Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

Compatible pair of actions for two same cyclic groups of 2-power order

To cite this article: Mohd Sham Mohamad *et al* 2017 *J. Phys.: Conf. Ser.* **890** 012120

IOP Conf. Series: Journal of Physics: Conf. Series **890** (2017) 012120 doi:10.1088/1742-6596/890/1/012120

Compatible pair of actions for two same cyclic groups of 2-power order

Mohd Sham Mohamad, Sahimel Azwal Sulaiman, Yuhani Yusof and Mohammed Khalid Shahoodh

Applied & Industrial Mathematics (AIMs) Research Cluster, Faculty of Industrial Sciences & Technology, Universiti Malaysia Pahang, Lebuhraya Tun Razak, 26300 Gambang, Kuantan, Pahang Darul Makmur

Email: mohdsham@ump.edu.my

Abstract. The nonabelian tensor product is defined for a pair of groups which act on each other provided the actions satisfying the compatibility conditions. In this paper, the compatible pairs of actions and cyclic groups of the 2-power order are considered. The aim of this study is to investigate the number of the compatible pair of actions for two same cyclic groups of 2-power order with same order of actions. In attained the objectives stated, the necessary and sufficient conditions of cyclic groups of 2-power order act compatibly on each other are used. At the end of this paper, the exact number of compatible pair of actions for two same cyclic groups of 2-power order with a same order of actions is given.

1. Introduction
In 1984, [1] introduced the nonabelian tensor product for group extended from the concept by [2]. It has its origins in the algebraic K-theory and homotopy theory. The nonabelian tensor product is defined in [1] as a group generated by the symbols $g \otimes h$ with relations $gg' \otimes h = \left({}^{g}g' \otimes {}^{g}h\right)\left(g \otimes h\right)$, $g \otimes hh' = \left(g \otimes h\right)\left({}^{h}g \otimes {}^{h}h'\right)$ and satisfy the compatibility conditions for all $g, g' \in G$ and $h, h' \in H$. However, the paper by [3] in 1987 became starting point for research of nonabelian tensor product. They provided a list of open problems regarding nonabelian tensor product and nonabelian tensor square. From the given problems by [3], many researchers had studied group theoretical condition of nonabelian tensor product.

This research focused on the concept of compatibility since it is the conditions in determining the nonabelian tensor product. [4] determined the compatible conditions of actions for the finite cyclic group when one of action is trivial and both are trivial. Then, [5] presented the new necessary and sufficient conditions of finite cyclic groups of 2-power order which included the order of the action as one of the conditions. [6] also provide the compatible conditions of actions for finite cyclic group of order p^2 with the order of action is p. In 2015, [7] gave specific pairs of actions which compatible for some finite cyclic 2-groups but focused on actions of order two and four. Then [8] studies on the compatible pair of nontrivial action for finite cyclic 2-groups. While, [9] focused on compatible pair of nontrivial actions for cyclic groups of 3-power order.

IOP Conf. Series: Journal of Physics: Conf. Series **890** (2017) 012120 doi:10.1088/1742-6596/890/1/012120

In this paper, the specific cases of the compatible pair of actions are determined which is when $G = H$ with the actions have same order. Next, the general exact numbers of compatible pairs of nontrivial actions for two same cyclic groups of 2-power order with same order of actions are given.

2. The Preliminary Results

In this section, all related definitions and previous results on compatible conditions that have been done before are stated. We start with the definition of actions which given in the following definition.

Definition 2.2 Action [4]

Let *G* and *H* be groups. An action of *G* on *H* is a mapping, $\Phi: G \to \mathrm{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)\big(\Phi(g')(h)\big)$$

for all $g, g' \in G$ and $h \in H$.

If *G* and *H* are cyclic groups, then the action of group *G* act on group *H* be required to have the property that the identity in *G* acts as the identity mapping on *H*. Thus, all elements in *G* act as automorphism *H* on *H*. Then, the definition of compatible action is given as follows.

Definition 2.1 Compatible Action [1]

Let *G* and *H* be groups which act on each other. These mutual action are said to be compatible with each other and with the actions of *G* and *H* on themselves by conjugation if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \text{ and } ^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

By Definition 2.1, if one of actions is trivial, then any actions of *H* act on *G* are compatible. Next, the result on action of *G* acts trivially on *H* is given in the following corollary.

Corollary 2.3 [4]

Let *G* and *H* be groups. Furthermore, let *G* act trivially on *H*. If *G* is abelian, then for any action of *H* on *G* the mutual actions are compatible.

