AN EFFICIENT RANKING ANALYSIS IN MULTI-CRITERIA DECISION MAKING

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Nomenclature

A_i	Alternative i
Y_j	Criterion j
Y_{ij}	Performance of alternative i in terms of criterion j
w_j	Weight of criterion j
w^*	New weight
$w^{'}$	Normalized weight
m	Number of criteria/objective functions
q	Number of alternatives/Pareto solutions
I^+	Ideal solution
I^-	Anti-ideal solution
A_k^*	k-th extreme solution
$\sim$	Uncertain parameter
σ_Y^2	Variance of variable Y
n	number of parameters which vary
ð	Value of change in the current weight
mp	Possibilistic mean
K	Number of Decision Makers
DM	Decision Maker
MCDM	Multi-criteria decision making
AHP	Analytic Hierarchy Process
ANP	Analytic Network Process
MAUT	Multi-Attribute Utility Theory

ELECTRE	Elimination and Choice Expressing Reality
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluations
GA	Genetic algorithm
PSO	Partial swarm optimization
DSD	Directed Search Domain
TOPSIS	Technique for order preference by similarity to ideal solution
VIKOR	Vise Kriterijumska Optimizacija I Kompromisno Resenje

An Efficient Ranking Analysis in Multi-criteria Decision Making

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The University of Manchester

Abstract

This study is conducted with the aims to develop a new ranking method for multi-criteria decision making problem with conflicting criteria. Such a problem has a set of Pareto solutions, where the act of improving a value of one solution will result in depreciating some of the others. Thus, in this type of problem, there is no unique solution. However, out of many available options, the Decision Maker eventually has to choose only one solution. With this problem as the motivation, the current study develops a compromise ranking algorithm, namely a trade-off ranking method. The trade-off ranking method able to give a trade-off solution with the least compromise compared to other choices as the best solution. The properties of the algorithm are studied in the thesis on several test cases. The proposed method is compared against several multi-criteria decision making methods with ranking based on the distance measure, which are the TOPSIS, relative distance and VIKOR. The sensitivity analysis and uncertainty test are carried out to examine the methods robustness. A critical criteria analysis is also done to test for the most critical criterion in a multi-criteria problem. The decision making method is considered further in a fuzzy environment problem where the fuzzy trade-off ranking is developed and compared against existing fuzzy decision making methods.

Keywords: Trade-off, ranking, multi-objective optimization, multi-criteria decision making, Pareto optimal solution, directed search domain algorithm

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning

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Author list of publications

- **Journal papers**

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- **Peer reviewed conferences**

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Sometimes in life,
you have to make DECISIONS
that are best for you,
not for everyone else

When someone makes a decision,
he is really diving into a strong current
that will carry him to places he had never dreamed of
when he first made the decision.
-Paul Coelho, *The Alchemist*

for my *Husband* and our two *daughters*

SECTION ONE

1 Introduction

1.1 Background, motivation and scope

Life is about making decisions. Most people attempt to make the best decision within a specified set of possible options. Thus, in a decision making process, the first task to do is to determine the criteria needed and the alternatives available. The alternative should satisfy the criteria constraints. In a multi-objective optimization problem, by presenting the criteria as the objective functions, a set of feasible solutions (alternatives) satisfying a set of constraints given can be generated. Once the Decision Maker (DM) is presented with the alternatives, he/she now have to choose the best option. Choosing the best solution might be difficult if there are many available options with almost the same quality. In particular, in the problem with conflicting multi-criteria, where there are no unique solutions and each solution is a trade-off of its criterion. For example, in order to decrease the purchase price of an item, the DM has to decrease its quality as well. However, in reality, DM always wants to opt for the best quality item in cheapest price available. Having the best quality of all things is the nature of human beings. In a situation where it is impossible to have all the best quality at once, the DM may opt for his/her preference. This is the part where the DM placed his/her preference value in each criterion. Referring the previous example of purchasing an item, a DM who prefers the purchase price over the quality may choose the item which is the cheapest, while a DM who opt for the quality may choose an item with the best quality regardless the price. There is no right or wrong in making choices. In the end, it would up to the DM. However, there are few methods available in helping the DM in this task. Each method stands by its own property and background. By this means, there is no single method that can cater for all types of decision making problems or all types of DMs.

The DM attribution in achieving the optimal solution is divided into four classes: no preference method, priori method, posteriori method and interactive method (Hwang and Masud, 2012). In no preference method, a neutral compromise solution is identified without the DM preference information. In priori method, the DM preference information is first asked and then the best solution satisfying the preference is found. While in posteriori, a representative set of Pareto optimal solutions is first found and then the DM must choose one of them. In interactive methods, the decision maker is allowed to iteratively search for the most preferred solution. In this study, no DM is involved physically, however we take into account the DM preferences by imposing the criteria weights. Hence, we can say that we imply the priori method in the study.

In helping the DM to choose the appropriate decision making methods, each method should be tested for its robustness and sensitivity towards a change in the problem. A multi-criteria decision making (MCDM) problem may have uncertainty towards its data. In this case, the change may occur in the input data, i.e. the criteria parameters or the DM preference. Once the sensitivity analysis and the robustness test are done, the DM is now able to distinguish each method in its strength and weakness towards the specific problem. Note that different problem may be better solved by the means of different methods. After the sensitivity analysis is done, the DM is now able to choose the most suitable method to seek for the best option according to him/her.

Once obtaining the best option, the DM may want to change his/her preference. Let say, instead of preferring the quality of an item, the DM now prefers the purchase price. The analysis of the preference change after getting the solution may give insight to the DM of how much of the change value that would affect the current solution. In a situation where there are few criteria considered, the DM can know whether the alternative stays as the best solution if he/she change the preference of one criterion over the others. If there is no change occurs, the DM may be delighted to know that the solution is not only satisfies his/her previous preference, but also his/her change of preference. If there is a change occurs, the DM may know by how much of the change that would affect the previous solution. With the knowledge of sensitivity of the methods, the uncertainty test of the problem and the change of a criterion preference, the DM is now able to make an efficient decision making process.

Now, consider a problem with conflicting criteria in a fuzzy environment. The fuzzy decision making consists of the fuzziness in the performance value of an alternative in each criterion and the fuzziness in the preference towards each criterion. It is the same as the uncertainty part mentioned in previous text, however in this problem the data are now consists of fuzzy numbers. A decision making method that can solve the fuzzy problem would be the help to the DM. In this problem, the fuzzy decision making method should be considered rather than the conventional method.

