1	1	0	1	0	1	0

The approximate solutions obtained by using the collocation points $x_s = \frac{s}{N}$, where $s = 0, 1, 2, 3, 4...$ and Value of N depend on the size of matrix. Table 2 shows the approximated solution produced from Boubeker polynomials which goes to very near to exact result when $N = 15$ in the interval [0,1]. From numerical computations, presented method converges quite fast.

4.3. Example 3

Now, if we consider the second order Pantograph DE with unbounded delay

$$u''(x) + \frac{1}{x}u'\left(\frac{1}{2}x\right) + \frac{1}{x^2}u'\left(\frac{1}{4}x\right) + \frac{1}{1-x}u(x)$$
$$= \frac{-(e^{\frac{x}{4}}(-1+x))}{4} - \frac{e^{\frac{x}{2}}(-1+x)x}{2} - e^x(-2+x)x^2, 0 \le x \le 1 \quad (23)$$

with initial condition

$$u(0) = 1, \; u(1) = e \quad (24)$$

True solution for the problem is e^x. Present solution with Boubeker polynomial give s better solution than reported in [15].

Table 3 shows the reduction in error by collocation method via Boubeker polynomial. Results shows that collocation method gives better result as compared to any other methodology. As we raise the value of N it may possible we can get more accurate result. Fig. 2 shows the comparison of the reported and the present method. It is clearly seen the present method gives the better and accurate answer as compare to [15].

Table 3: Comparison between the Reported result [15] and collocation method of Example 3

		Exact	Reported result [15]		Present Method	
i	x	$u(x_i) = e^{x_i}$	$u(x_i)$	$Er(x_i)$	$u(x_i)$	$Er(x_i)$
1	0.001	1.0010005	1.001	3.8E-09	1.0010005	2.8E-11
2	0.08	1.0832870	1.08328	4.8E-06	1.08328702	2E-08
3	0.16	1.1735108	1.17349	2.1E-05	1.17351089	9E-08
4	0.32	1.3771277	1.37709	3.4E-05	1.37712773	3E-08
5	0.48	1.6160744	1.61605	2.5E-05	1.61607445	5E-08
6	0.64	1.8964808	1.89647	7.3E-05	1.89648086	6E-08
7	0.80	2.2255409	2.22555	1.1E-05	2.22554091	1E-08
8	0.96	2.6116964	2.61172	2.6E-05	2.61169271	7E-06
9	1.00	2.7182818	2.71828	0	2.71828252	2E-06

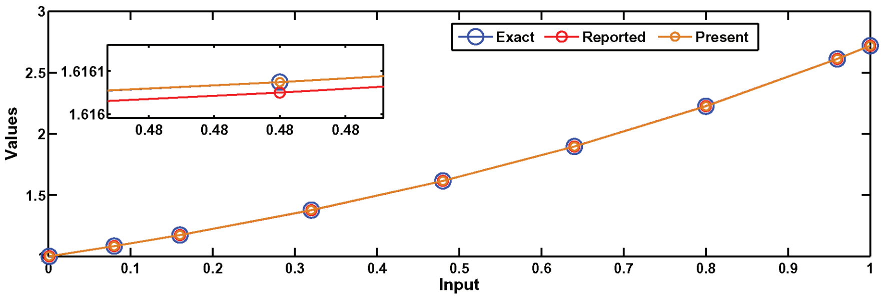

Figure 2: Figure shows the comparison between the exact, reported and present method results of Example 3.

5. Conclusion

A collocation method is basically based on the truncated Boubeker series expansion is construct to computationally solve second order pantograph equations with mixed condition. Present method is also applied to any finite interval. It is clear that Nth order Boubeker polynomial gives the exact result when the result is polynomial of degree equal or less than N. However, more terms of the Boubeker polynomial are required for accurate calculation for big value of x, if the result is not polynomial. Boubeker approximation converges to the exact solution as N increases, but the truncation limit N must be chosen large enough. Collocation method gives two main advantages: it is very simple to construct the main matrix equation and it is quite easy for MATLAB programming. Another considerable advantage is that convergence time of the method is too short. Besides, our method produces much better results than the other methods in the examples.

Acknowledgement

We would like to thank the Ministry of Education (MOE)and Research and Innovation Department, University Malaysia Pahang (UMP) for their financial supports through FRGS Vote No: RDU130122.