Next, the presentation of all automorphisms for cyclic groups of 2-power order is given in the following theorem.

Theorem 2.4 [5]

Let $G = \langle g \rangle \cong C_{2^n}, n \geq 3$. Then, Aut(*G*) $= \langle \tau \rangle \times \langle \rho \rangle$, where $\tau(g) = g^{-1}$ and $\rho(g) = g^5$ and every $\sigma \in$ Aut(*G*) can be represented as $\sigma = \tau^i \rho^j$ with $i = 0, 1$ and $j = 0, 1, \ldots, 2^{n-2} - 1$ and $\tau^i \rho^j (g) = g^t$ with $t \equiv (-1)^i \bullet 5^j \bmod 2^n$.

[5] provided the necessary and sufficient conditions for a compatible pair of actions for two cyclic groups of 2-power order to act compatibly on each other as stated in Theorem 2.5 and Theorem 2.6. The compatible pair of actions when one of the actions has order two are provided in the following theorem.

IOP Conf. Series: Journal of Physics: Conf. Series **890** (2017) 012120 doi:10.1088/1742-6596/890/1/012120

Theorem 2.5 [5]
Let $G=\langle x\rangle \cong C_{2^m}$ and $H=\langle y\rangle \cong C_{2^n}$ .Let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma|=2$ and $\sigma' \in \mathrm{Aut}(H)$ $m\geq 1, n\geq 3$.

i. If $\sigma(x)=x^t$ with $t\equiv -1 \mod 2^m$ or $t\equiv 2^{m-1}-1 \mod 2^m$, then (σ,σ') is a compatible pair if and only if σ' is trivial automorphism or $|\sigma'|=2$.

ii. If $G=\langle x\rangle \cong C_{2^m}$ with $t\equiv 2^{m-1}+1 \mod 2^m$, then (σ,σ') is a compatible pair if and only if $|\sigma'|=2^{s'}$ with $s'\leq m-1$, in particular σ is compatible with all $\sigma'\in \mathrm{Aut}(H)$ provided $n\leq m+1$.

Next, the necessary and sufficient conditions when one of the actions has order greater than two are given.

Theorem 2.6 [5]
Let $G=\langle x\rangle \cong C_{2^m}$ and $H=\langle y\rangle \cong C_{2^n}$ .Let $\sigma\in \mathrm{Aut}(G)$ with $|\sigma|=2^s$, $s\geq 2$ and $\sigma'\in \mathrm{Aut}(H)$ $m\geq 4, n\geq 1$.

i If $\sigma(x)=x^t$ with $t\equiv (-1)^i 5^j \mod 2^m$ and i = 1, then (σ,σ') is a compatible pair if and only if $\sigma'(y)=y^{t'}$ with $t\equiv 1 \mod 2^n$ and $t\equiv 2^{n-1}+1 \mod 2^n$.

ii If $\sigma(x)=x^t$ with $t\equiv (-1)^i 5^j \mod 2^m$ and i = 0, then (σ,σ') is a compatible pair if and only if $|\sigma'|\leq 2^{m-s}$ provided $n\leq m-s+2$.

[8] presented the necessary and sufficient conditions of two cyclic groups of 2-power order that actions compatible on each other when $G=H$ as given in the following proposition.

Proposition 2.7 [8]
Let $G=H=\langle g\rangle \cong C_{2^m}$. Furthermore, let $\sigma\in \mathrm{Aut}(G)$ and $\sigma'\in \mathrm{Aut}(H)$ with $|\sigma|=|\sigma'|=2^k$. (σ,σ') is compatible pair of actions if $k=0, k=1$ and $k\geq 2$ with $\sigma(g)=g^t$ where $t\equiv 5^j \mod 2^m$.

In the next section, the total number of the compatible pairs of actions with same order for two same cyclic groups of 2-power order is determined.

3. Results
The necessary and sufficient conditions of two cyclic groups of 2-power order given in the previous section are used to find the number of compatible pairs of actions. The general exact numbers of compatible pairs of nontrivial actions for two same cyclic groups of 2-power order with $|\sigma|=|\sigma'|=2^k$ are determined. The following proposition shows the number of compatible pair of actions when $|\sigma|=|\sigma'|=2^k$ and $G=H$.

Proposition 3.1
Let $G=H=\langle g\rangle \cong C_{2^m}$ be cyclic groups, $m\geq 1$. Furthermore, let $\sigma\in \mathrm{Aut}(G)$ and $\sigma'\in \mathrm{Aut}(H)$ with $|\sigma|=|\sigma'|=2^k$.

i. If $k=0$, then the number of compatible pair of actions is one.
ii. If $k=1$, then the number of compatible pair of actions are nine.
iii. If $k\geq 2$, then the number of compatible pair of actions are 2^k.