The problem, that is considered is this study, is the conflicting multi-criteria problem. This study is focused on the distance-based ranking techniques for the decision making tools. Either it is the deterministic decision making problem or the fuzzy decision making problem, the DM would be able to choose the best solution in an efficient way. That is the ultimate aim of this study. To achieve this aim, the objectives of this work are:

1. develop an efficient ranking algorithm based on a set of Pareto solutions,

2. test the robustness of the ranking algorithm for uncertainties and sensitivity analysis,

3. develop the ranking algorithm for a fuzzy multi-criteria problem,

4. apply the ranking algorithm to a design problem.

1.2 Multi-Criteria decision making method

In the real-life design it is required to improve different objectives simultaneously. A trade-off between the objectives is usually unavoidable because of the conflicting objectives as well as the constraints. As a result, the optimal solution is not unique and corresponds to a so-called Pareto solution. Each Pareto solution is defined as a trade-off between the conflicting criteria, where it is not possible to achieve the best score of a criterion without downgrading the score of some other criteria. In the objective space all Pareto solutions create a Pareto frontier. For a practical decision making analysis the Pareto frontier is represented by a Pareto set that contains a finite number of optimal solutions. Eventually, the DM has to choose only one solution. This leads to the problem of ranking because the definition of the Pareto frontier does not presume any preferences. The Decision Making analysis can be based on a ranking procedure to select the best solution among formal candidates representing the Pareto set. An additional algorithm is required to introduce the ranking.

In 1881, Edgeworth is the first to define an optimum for multi-criteria economic decision making (Edgeworth, 1881). He does so for the multi-utility problem within the context of two consumers, P and π, where it is required to find a point (x, y) such that in whatever direction we take, an infinitely small step, P and π do not increase together but that, while one increases, the other decreases. In 1906, a civil engineer turns economist, Pareto, creates his infamous theory, the Pareto optimum (Pareto, 1906). The theory states that the optimum allocation of the resources of a society is not attained so long as it is possible to make at least one individual better off in his own estimation while keeping others as well off as before in their own estimation. After the translation of Pareto's Manual of Political Economy into English, the notion of Pareto optimality begins to be applied to the fields of engineering and science (Pareto, 1971).

In the multi-criteria decision analysis, the decision making methods have been developed for more than 50 years (Triantaphyllou, 2013). However, there are no universal approaches. Each method stands on its own background and principles. The most natural approach is to introduce individual preferences. One of the basic and simplest multi-criteria decision analysis techniques is the sum of weight calculation model. In this technique, a weight is assigned to each criterion to denote its importance. Each aggregate function is then calculated as the sum of weight criteria. A classic work on the weight determination is by Eckenrode (1965). Eckenrode worked with twenty-four expert judges, who were required to put a weight on six criteria in a specified experiment related to an air-defence system.

Another well-known decision making method is the Analytic Hierarchy Process (AHP). AHP was proposed by Saaty (1980). The essence of this method is that a human judgement is used in performing evaluations. AHP structures a decision problem into an hierarchy with the goal, decision criteria, and alternatives. Then, it uses the pairwise comparison and the expert judgement, where these judgements are converted into a numerical evaluation. However, a human can lack of consistency in judging (influenced by emotional, experience, etc.) and different people have different preferences. AHP works best for decision making process in a group of people having consensus. Many authors used AHP in the decision making process (e.g. Kablan, 2004; Herath, 2004; Randall et al., 2004; Bascetin, 2007; Brent et al., 2007; Iwanejko, 2007; Wu et al., 2007; Srdjevic, 2007; Contreras et al., 2008; Dabaghian et al., 2008; Ercanoglu et al., 2008; Thapa and Murayama, 2008; Chatzimouraddis and Pilavachi, 2009; Chen, 2009). Current work on AHP is by Zaidan et al. (2015). They imposed the AHP method, integrated it with other MCDM techniques, to select the right software for open-source electronic medical record.

The Analytic Network Process (ANP) is an extension of AHP also proposed by Saaty (1996). Apart from structuring the multi-objective problem as an hierarchy, ANP treats it as a network. The decision criteria in AHP assume to be independent from each another, while ANP allows interdependence of those criteria. Several authors used ANP in their research (e.g. Levy, 2005; Cheng and Li, 2007; Banar et al., 2007; Khan and Faisal, 2008; Tseng et al., 2008; Gomez-Navarro et al., 2009; Boj et al., 2014).

The Multi-Attribute Utility Theory (MAUT) by Keeney and Raiffa (1976) is among the classical methods of multi-criteria decision analysis. It follows the utility axioms of Von Neumann and Morgenstern (1944). MAUT is a structured methodology designed to handle the trade-off among multiple objectives. MAUT assigns a utility value to each action and its quantifying individual's preferences. The result of using this method is a set of choices that represents the decision maker's preferences. MAUT was employed in the decision making by Ananda and Herath (2005).

The Elimination and Choice Expressing Reality (ELECTRE) was proposed by Bernard Roy in 1960s. There are several extensions of the method (ELECTRE I, II, III, IV, IS and TRI). The original version of ELECTRE, ELECTRE I, is an outranking method that discards unacceptable alternatives using a binary relation. It was designed to lead to "choice-type" results (Bouyssou, 2008). A limited set of alternatives that are obtained saves much of selecting time. Another outranking method is PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations), which is a modified approach of ELECTRE proposed by Brans and Vincke (1985). PROMETHEE is a much simpler version of the outranking technique that uses pairwise comparison of alternatives via a preference index. PROMETHEE consists of three tools: the PROMETHEE I (partial ranking), the PROMETHEE II (complete ranking) and the PROMETHEE-GAIA (geometrical analysis for interactive aid). Several authors applied the outranking method to multi-criteria decision problems (e.g. Goumas and Lygerou, 2000; De Leeneer and Pastijn, 2002; Soltanmohammadi et al., 2009; Oberschmidt et al., 2010; Petrović et al., 2014).

The genetic algorithm (GA) has also been used as a MCDM method.

