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Mathematical modelling of wave impacts on seaward-inclined seawall

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Abstract. In this study, the numerical investigation of a mathematical model of wave impacts on an inclined seawall is considered, using an extension of Cooker's model for vertical seawalls due to Greenhow, who considered a perturbation model to study the effect of small wall inclinations from the vertical. Cooker used pressure impulse theory to simplify the highly time-dependent and very nonlinear boundary-value problem (bvp) by considering the time integral of the pressure during the duration of the impact pressure-impulse. The solution to this bvp is found by solving Laplace's Equation for simplified boundary conditions. The perturbation theory gives a series of bvps which are solved analytically by MATLAB. The relation between the pressure impulse and the inclination angle of a wall is investigated. We find that as the seaward or landward inclination of the seawall increases, the pressure impulse increases.

1. Introduction

An important problem in coastal engineering is wave impact on coastal structures. Experiments at model- and full-scale can be difficult to replicate due the the uncertain nature of the impacting wave, whilst theory requires the numerical solution of an analytically-intractable bvp. However, Bagnold [1] found that whilst the peak pressures were highly variable, the pressure impact of equation 1) was much more stable and robust. Thus Cooker and Peregrine [2,3] formulated the bvp for the pressure impulse to investigate the impact of waves breaking on coastal structures. Since the pressure impulse $P(x,y)$ is equal to the integral of time of the pressure within the duration of the impact, time is therefore integrated out of the problem, i.e.

$$P(x,y)=\int_{t_b}^{t_a} p(x,y,t)dt \tag{1}$$

Cooker [4] idealised the real situation as a rectangular shaped region filled by fluid and the coastal structure was a vertical wall. The bvp is given as in Figure 1. Noar and Greenhow [5,6] extended this simple analytical model to a vertical seawall with a berm or ditch, and a vertical seawall with a missing block. They claimed the berm has only small effect in the pressure impulse on the seawall while within the ditch, high pressure impulses were found; if repeated throughout a storm, they may liquefy the seabed there and may destabilize the wall. The effect of a damaged seawall with a missing block, might be less serious since the pressure impulse decreases as the width of increases. Noar and Greenhow [7,8] also studied wave impact on a deck and baffle using a theoretical approach. Recently, Ramli MS et al. [9] modified the mathematical modeling of Cooker's model from vertical wall into landward-inclined seawall. They found the pressure impulse on landward-inclined seawall increases as the angle of an inclined seawall increases.

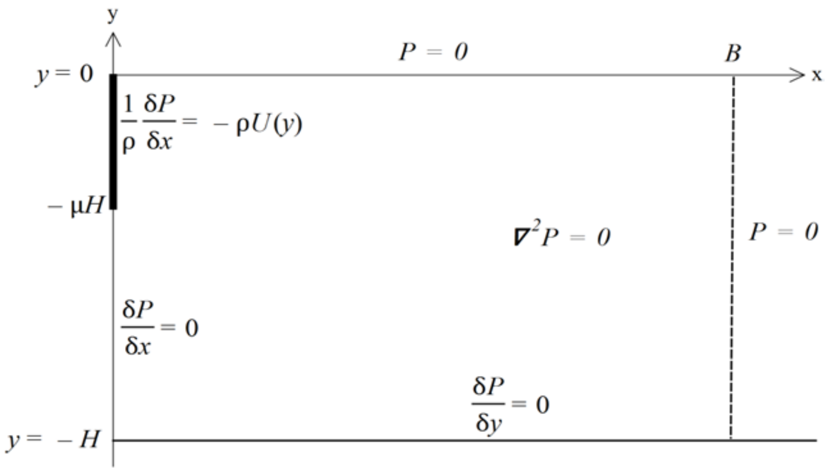


Figure 1. The boundary-value problem for pressure impulse.

Richert [10] conducted laboratory experiments for breaking wave impact on a 30° landward sloping wall while Whillock [11] conducted breaking wave impact experiments on a 15° landward-sloping wall. Later on, Kirkgoz [12] studied the impact pressure of breaking waves on vertical and landward sloping wall for 5°, 10°, 20°, 30° and 45° slopes. He stated that a landward sloping wall experienced greater impact pressure compared to vertical wall.