References

[1] Ockendon J, Tayler AB. The dynamics of a current collection system for an electric locomotive. InProceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 1971 May 4 (Vol. 322, No. 1551, pp. 447–468). The Royal Society.

[2] Boubaker K. On modified Boubaker polynomials: some differential and analytical properties of the new polynomials issued from an attempt for solving bi-varied heat equation. Trends in Applied Sciences Research. 2007;2(6):540–4.

[3] Ghanouchi J, Labiadh H. An Attempt to solve the heat transfer equation in a model of pyrolysis spray using 4q-Order Boubaker polynomials. International Journal of Heat and Technology. 2008;26(1):49–53.

[4] Awojoyogbe OB, Boubaker K. A solution to Bloch NMR flow equations for the analysis of hemodynamic functions of blood flow system using m-Boubaker polynomials. Current Applied Physics. 2009 Jan 31;9(1):278–83.

[5] Labiadh H. A Sturm-Liouville shaped characteristic differential equation as a guide to establish a quasi-polynomial expression to the Boubaker polynomials. Differ. Uravn. Protsessy Upr. 2007 Apr:117–33.

[6] Liu MZ, Li D. Properties of analytic solution and numerical solution of multi-pantograph equation. Applied Mathematics and Computation. 2004 Aug 16;155(3):853–71.

[7] Derfel G, Iserles A. The pantograph equation in the complex plane. Journal of Mathematical Analysis and Applications. 1997 Sep 1;213(1):117–32.

[8] Muroya Y, Ishiwata E, Brunner H. On the attainable order of collocation methods for pantograph integro-differential equations. Journal of Computational and Applied Mathematics. 2003 Mar 1;152(1):347–66.

[9] Aiello WG, Freedman HI, Wu J. Analysis of a model representing stage-structured population growth with state-dependent time delay. SIAM Journal on Applied Mathematics. 1992 Jun;52(3):855–69.

[10] Buhmann M, Iserles A. Stability of the discretized pantograph differential equation. Mathematics of Computation. 1993;60(202):575–89.

[11] Li D, Liu MZ. Runge–Kutta methods for the multi-pantograph delay equation. Applied mathematics and computation. 2005 Apr 5;163(1):383–95.

[12] Sezer M, Sahin N. Approximate solution of multi-pantograph equation with variable coefficients. Journal of Computational and Applied Mathematics. 2008 May 1;214(2):406–16.

[13] Sezer M, Akyüz-Dascioglu A. A Taylor method for numerical solution of generalized pantograph equations with linear functional argument. Journal of Computational and Applied Mathematics. 2007 Mar 1;200(1):217–25.

[14] Sezer M, Yalçinbas S, Gülsu M. A Taylor polynomial approach for solving generalized pantograph equations with nonhomogenous term. International Journal of Computer Mathematics. 2008 Jul 1;85(7):1055–63.

[15] Du P, Geng F. A new method of solving singular multi-pantograph delay differential equation in reproducing kernel space. Appl. Math. Sci. 2008;2(27):1299–305.

[16] RAJA MA, AHMAD I, Imtiaz KH, Muhammed Ibrahem SY, WAZWAZ AM. Neuro-heuristic computational intelligence for solving nonlinear pantograph systems. Frontiers. 2016;1.

[17] Ahmad I, Mukhtar A. Stochastic approach for the solution of multi-pantograph differential equation arising in cell-growth model. Applied Mathematics and Computation. 2015 Jun 15;261:360–72.

[18] Jiang W, Chen Z, Zhang C. Chebyshev collocation method for solving singular integral equation with cosecant kernel. International Journal of Computer Mathematics. 2012 May 1;89(8):975–82.

[19] Isik OR, Güney Z, Sezer M. Bernstein series solutions of pantograph equations using polynomial interpolation. Journal of Difference Equations and Applications. 2012 Mar 1;18(3):357–74.

[20] Ahmed, H.M., 2016. Solution of nonlinear fractional stochastic integro-differential equation. Lobachevskii Journal of Mathematics, 37(2), pp. 105–113.

[21] Bibikov, P., 2015. Differential invariants and contact classification of ordinary differential equations. Lobachevskii Journal of Mathematics, 36(3), pp. 245–249.