Several authors employed GA for this purpose (e.g. Fonseca and Fleming, 1993; Tanaka et al., 1995; Feng et al., 1997; Hegazy, 1999; Zheng et al., 2005). They used GA prior to the decision making process to obtain the Pareto solutions. The foundation of GA lies in the survival of fitted individuals that mimics the process of the natural selection. Several natural selection techniques such as mutation, selection and crossover are implemented. This approach proved to be efficient. However, in the algorithm, the solutions can bias towards some regions and the method also produces non Pareto solutions. The algorithm generates a large number of solutions. Eventually, most of them appear to be redundant. Massive number of solutions, including the redundant ones, make the ranking procedure problematic.

Wang and Yang (2009) used another natural behaviour algorithm, the particle swarm optimization (PSO), combined with the preference order procedure to determine a ranking order for the MCDM problem. The PSO was inspired by the movement of bird flock or fish school. Particle swarm improves the search ability of GA for the best alternatives by having a better convergence to the Pareto frontier. However, as shown by Wang and Yang (2009), PSO requires up to 30,000 iterations to solve the problem. Therefore, it might be time consuming.

The Technique for Order Preference by Similarity to the Ideal Solution (TOPSIS) was first proposed by Hwang and Yoon (1981). The TOPSIS method embed the priori weights which are specified beforehand by the DM. The core of the ranking for this method lies in the distance of alternatives to the ideal and anti-ideal solutions. An alternative that is "closer to ideal" and "farther from anti-ideal" holds a higher ranking. However, the TOPSIS method produces an inconsistent ranking between the "closer to ideal" and "farther from anti-ideal". Many authors have used the TOPSIS method as a decision making method (e.g. Chen, 2000; Chu and Lin, 2003; Jahanshahloo et al., 2006; Liu et al., 2006; Yong, 2006; Shih et al., 2007; Wang and Chang, 2007; Gumus, 2009; Kilic et al., 2014).

Kao (2010) addresses the disadvantages of inconsistency ranking in the TOPSIS and proposes a consistent ranking between the "closer to ideal" and "farther from anti-ideal". In contrast to the TOPSIS, Kao suggests a relative distance ranking method and introduces the posteriori weights obtained from the data.

All the existing methods consider the value of each Pareto solution separately without its position with respect to the others in the objective space. The ranking is obtained as the result of such individual evaluations. Meanwhile, any Pareto solution is a trade-off solution. It seems natural to minimize the level of trade-off to identify "the best" design. In this study, a new ranking method, namely trade-off ranking, that reflects the level of compromise between different Pareto solutions is developed. It is clear that it is not practical and even unrealistic to consider the trade-off with all Pareto solutions. However, it is quite realistic to minimize the level of compromise for a selected Pareto set that represents the entire Pareto frontier well enough. It is worth noting that this kind of ranking is non-local because the value of each Pareto solution depends on its position with respect to the others in the objective space. In this way, the task is reduced to two problems. First, the Pareto set to be analyzed should represent the entire Pareto frontier. Second, a ranking algorithm should be identified to rearrange the Pareto set according to preferences that are beyond the original formulation of the problem. The former problem can be resolved via generating an evenly distributed Pareto set. It is well known that such a task is far from trivial. However, there are a few techniques that are able to tackle this problem such as the Normal Boundary Intersection (NBI) method (Das and Dennis, 1998), the Normal Constraint (NC) method (Messac et al., 2003; Messac and Mattson, 2004) and the Directed Search Domain (DSD) algorithm (Utyuzhnikov et al., 2005, 2009; Erfani and Utyuzhnikov, 2011; Erfani et al., 2013). The DSD algorithm is capable of generating a well distributed Pareto set on the entire Pareto frontier in a quite general formulation. Eventually, it provides a set of limited optimal choices for the DM for handling trade-off between multiple criteria.

1.3 Uncertainty analysis in multi-criteria decision making process

There is a considerable existing research on sensitivity analysis in the deterministic multi-criteria decision making (MCDM) methods. Barron and Schmidt (1988) proposed two procedures - an entropy based procedure and a least square technique - to test the sensitivity of the attributes (criteria) weights in the multi-attribute value theory (MAVT) method. It is assumed that in the former approach the weights are nearly equal, whilst the latter requires a set of arbitrary weights of the criteria.

Von Winterfeldt and Edwards (1986) defined the Flat Maxima Principle to test sensitivity analysis on the multi-attribute utility theory (MAUT) method. Rios Insua (1990) described a sensitivity analysis in the traditional MCDM Bayesian model.

In addition, there also exist several sensitivity analyses on the Analytic Hierarchy Process (AHP). The AHP is developed by Saaty (Saaty, 1980). Masuda (1990) studied the effect of changes in the entire decision matrix vectors on the ranking of the alternatives in the AHP method. Further research was done by Armacost and Hosseini (1994), who presented a procedure for determining the most critical criterion for the AHP problem. There is also a software package for the AHP, named Expert Choice, developed in 1990, to carry out the sensitivity analysis of the method where the user can alter the weights of the decision criteria and see how the ranking changes.

Triantaphyllou and Sanchez (1997) carried out a sensitivity analysis on the weights of the decision criteria and the performance values of the alternatives to three MCDM methods: Weighted Sum Model, Weighted Product Model and AHP. They determined the smallest changes of the current weights that would affect the existing ranking.

Alinezhad and Amini (2011) carried out a sensitivity analysis on the TOPSIS method. They changed the weight of a criterion and observed its effect on the final score of the alternatives. Simanaviciene and Ustinovichius (2010) also presented a sensitivity analysis on the TOPSIS method. They carried out a comparison with the simple additive weighting (SAW) method. They found out that the TOPSIS method is more sensitive to the differ in criteria value than the SAW method.

The existing papers, which are related to uncertainty in the MCDM process, address the sensitivity of MCDM models to the change of criteria weights. However, the first task in any decision making process is to identify the set of alternatives for the DM to make the choice. Thus, it is also essential to examine the uncertainty in this task of the decision making process, and to determine how it affects the ranking. This thesis examines both uncertainties in the MCDM process using the new ranking method, trade-off ranking, as well as other MCDM methods classified as the distance-based ranking techniques.

1.4 Critical criterion in multi-criteria decision making

In MCDM process, the DM may prefer one criterion more than the others. The preferences can be reflected via the weights of criteria. Once a decision ranking has been obtained, the DM may want to change their preferences. The change may or may not affect the current decision ranking. The smallest change in the preferences value that affects the current ranking may determine the critical criterion. To seek for the critical criterion, the sensitivity of ranking to various criteria weights is analysed.