The purpose of the present study is to investigate the effect of the small angle of inclination from the vertical as Greenhow [13] and calculate its effect on the pressure impulses on the wall by using pressure-impulse theory. The perturbation method is applied here and the results are analysed and discussed in details.

2. Mathematical model and formulation

By referring figure 2, the base of the wall is positioned at $(0, -\mathrm{H})$ and the top of the wall is at $(\mathrm{x_w}, 0)$. An angle, ε is chosen as a parameter, where ε is the angle of the wall from vertical line in an opposite clockwise direction and it is greater than zero (positive value) in this case. Now, we have

$$P(x, y; \varepsilon) = \sum_{k=0}^{\infty} P^{(k)}(x, y)\varepsilon^k \tag{2}$$

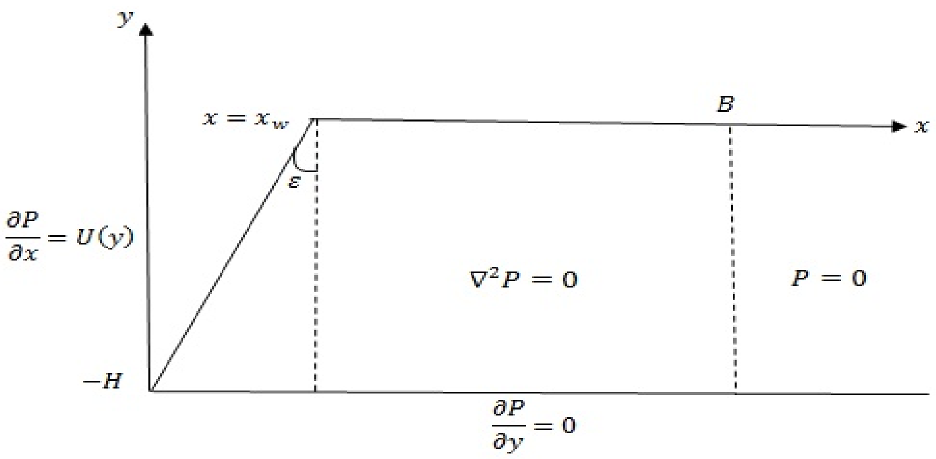


Figure 2. The boundary condition for pressure impulse of wave impact on seaward-inclined seawall.

This problem becomes a series of problems for various type of order of ε, with each new of order depends on the previous one. Each of $P^{(k)}$ term is given by

$$P^{(k)}(x, y) = P^{(k)} e^{-\gamma_n x} \sin \gamma_n y \tag{3}$$

We assume the wall is flat, and it can be explained by

$$x = \frac{x_w}{H} y + x_w = (y + H) \tan \varepsilon \tag{4}$$

The normal for this type of wall is given by:

$$\underline{n} = \cos \varepsilon \underline{i} - \sin \varepsilon \underline{j} \tag{5}$$

The boundary condition of the wall impacted by the water waves become

$$\frac{\partial P}{\partial n} = \nabla P . \underline{n} = \cos \varepsilon \frac{\partial P}{\partial x} - \sin \varepsilon \ \frac{\partial P}{\partial x} = -\rho U \cos \varepsilon \tag{6}$$

For the impacted region, the boundary condition is

$$\frac{\partial P}{\partial x} = \tan \varepsilon \frac{\partial P}{\partial y} \tag{7}$$

For small ε we have the following approximations:

$$\sin \varepsilon \approx \varepsilon - \frac{\varepsilon^3}{6} + \cdots \quad , \quad \cos \varepsilon \approx 1 - \frac{\varepsilon^2}{2} + \cdots \quad , \quad \tan \varepsilon \approx \varepsilon + \frac{\varepsilon^3}{3} + \cdots \tag{8}$$

By expanding in powers of ε and applying perturbation theory we get:

$$P=\sum_{k=0}^{\infty}P^{(k)}\varepsilon^{k}=P^{(k)}+\varepsilon P^{(1)}+\varepsilon^{2}P^{(2)}+\varepsilon^{3}P^{(3)}+\cdots \tag{9}$$