Proof Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 1$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$.

i. Let $k = 0$. There is one automorphism of order one for each σ and σ'. Thus, there is one compatible pair of actions only.
ii. Let $k = 1$. There are three actions have order two and by Proposition 2.5 and Proposition 2.7, all actions are always compatible with both actions have order two. Thus, there are nine compatible pair of actions.
iii. Let $k \geq 2$. By Theorem 2.4, there exist 2^k automorphisms of order 2^k and (σ, σ') is compatible pair of actions when $|\sigma'| \leq 2^{m-k}$ in Theorem 2.6. Thus, the number of compatible pair of actions are 2^k. □

Particularly, the following lemma gives the number of the compatible pair of actions for two same cyclic groups 2-power order with $|\sigma| = |\sigma'| = 2^k$ for $k \geq 2$ is presented.

Lemma 3.2
Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 1$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k \geq 2$. The number of compatible pair of actions are $2^{m-2} - 4$.

Proof Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 1$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k \geq 2$. By Proposition 3.1, there are 2^k compatible pair of actions when $k \geq 2$. Without loss of generality, consider the highest order of σ in C_{2^m} is 2^{m-3}. Then, (σ, σ') is compatible pair of actions when $|\sigma'| \leq 2^{m-k}$ by Theorem 2.6 (ii). Hence, the number of the compatible pair of actions are

$$2^2 + 2^3 + 2^4 + \ldots + 2^{m-3} = 2^{m-2} - 4$$

Therefore, the number of compatible pair of actions that has order greater than two are $2^{m-2} - 4$. □

Consequently, the total number of the compatible pair of actions for two same cyclic groups 2-power order with $|\sigma| = |\sigma'| = 2^k$ for $k \geq 0$ is given as follows.

Theorem 3.3
Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 1$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k = 0, 1, \ldots, 2^{m-3}$. Hence, there exist $2^{m-2} + 6$ compatible pair of actions.

Proof Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 1$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k = 0, 1, \ldots, 2^{m-3}$. By Proposition 3.1, there are three cases represented by $k = 0$, $k = 1$ and $k \geq 2$.

i. By Proposition 3.1(i), only one compatible pair of actions exist when $k = 0$.
ii. By Proposition 3.1(ii), there are nine compatible pair of actions when $k = 1$.
iii. By Lemma 3.2, the number of compatible pair of actions are $2^{m-2} - 4$ when $k \geq 2$,

Hence, the number of the compatible pair of actions for the same group and both actions have same order is $1+9+2^{m-2}-4=2^{m-2}+6$. □

4. Conclusion

The number of the compatible pairs of actions for two cyclic groups has been determined. There are $2^{m-2}+6$ compatible pair of actions with same order of actions for two same cyclic groups of 2-power order.

Acknowledgments

This research is supported by UMP Research Grant, RDU1703265.

References

[1] Brown R and Loday J L 1984 *Acad. Sci. Paris SI Math* **298**(15) pp 353-356

[2] Whitehead J A 1950 *Ann. of Math* **51 pp** 51-110

[3] Brown R, Johnson D L and Robertson E F 1987 *Journal of Algebra* **111**(1) pp 177-202

[4] Visscher M P 1998 *On the nonabelian tensor product of groups* Dissertation State University of New York

[5] Mohamad M S 2012 *Compatibility conditions and nonabelian tensor products of finite cyclic groups of p-power order* Dissertation Universiti Teknologi Malaysia

[6] Mohamad M S, Sarmin N S, Ali N M M, and Kappe L C 2012 *Jurnal Teknologi* **57**(1)

[7] Sulaiman S A, Mohamad M S,Yusof Y, Sarmin N H, Ali N M M, Ken T L and Ahmad T 2014 *The 2nd ISM International Statistical Conference 2014 (ISM-II): Empowering the Applications of Statistical and Mathematical Sciences AIP Proceeding Journal* **1643** pp 700-705

[8] Sulaiman S A, Mohamad M S, Yusof Y and Shahoodh M K 2016 *National Conference for Postgraduate Research (NCON-PGR 2016)* pp 39-42

[9] Shahoodh M K, Mohamad M S, Yusof Y, and Sulaiman S A 2016 *National Conference for Postgraduate Research (NCON-PGR 2016)* pp 792-797