The weights, which represent the importance of each criterion in terms of the DMs preferences, are used in the ranking calculation process. To date, there are many procedures proposed in the determination of the weights. For instance, Von Winterfeldt and Edwards (1986) have proposed the ratio method and the swing method to determine the average weights. Meanwhile, Butler et al (1997) have suggested three types of weights; random weight, rank order weight and response distribution weight. On the other hand, Olson (2004) introduced the equal weights, the weights generated by ordinal rank and the weights generated by a regression technique. Moreover, Kao (2010) calculated the weights by minimising the sum of squared distances from the alternatives to the ideal solution. The terminology of the ideal solution is explained further in the thesis. However, the most popular approach used to obtain the weights is the one carried out by the DMs themselves (e.g. Eckenrode, 1965; Saaty, 1980; Hwang and Yoon, 1981; Saaty, 1996). Once a ranking is obtained, the DMs may be interested in the sensitivity of the ranking to the criteria weights. The analysis of the weight changing versus the current ranking is considered in this thesis. The idea of the analysis came from the work of Triantaphyllou & Sanchez (1997) in which they carried out a sensitivity analysis for three decision making methods; the weighted sum model, the weighted product model and the analytic hierarchy process.

1.5 Fuzzy multi-criteria decision making method

The real-world design is usually related to the inevitable uncertainties in the input data, parameters, etc. The uncertainty in the MCDM (MCDM) problem includes the imprecision of criteria values, vagueness in the importance of criteria (weights), and dealing with qualitative, linguistic or incomplete information.

The concept of fuzziness, first introduced by Zadeh (1965), has proved to be an efficient tool to include uncertainties in MCDM problems. Numerous fuzzy MCDM methods have been developed, including the fuzzy TOPSIS (Chen, 2000; Wang and Elhag, 2006; Wang and Lee, 2007; Krohling and Campanharo, 2011) and fuzzy VIKOR (Vise Kriterijumska Optimizacija I Kompromisno Resenje) (Opricovic and Tzeng, 2004; Opricovic, 2007, 2011) - they utilize the fuzzy numbers in the formulation of their fuzzy MCDM methods.

Apart from the fuzzy TOPSIS and fuzzy VIKOR methods, several authors have implemented the fuzzy theory in other MCDM methods and application problems (e.g. Cakir and Canbolat, 2008; Gungor et al., 2009; Amiri, 2010; Buyukozkan et al., 2011; Kilincci and Onal, 2011; Torlak et al., 2011; Buyukozkan and Cifci, 2012; Rouhani et al., 2012).

There are two options to solve the fuzzy MCDM problem (Perny and Roubens, 1998): (i) utilizing the fuzzy MCDM method, and (ii) pre-defuzzifying the fuzzy MCDM problem and solving it by a conventional MCDM method. The defuzzification process converts the fuzzy numbers into crisp values; in both options, the defuzzification process is essential, since the MCDM solution must provide a crisp result. Many defuzzification methods can be used, including the center of sum and the center of gravity (Van Leekwijck and Kerre, 1999; Wang and Luoh, 2000). Both options to solve the fuzzy MCDM problem are used in this thesis for the proposed method, namely a fuzzy trade-off ranking method, for solving the fuzzy MCDM problem.

1.6 Research contributions and thesis structure

This thesis is divided into three main parts concerning the conventional multi-criteria decision making methods, uncertainty and sensitivity analysis in the multi-criteria decision making process and multi-criteria decision making in a fuzzy environment. In the next section, Section 2, the focus is on distance-based ranking techniques. In particular, a new ranking method, called trade-off ranking is introduced. In the beginning of Section 2.1, a brief introduction to the Pareto optimality is presented. Next, the main principles of two distance-based ranking methods, the TOPSIS and the relative distance ranking are described in Section 2.2 and Section 2.3, respectively. After that, a proposed algorithm of the trade-off ranking method is discussed in Section 2.4. Lastly, in Section 2.5, different test cases are considered with analysis and comparison between the methods.

In Section 3 of the thesis, the uncertainty and sensitivity analysis in multi-criteria decision making process are discussed. In particular, two types of uncertainty are considered. The beginning of Section 3 starts with the trade-off ranking modification in Section 3.1. Next, the first type of uncertainty, the uncertainty in the input data, is discussed in Section 3.2. In this first type, a robust set of alternatives is obtained by adding a new robustness function into the multi-objective optimization problem. The second type of uncertainty - in the decision makers preference is presented in Section 3.3. Section 3.4 identifies the critical critical in multi-criteria decision making. Both sections 3.3 and 3.4 involve the analysis in the criterion weight, i.e. the decision makers preference.

Apart from conventional decision making method, the fuzzy method is also considered in Section 4 of the thesis. The beginning of the section reviews the properties of fuzzy numbers that are used in the fuzzy decision making method. The fuzzy decision making methods begin with the proposed methods, the trade-off ranking with defuzzification and the fuzzy trade-off ranking in Section 4.2 and Section 4.3, respectively. Two fuzzy decision making methods, the fuzzy TOPSIS and the fuzzy VIKOR are reviewed in Section 4.4 and Section 4.5, respectively for a comparison purpose and the validation of the proposed methods. The analysis and comparison are discussed in Section 4.6.

The last section wraps up the thesis, where the summary of research findings are given in Section 5.1 and the future research implications are recommended in Section 5.2.

May your choices reflect your hopes,
not your fears
-Nelson Mandela.

SECTION TWO

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2 Distance-based ranking methods in multi-criteria decision making

Multi-criteria decision analysis presumes trade-off between different criteria. As a result, the optimal solution is not usually unique. A trade-off between the objectives is usually unavoidable because of the constraints. If the Decision Making preferences are not priori formulated, then the optimal solution is not unique. It is usually represented by a so-called Pareto solution. A Pareto solution usually represents a trade-off between different objectives. Despite there are unlimited number of Pareto optimal options, the DM eventually has to choose only one solution. Such a choice has to be made with the use of additional preferences not included in the original formulation of the optimization problem.

This section of the thesis represents a new approach to an automatic ranking that can help the DM. In contrast to the other methodologies, the proposed method is based on the trade-off minimization between different Pareto solutions. To be realized, the approach presumes the existence of a well-distributed Pareto set representing the entire Pareto frontier. In the thesis, such a set is generated with the use of the Directed Search Domain algorithm (Erfani and Utyuzhnikov, 2011). The proposed method is applied to a number of test cases and compared against two existing alternative approaches, the relative distance ranking and the TOPSIS methods. Both methods are used for comparison with the new proposed method as they all imply the distance formula in their ranking algorithm which lead to an automatic ranking that does not presume an immediate selection based on subjective experts opinion.