Equation of (9) will be truncated for small ε, giving the sought perturbation solution to the boundary conditions as

$$\begin{aligned}&P_x^{(0)}+\varepsilon\left(P_x^{(1)}-P_y^{(0)}\right)+\varepsilon^2\left(P_x^{(2)}-P_y^{(1)}-\frac{P_x^{(0)}}{2}\right)+\varepsilon^3\left(P_x^{(3)}-P_y^{(2)}-\frac{P_x^{(1)}}{2}-\frac{P_y^{(0)}}{6}\right)\\&\qquad+\cdots\\&=-\rho U_\circ\cos\varepsilon=-\rho U_\circ\left(1-\frac{\varepsilon^2}{2}\right)+O(\varepsilon^4)\end{aligned} \tag{10}$$

Each of $P_x^{(k)}$ and $P_y^{(k)}$ term is expanded by using Taylor Series at $x=0$; thus

$$P_x^{(k)}=P_x^{(k)}(0,y)+xP_{xx}^{(k)}(0,y)+\frac{x^2}{2}P_{xxx}^{(k)}(0,y)+\frac{x^3}{6}P_{xxx}^{(k)}(0,y)+\cdots \tag{11}$$

$$P_y^{(k)}=P_y^{(k)}(0,y)+xP_{yx}^{(k)}(0,y)+\frac{x^2}{2}P_{yxx}^{(k)}(0,y)+\frac{x^3}{6}P_{yxx}^{(k)}(0,y)+\cdots \tag{12}$$

Noticed from (8): $(y+H)\tan\varepsilon\approx(y+H)\left(\varepsilon+\frac{\varepsilon^3}{3}\right)$, so equation of (11) and (12) become:

$$\begin{aligned}P_x^{(k)}&=P_x^{(k)}(0,y)+\varepsilon.(y+H)P_{xx}^{(k)}(0,y)+\frac{\varepsilon^2(y+H)^2}{2}P_{xxx}^{(k)}(0,y)\\&\quad+\frac{\varepsilon^3}{3}\left((y+H)P_{xx}^{(k)}(0,y)+\frac{(y+H)^3}{2}P_{xxx}^{(k)}(0,y)\right)+\cdots\end{aligned} \tag{13}$$

$$\begin{aligned}P_y^{(k)}&=P_y^{(k)}(0,y)+\varepsilon.(y+H)P_{yx}^{(k)}(0,y)+\frac{\varepsilon^2(y+H)^2}{2}P_{yxx}^{(k)}(0,y)\\&\quad+\frac{\varepsilon^3}{3}\left((y+H)P_{yx}^{(k)}(0,y)+\frac{(y+H)^3}{2}P_{yxx}^{(k)}(0,y)\right)+\cdots\end{aligned} \tag{14}$$

If we neglect terms of $O(\varepsilon^4)$ and above, after substituting equation of (13) and (14) into (10) the boundary conditions on the wall take the form:

$$\begin{aligned}&P_x^{(0)}+\varepsilon(y+H)P_{xx}^{(0)}+\frac{\varepsilon^2(y+H)^2}{2}P_{xxx}^{(0)}+\frac{\varepsilon^3}{3}\left((y+H)P_{xx}^{(0)}+\frac{(y+H)^3}{2}P_{xxx}^{(0)}\right)+\cdots\\&+\varepsilon\left(P_x^{(1)}+\varepsilon(y+H)P_{xx}^{(1)}+\frac{\varepsilon^2(y+H)^2}{2}P_{xxx}^{(1)}-P_y^{(0)}-\varepsilon(y+H)P_{yx}^{(0)}-\frac{\varepsilon^2(y+H)^2}{2}P_{yxx}^{(0)}\right)+\cdots\\&+\varepsilon^2\left(P_x^{(1)}+\varepsilon(y+H)P_{xx}^{(2)}-P_y^{(1)}-\varepsilon(y+H)P_{yx}^{(1)}-\frac{P_x^{(0)}}{2}-\frac{\varepsilon(y+H)P_{xx}^{(0)}}{2}\right)+\cdots\\&+\varepsilon^3\left(P_x^{(3)}-P_y^{(2)}-\frac{P_x^{(1)}}{2}-\frac{P_y^{(0)}}{6}\right)=-\rho U_\circ\left(1-\frac{\varepsilon^2}{2}\right)\end{aligned} \tag{15}$$

which is applied at $x=0$. Equating the coefficients of the powers of ε for both sides in equation (15) gives:

$$\varepsilon^0: \; P_x^{(0)} = \begin{cases} 0, & -H < y < -\mu H \\ -\rho U, & -\mu H < y < 0 \end{cases} \tag{16}$$