Both methods, the TOPSIS and the relative distance, imply the same idea of having the alternative that is "closer to an ideal" and "farther from an anti-ideal". One difference is in the calculation of the distance. The TOPSIS uses the Euclidean distance L_2-metric from an alternative to the ideal and the anti-ideal solutions. In contrast, the relative distance ranking is based on the measure that represents a relative position of an alternative from the origin to the ideal, deduced into the L_1-metric. Another difference is in the calculation of the weights. The TOPSIS uses the priori weights obtained beforehand by the DM, whilst the relative distance ranking exploiting the posteriori weights obtained from the data.

For further consideration, assume that there are q alternatives. Then, a multi-criteria decision analysis problem can be expressed via a trade-off matrix form as

	Criterion				
Alternative	Y_1	Y_2	Y_3	...	Y_m
A_1	Y_{11}	Y_{12}	Y_{13}	...	Y_{1m}
A_2	Y_{21}	Y_{22}	Y_{23}	...	Y_{2m}
A_3	Y_{31}	Y_{32}	Y_{33}	...	Y_{3m}
:	:	:	:	:	:
A_q	Y_{q1}	Y_{q2}	Y_{q3}	...	Y_{qm}

where the value of Y_{ij} denotes the performance of alternative i in terms of criterion j.

2.1 Pareto optimality

Let the design space be presented by $X \subset R^n$. Consider m objective functions, forming an objective space $Y \subset R^m$. For each $x \in X$, there exists a point in Y corresponding to mapping $R^n \mapsto R^m$.

Multi-objective optimization problem is formulated by

$$\begin{aligned}&\text{Minimize } Y = \{Y_1(x), Y_2(x), ..., Y_m(x)\},\\ &\text{subject to } x \in X^*.\end{aligned} \tag{2.1}$$

Here, $X^* \subseteq X$ is the feasible design space defined as the set of elements $x \in X^*$ satisfying all the constraints. The feasible objective space Y^* is defined as the set $\{Y(x) \mid x \in X^*\}$.

A design vector $x \in X^*$ is called Pareto optimal iff there does not exist any $a \in X^*$ such that

$$Y(a) \leq Y(x) \text{ and exists } k \in 1, ..., m : Y_k(a) < Y_k(x).$$

2.2 Relative distance ranking

In the relative distance approach, the first task is to identify the ideal solution I_R^+, and the anti-ideal solution I_R^-. In general, the ideal solution is the solution with the best score in all criteria. In turn, the anti-ideal solution is the solution with the worst score in every criteria. Consider a minimization problem. Thus, the ideal I_R^+ and the anti-ideal I_R^- solutions in the relative distance method are determined as follows:

$$I_R^+ = (Y_1^+, Y_2^+, ..., Y_m^+), \tag{2.2}$$

$$I_R^- = (Y_1^-, Y_2^-, ..., Y_m^-), \tag{2.3}$$

where

$$Y_j^+ = \min\{Y_{ij}, i = 1, ..., q\},$$

$$Y_j^- = \max\{Y_{ij}, i = 1, ..., q\}, \ (j = 1, ..., m).$$

In turn, for the maximization problem, the ideal I_R^+ and the anti-ideal I_R^- solutions are defined as:

$$I_R^+ = (Y_1^+, Y_2^+, ..., Y_m^+), \tag{2.4}$$

$$I_R^- = (Y_1^-, Y_2^-, ..., Y_m^-), \tag{2.5}$$

where

$$Y_j^+ = \max\{Y_{ij}, i = 1, ..., q\},$$

$$Y_j^- = \min\{Y_{ij}, i = 1, ..., q\}, \ (j = 1, ..., m).$$

The next task of the algorithm is to determine the weights for each criterion. According to Kao (2010), the weight is determined by minimizing the quadratic problem:

$$\begin{aligned} \text{Minimize } & \sum_{i=1}^{q} \left[\sum_{j=1}^{m} w_j |Y_j^+ - Y_{ij}| \right]^2 \\ \text{subject to } & \sum_{j=1}^{m} w_j |Y_j^+ - Y_j^-| = 1 \\ & w_j |Y_j^+ - Y_j^-| \geq \varepsilon, j = 1, ..., m, \\ & \varepsilon > 0. \end{aligned} \tag{2.6}$$

where w_j is the weight or importance of the *j-th* criterion. The objective function is the total distances between the ideal solution and each alternative in the objective space. The aim is to obtain a set of optimal weights that minimizes the distances. The small quantity ε is suggested to avoid any criterion being neglected.

Using the weights obtained from formula (2.6), the distance of each alternative to the ideal solution and the anti-ideal solution is then calculated respectively by the formulas (Kao, 2010)

$$dR_i^+ = \sum_{j=1}^{m} w_j |Y_j^+ - Y_{ij}|, \; i = 1, ..., q, \tag{2.7}$$

$$dR_i^- = \sum_{j=1}^{m} w_j |Y_j^- - Y_{ij}|, \; i = 1, ..., q. \tag{2.8}$$

The alternative with the shortest distance to the ideal and the longest distance to the anti-ideal is ranked the highest.

2.3 TOPSIS

In the TOPSIS (technique for order preference by similarity to ideal solution), the first step is to standardize the data set. The step can be skipped if the data are already in the standard form. The data standardization is done by the formula:

$$r_{ij} = \frac{Y_{ij}}{\sqrt{\sum_{j=1}^{m} Y_{ij}^2}}, \; i = 1, ..., q, j = 1, ..., m.$$

Next task is the data weighting process using the formula:

$$v_{ij} = w_j r_{ij}, \text{ where } \sum_{j=1}^{m} w_j = 1.$$

As mentioned earlier, the weights for each criterion in the TOPSIS method might be determined by the DM. However, in this study the DM is not involved. Therefore, the same approach to calculate the weights as in the relative distance ranking method is imposed.

The ideal and the anti-ideal solutions in the TOPSIS are then determined by:

$$I_T^+ = (v_1^+, v_2^+, ..., v_m^+), \tag{2.9}$$

$$I_T^- = (v_1^-, v_2^-, ..., v_m^-), \tag{2.10}$$

where

$$v_j^+ = \min \{v_{ij}, i = 1, ..., q\},$$

$$v_j^- = \max \{v_{ij}, i = 1, ..., q\}, \ (j = 1, ..., m).$$

In a similar way with the relative distance approach, the ideal I_T^+ and the anti-ideal I_T^- solutions are defined as reverse from the above definitions for maximization problem.

The distance from an alternative solution to the ideal solution is then calculated using the Euclidean distance as follows:

$$dT_i^+ = \sqrt{\sum_{j=1}^{m} w_j^2 (v_j^+ - v_{ij})^2}, \ i = 1, ..., q. \tag{2.11}$$

In turn, the distance from an alternative solution to the anti-ideal solution is calculated by formula (2.12):

$$dT_i^- = \sqrt{\sum_{j=1}^{m} w_j^2 (v_j^- - v_{ij})^2}, \ i = 1, ..., q. \tag{2.12}$$

As proven by Kao (2010), the TOPSIS ranking with respect to the ideal solution is different from the ranking with respect to the anti-ideal solution.

The full ranking in the TOPSIS is expressed by formula (Hwang and Yoon, 1981):

$$D_i^+ = \frac{dT_i^-}{dT_i^+ + dT_i^-}. \tag{2.13}$$

The largest value of D_i^+ is accepted as the best solution, while the smallest value is regarded as the worst solution.

In the next section, the trade-off ranking approach is introduced. The method is then compared against the TOPSIS and the relative distance ranking.

2.4 Trade-off ranking method

In this section, the key steps of the proposed method are described. The trade-off ranking is based on the property that the set of Pareto points is a set of trade-off solutions.

In the default trade-off ranking, there is no weights calculation that saves much calculation time. The importance of each criterion is assumed to be equal.

To demonstrate some justifications to the approach, consider a simple example with two sets of Pareto solutions, as shown in Figure 2.1.

The first set consists of points F, G and H while the other set contains points F, I and J. The lines FH and FJ are two different Pareto frontiers,

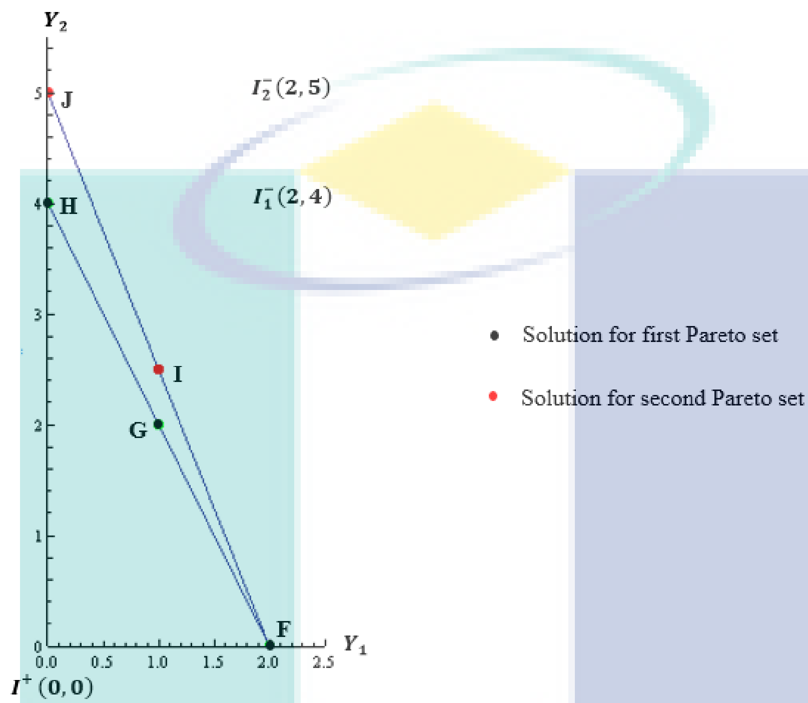

Figure 2.1: Two sets of Pareto solutions

but both contain the same point F as one of the three alternatives. Consider the minimization problem. Then, the ideal solution for the example is $I^+ = (0, 0)$. The anti-ideal solution for the first set is $I^- = (2, 4)$ and for the second set is $I^- = (2, 5)$. In the first Pareto frontier, FH, point F is the closest to the ideal solution and farthest from the anti-ideal solution. Hence, in the two ranking approaches considered above, point F is the most preferable solution out of the alternatives G and H. Consider now another Pareto frontier, FJ. Point F holds the shortest distance to the ideal solution and the longest distance to the anti-ideal solution. Thus, point F still holds the highest ranking versus the other points I and J. As a consequence, the ranking captures the same solution regardless the entire Pareto frontier.

The key principle of the trade-off ranking is to prefer the solutions with less compromise with the others. The trade-off minimization can be achieved by calculating the distance from one point to all other points in the objective space. The distance reflects the degree of trade-off between the solutions.

The general formula for the distance between point (alternative) $A_1 = (A_1^{(1)}, A_2^{(1)}, ..., A_m^{(1)})$ and point $A_2 = (A_1^{(2)}, A_2^{(2)}, ..., A_m^{(2)})$ is:

$$d(A_1, A_2) = \left[\sum_{j=1}^{m} \left(A_j^{(1)} - A_j^{(2)} \right)^2 \right]^{1/2} \tag{2.14}$$

Then, the sum of distances from one point to the other points is considered as the degree of trade-off:

$$DT_k = \sum_{i=1}^{q} \left[d(A_k, A_i) \right], k = 1, 2, ..., q \tag{2.15}$$

The trade-off ranking of each solution is determined by the value of DT with respect to the others. The least value of DT holds the highest ranking.

For the trade-off analysis, it is efficient to have an evenly distributed set representing the entire Pareto frontier. Thus, the first step in the trade-off ranking is generating an evenly distributed Pareto set. Evenly distributed solutions give the maximum information of the Pareto frontier to the DM.

As an example, consider Figure 2.1 again with two different sets of evenly distributed Pareto solutions F(2,0), G(1,2), H(0,4), I(1,2.5) and J(0,5). The results of the trade-off ranking are given in Table 2.4 and Table 2.4. In this simple example, solutions G and I seem more preferable because they better represent the entire Pareto frontier. In addition, it is easy to see that they correspond to the minimized trade-off among the other Pareto solutions.