This is identical to Cooker's bvp which then has the solution:

$$P(x,y) = \sum_{n=0}^{\infty} a_n \sin\left(\gamma_n \frac{y}{H}\right) \frac{\sinh\left(\frac{\gamma_n (x-B)}{H}\right)}{\cosh\left(\frac{\gamma_n B}{H}\right)} \tag{17}$$

$$\varepsilon^1 \; : \quad P_x^{(1)} = P_y^{(0)} - (y+H) P_{xx}^{(0)}, \qquad -H < y < 0 \tag{18}$$

$$\varepsilon^2 \; : \quad P_x^{(2)} = \begin{cases} P_y^{(1)} + \dfrac{P_x^{(0)}}{2} - (y+H)\left(P_{xx}^{(1)} - P_{yx}^{(0)}\right) - \dfrac{(y+H)^2 P_{xxx}^{(0)}}{2}, & -H < y < -\mu H \\ P_y^{(1)} + \dfrac{P_x^{(0)}}{2} - (y+H)\left(P_{xx}^{(1)} - P_{yx}^{(0)}\right) - \dfrac{(y+H)^2 P_{xxx}^{(0)}}{2} + \dfrac{\rho U_\circ}{2}, & -\mu H < y < 0 \end{cases} \tag{19}$$

$$\varepsilon^3 \; : \quad P_x^{(3)} = P_y^{(2)} - \frac{P_x^{(1)}}{2} - \frac{P_y^{(0)}}{6} - (y+H)\left(P_{xx}^{(2)} - P_{yx}^{(1)} - \frac{P_{xx}^{(0)}}{6}\right) - \frac{(y+H)^2}{2}\left(P_{xxx}^{(1)} - P_{yxx}^{(0)}\right) - \frac{(y+H)^3 P_{xxxx}^{(0)}}{6}, \qquad -H < y < 0 \tag{20}$$

Next, the first-order solution $O(\varepsilon^1)$ is solved. The nth term of $P_n^{(k)}$ can be separated on the left-hand side by multiply each condition by $\sin\left(\frac{\gamma_n}{H}\right)$, integrating from $(-H, 0)$ and applying orthogonality of the functions $\left(\sin\left(\frac{\gamma_n}{H}\right), n \in N\right)$. On the right hand side, we have an infinite sum. Hence, each $P_n^{(k-1)}$, $P_n^{(k-2)}$ and so on are defined by the solution to the previous bvp. The required coefficients are determined by integral of products from the set $\left(\sin\left(\frac{\gamma_n y}{H}\right), \cos\left(\frac{\gamma_n}{H}\right), y^n, n \in N\right)$. Now, in order to find the first order solution, the equation in ε^1 will be multiplied by $\sin\left(\frac{\gamma_n y}{H}\right)$ and integrated from $(-H, 0)$, giving:

$$\int_{-H}^{0} P_x^{(1)} \sin\left(\frac{\gamma_n y}{H}\right) dy = \int_{-H}^{0} \left(P_y^{(0)} - (y+H) P_{xx}^{(0)}\right) \sin\left(\frac{\gamma_n y}{H}\right) dy \tag{21}$$

Calculating the derivatives gives,

$$P_x^{(1)}(0,y) = -\sum_{n=0}^{\infty} \gamma_n P_n^{(1)} \sin\left(\frac{\gamma_n y}{H}\right) \tag{22}$$

$$P_y^{(0)}(0,y) = \sum_{n=0}^{\infty} \gamma_n P_n^{(0)} \cos\left(\frac{\gamma_n y}{H}\right) \tag{23}$$

$$P_{xx}^{(0)}(0,y) = -\sum_{n=0}^{\infty} \gamma_n^2 P_n^{(0)} \sin\left(\frac{\gamma_n y}{H}\right) \tag{24}$$

Next, we substitute the equation of (22), (23) and (24) into (21) giving the following approximation for the first-order solution:

$$\begin{aligned} P(x,y) &= P^{(0)}(x,y) + \varepsilon P^{(1)}(x,y \\ &= \textstyle\sum_{n=0}^{\infty} P_n^{(0)} e^{-\gamma_n x} \sin\left(\frac{\gamma_n y}{H}\right) + \varepsilon \sum_{n=0}^{\infty} P_n^{(1)} e^{-\gamma_n x} \sin\left(\frac{\gamma_n y}{H}\right) \\ &= \textstyle\sum_0^{\infty} \left(P_n^{(0)} + \varepsilon P_n^{(1)}\right) e^{-\gamma_n x} \sin\left(\frac{\gamma_n y}{H}\right) \\ &\approx \textstyle\sum_0^{N} \left(P_n^{(0)} + \varepsilon P_n^{(1)}\right) e^{-\gamma_n x} \sin\left(\frac{\gamma_n y}{H}\right) \end{aligned} \tag{25}$$

A solution of $P^{(2)}(x,y)$ of $O(\varepsilon^2)$ will be found by using similar method as $P^{(1)}(x,y)$ so equation of (19) will be multiplied by $\sin\left(\frac{\gamma_n y}{H}\right)$ and integrated over $(-H,0)$, giving:

$$\int_{-H}^{0} P_x^{(2)} \sin\left(\frac{\gamma_n y}{H}\right) dy = \int_{-H}^{0} \left(P_y^{(1)} - (y+H)\left(P_{xx}^{(1)} - P_{yx}^{(0)}\right) - \frac{(y+H)^2}{2} P_{xxx}^{(0)}\right) \sin\left(\frac{\gamma_n y}{H}\right) dy \tag{26}$$

Calculating the relevant derivatives and substituting them into equation 26) leads to the following approximation for the second-order term $P^{(2)}(x,y)$ and therefore the second-order solution:

$$\begin{aligned} P(x,y) &= P^{(0)}(x,y) + \varepsilon P^{(1)}(x,y) + \varepsilon^2 P^{(2)}(x,y) \\ &= \sum_0^{\infty} \left(P_n^{(0)} + \varepsilon P_n^{(1)} + \varepsilon^2\right) e^{-\gamma_n x} \sin\left(\frac{\gamma_n y}{H}\right) \\ &\approx \sum_0^{N} \left(P_n^{(0)} + \varepsilon P_n^{(1)} + \varepsilon^2\right) e^{-\gamma_n x} \sin\left(\frac{\gamma_n y}{H}\right) \end{aligned} \tag{27}$$

Equations (16), (18) and (19), equation (27) are solved using MATLAB.