The trade-off ranking method can be applied to find the best compromise solution in any set of Pareto alternatives. However, it is better to have a set of whole and evenly distributed Pareto solutions as it is the best set of solutions

Table 2.1: Trade-off ranking for Pareto frontier FH

Distance between each Pareto points	F	G	H	DT	Ranking
F	0	$\sqrt{5}$	$\sqrt{20}$	$3\sqrt{5}$	2
G	$\sqrt{5}$	0	$\sqrt{5}$	$2\sqrt{5}$	1
H	$\sqrt{20}$	$\sqrt{5}$	0	$3\sqrt{5}$	2

Table 2.2: Trade-off ranking for Pareto frontier FJ

Distance between each Pareto points	F	I	J	DT	Ranking
F	0	$\sqrt{29/4}$	$\sqrt{29}$	$\sqrt{145/4}$	2
I	$\sqrt{29/4}$	0	$\sqrt{29/4}$	$\sqrt{29/2}$	1
J	$\sqrt{29}$	$\sqrt{29/4}$	0	$\sqrt{145/4}$	2

in helping the DM to make an efficient decision in time-saving environment. The integration of the trade-off ranking method with the property of an even distributed alternatives give the best solution with the most balance value in all criteria.

As a practical analogy, consider the examples shown in Figure 2.2.

Figure 2.2 represents two different real-life trade-off problems: (a) risk over return in a share investment, and (b) price over quality in a car purchase.

For a share investment, DM usually wants a low risk investment that generates a high return. However, such a situation seems almost unrealistic. As shown in Figure 2.2(a), investments offer possibility I and II as the extreme solutions, a low risk with a low return or a high risk with a high return. Apart from these extreme solutions, there is a yellow area which gives acceptable solutions with a return higher than in I and a risk lower than in

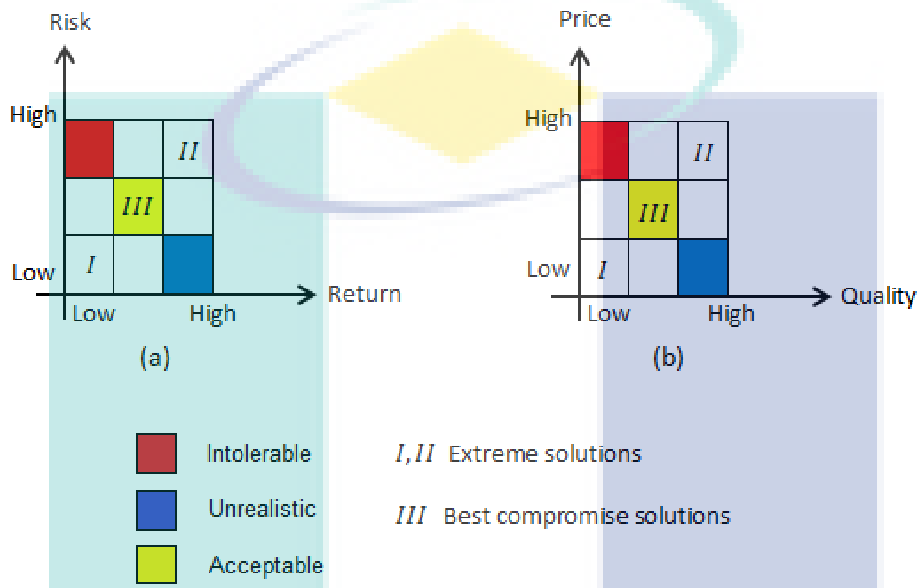

Figure 2.2: Practical examples for the trade-off ranking

II. The trade-off ranking method provides the best compromise solution III as the best choice.

The same situation occurs in problem (b). In case (b), it is almost impossible to buy a high quality car with a minimal price. On the market there is a choice in a wide range between cheap second-hand cars I and expensive luxury cars II. Many buyers prefer intermediate solutions that correspond to options in III, cheaper than II and higher quality than I. The trade-of ranking method can ensure that the optimal solution is in the yellow area III.

Now, for brief comparison between the methods, consider two arcs, A and B, represent two different Pareto frontiers as shown in Figure 2.3. In both Pareto frontiers, the top solutions for the trade-off ranking are situated in the middle of each frontier. For the TOPSIS and the relative distance method, the top solutions in B situated in the middle of the frontier as well, as they are closer to the ideal solution. However, their top solutions in A disperse between two extreme cases (areas I and II in Figure 2.2) as the middle area is no longer closer to the ideal solution.

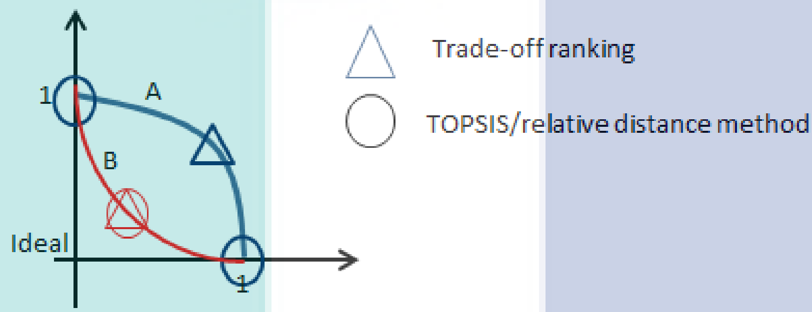

Figure 2.3: Results for the top ranking with the TOPSIS, relative distance and trade-off methods

Next, consider another two different graphs (a) and (b) as shown in Figure 2.4. In both graphs, the line C is the original Pareto frontier. In Figure 2.4(a), as $F1$ changes from the original value $F1 = 4$, the top solutions for the TOPSIS and the relative distance approach retain. In this example, the best choice in the TOPSIS and relative distance methods might not reflect the other alternatives. For the trade-off ranking method, the top solutions change according to the new Pareto frontier, nevertheless maintain in the middle of the frontiers. In Figure 2.4(b), as the value of $F2$ changes from $F2 = 2$, the top solutions for the TOPSIS and the relative distance method change as well. However, the solutions merely situated on the extreme cases (areas I and II in Figure 2.2). The example demonstrate that a small variations of the Pareto frontier can lead to a sharp replacement of the best choice in the TOPSIS/relative distance methods. For the trade-off ranking method, the top solutions retain in the middle of each frontier (area III in Figure 2.2).