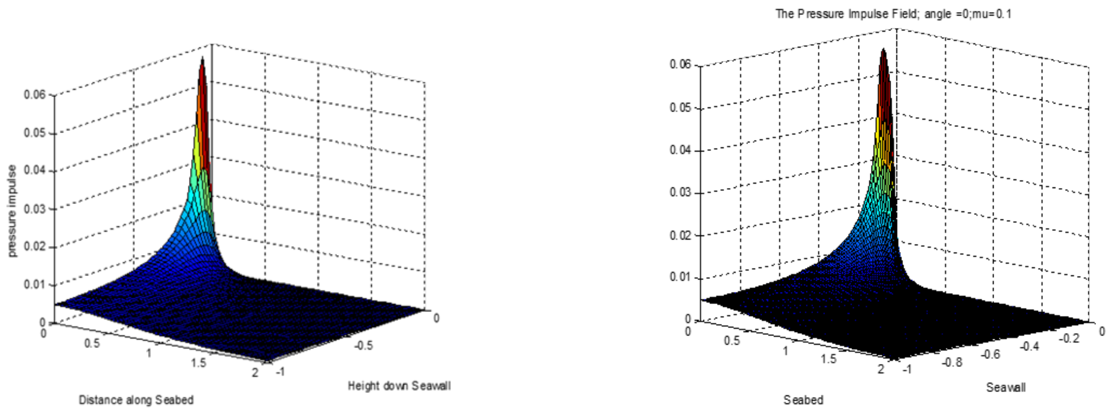
3. Result and discussion

Table 1 shows the comparison results of pressure impulse from Cooker with vertical wall and the present seaward-inclined-impact problem. We validated our method by comparing the results from Cooker [4] with present seaward-inclined-impact problem at $\varepsilon = 0^0$. Results for μ of 0.1, 0.5 and 0.8, are consistent with Cooker.

Table 1. Comparison values of pressure-impulse with Cooker for various value of μ when $\varepsilon = 0^0$.

μ	**Cooker**	**Present (at $\varepsilon = 0°$)**
0.1	0.058	0.058
0.5	0.290	0.290
0.8	0.510	0.510

Figure 3 shows good agreement with results for pressure impulse at $\mu = 0.1$ of Cooker and present formulation, respectively.



(a) (b)

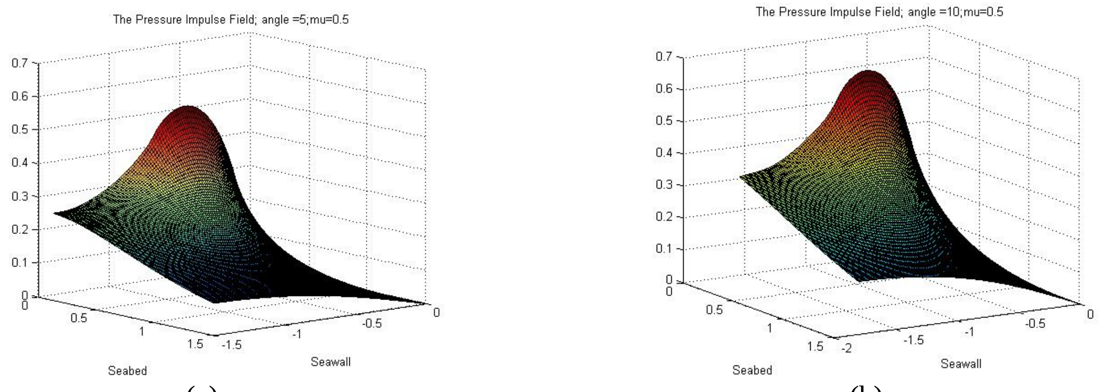
Figure 3. Comparison values of pressure-impulse between Cooker, (a) and present, (b) at $\mu = 0.1$.

Table 2 shows results for pressure impulse on seaward-inclined seawalls for different angle of inclination. We can see that as the angle of a seaward-inclined seawall increases, the pressure impulse also increases. Furthermore, when the value of μ is increased, we can see that the pressure impulse also increases and becomes higher when inclination is increased.

Table 2. Comparison values of pressure-impulse for various value of μ when $\varepsilon = 5^0$ and $\varepsilon = 10^0$.

μ	**$\varepsilon = 5°$**	**$\varepsilon = 10°$**
0.1	0.076	0.080
0.5	0.510	0.590
0.8	0.910	1.010

Figure 4 shows that comparison between pressure impulse for $\varepsilon = 5^0$ and $\varepsilon = 10^0$ at $\mu = 0.5$. We note that pressure impulse increases on the seaward-inclined seawall as the angle increases. From the figure, we also can see the peak for pressure point of the graphs become more rounded as μ is increased. This is similar to the results of Kirkgoz [12] who studied the pressure impulse on vertical and seaward-inclined walls.



(a) (b)

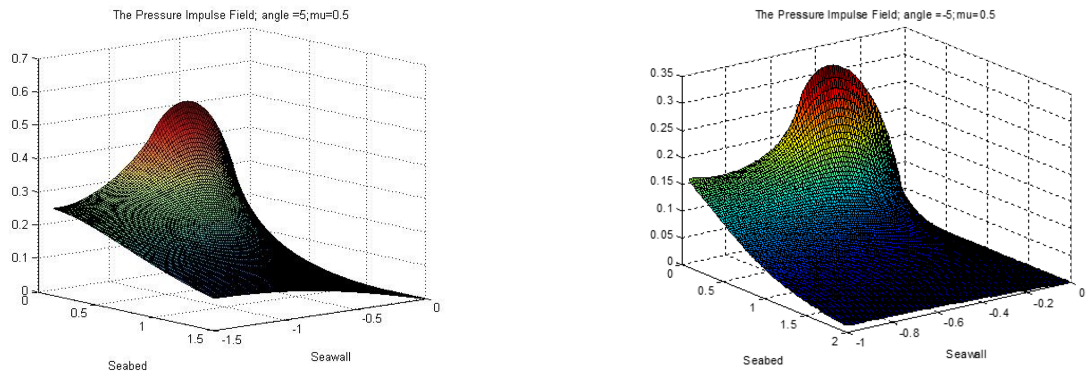
Figure 4. Comparison between angle of 5°, (a) and 10°, (b) at $\mu = 0.5$.

Table 3 shows results for pressure impulse on a seaward-inclined seawall at an angle of 5° for different values of μ. We can see that seaward-inclined seawall has much higher value of pressure impulse compared to landward-inclined wall at similar μ.

Table 3. Comparison values of pressure-impulse at different μ and $\varepsilon = 5^0$ of seaward-inclined and landward-inclined wall.

μ / ε	**Seaward-inclined seawall, 5°**	**Landward-inclined seawall, 5°**
0.1	0.076	0.068
0.5	0.510	0.310
0.8	0.910	0.530

Figure 5 shows the comparison between pressure impulse of seaward-inclined seawall of $\varepsilon = 5^0$ at $\mu = 0.5$. We note that seaward-inclined seawall experiences more pressure impulse compared to landward-inclined seawall at the same angle of inclination to the vertical.



(a) (b)

Figure 5. Comparison between angle of 5°, seaward-inclined wall (a) and landward-inclined wall (b) at $\mu = 0.5$.

4. Conclusion

A pressure-impulse model is presented for impulsive wave impact on a seaward-inclined seawall. From the numerical solution we found inclination has a significant effect on the pressure impulse on the wall. The effect of inclination, ε is largest when μ is largest. This conclusion means that mathematical models can be used to estimate or at least show trends in the wave impact pressures on coastal structures with inclined structures and are in agreement with Kirkgoz [12] who studied the impact pressure of breaking waves on landward sloping seawalls in laboratory experiments. By comparing with Ramli MS et al. [9], we see that seaward-inclined seawalls experience more pressure impulse compared to landward-inclined seawalls. Further comparisons with experiment data would be useful.

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References
[1] Bagnold R A 1939 *Journal of the Institute of Civil Engineer* **12** pp 202-206
[2] Cooker M J and Peregrine D H 1990 *Proc. 22nd Intl. Conf. Coastal Engineering* pp. 1473-1486
[3] Cooker M J and Peregrine D H 1995 *J. Fluid Mech* **297** pp 193-214
[4] Cooker M J 1990 *Ph.D Thesis* University of Bristol, United Kingdom
[5] Noar N A Z M 2012 *Ph.D. Thesis* Brunel University, United Kingdom
[6] Noar N A Z M and Greenhow M 2015 *AIP Conference Proceedings* **1643**, 670
[7] Noar N A Z M and Greenhow M 2016 *ARPN Journal of Engineering and Applied Sciences* **11** pp 7269-7273
[8] Noar N A Z M and Greenhow M 2015 *Applied Ocean Research* **53** pp 132-141
[9] Ramli MS et. al 2017 *AIP Conference Proceedings* **1842**, 030009
[10] Richert G 1968 *Proc. llth Conf. Coastal Engng, ASCE* **1** pp 954-973
[11] Whillock A F 1987 *Coastal Engng* **ll** pp 297-308
[12] Kirkgoz M S 1991 *Ocean Engineering* **18** pp 45-59
[13] Greenhow M 2006 *CoastalLabs06 Proceedings* pp 517-524