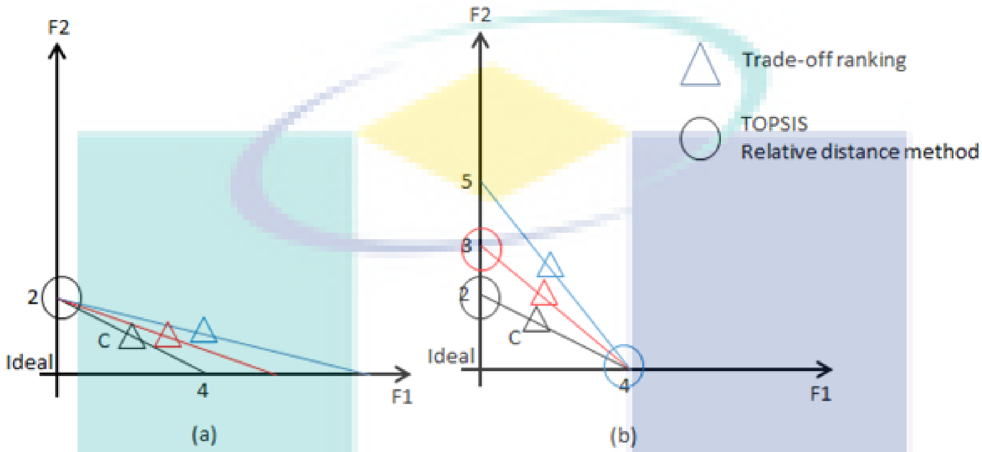

Figure 2.4: Results for the top ranking with the TOPSIS, relative distance and trade-off methods

In each example shown, the trade-off ranking method able to capture the best compromise solution of all alternatives provided.

In general, the steps for the trade-off ranking are as follows:

1. Generate an evenly distributed Pareto set.

2. Calculate the distance from one alternative to the others.

3. Calculate the degree of trade-off, DT.

4. Repeat steps 1 and 2 for all other alternatives from the Pareto set.

5. Alternative with less value of DT holds a higher ranking.

The concepts of distance measure in the trade-off ranking and TOPSIS methods are further illustrated in Figure 2.5.

Figure 2.5 shows the difference in the distance measure between the TOPSIS and the trade-off ranking method in evaluating three Pareto alternatives $A1, A2$, and $A3$. The TOPSIS uses the distance between an alternative to

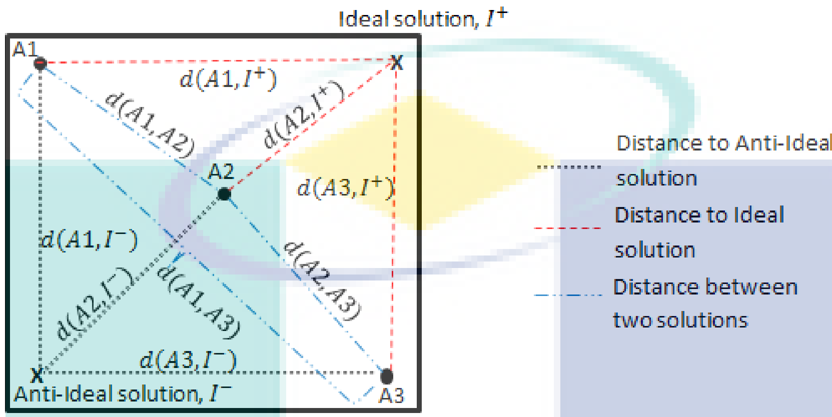

Figure 2.5: Distance measures in TOPSIS and trade-off ranking

the ideal/anti-ideal solutions as a ranking measure, which are denoted in Figure 2.5 as $d(Ai, I^+)$ and $d(Ai, I^-)$, respectively, for $i = 1, 2, 3$. In turn, the trade-off ranking method uses the distance from an alternative to the other alternatives to determine the ranking. In Figure 2.5, such distances are marked as $d(A1, A2), d(A2, A3)$ and $d(A1, A3)$. The ranking determination in the trade-off ranking method depends on the sum of the distances between those alternatives. The example in Figure 2.5 also shows that alternative $A2$, which is the closest to the ideal solution, is also the closest to the anti-ideal solution, compared to alternatives $A1$ and $A3$. In this case, a violation occurs in the aim of the TOPSIS method to have the best solution as the closest to the ideal solution and the farthest to the anti-ideal solution.

2.5 Test cases: analysis and comparison

In this section, the trade-off ranking is applied to six test cases. The TOPSIS and the relative distance ranking approaches are also used for a comparison. The trade-off ranking is applicable to n-dimensional problems. It is important to have a set that represents well enough the entire Pareto frontier. This problem can be talked by the DSD algorithm (Erfani and Utyuzhnikov, 2011). The number of candidate-solutions should depend on the problem. In this general test case, the DSD approach (Erfani and Utyuzhnikov, 2011) was used for generation of an evenly distributed Pareto set. The evenness property ensures that the limited optimal set obtained represents the whole Pareto frontier. The problem formulations for the test cases are given in the Appendix A.

2.5.1 General Test Cases

TNK problem: this test case is introduced by Tanaka et al. (1995). The test case considers a discontinuous Pareto frontier with significant gaps. Despite the discontinuity, DSD algorithm gives an evenly distributed Pareto solutions (Erfani and Utyuzhnikov, 2011). The results for the most preferable Pareto solutions in TNK problem, identified by each method, are shown in Figure 2.6.

For this test case, the ideal solution for the TOPSIS and the relative distance method is $I^+ = (0, 0)$ and the anti-ideal solution is $I^- = (1.1, 1.1)$. As can be seen in Figure 2.6, the trade-off ranking approach gives preferable solutions in the middle range of both criteria x_1 : $[0.55, 0.60]$ and x_2 : $[0.75, 0.80]$. The other two methods, the TOPSIS and the relative distance ranking, give the same level of ranking with much higher values for criterion x_2 : $[0.95, 1.00]$. The weights obtained in this test case using formula (2.6) are $w_1 = 0.71$ and $w_2 = 0.29$.

ZDT1 problem: this test case is introduced by Zitzler et al. (2000). The test case has a convex Pareto optimal frontier. The results of each ranking method for ZDT1 problem are shown in Figure 2.7.

The ideal solution for the TOPSIS and relative distance method in ZDT1 is $I^+ = (0, 0)$ and the anti-ideal solution is $I^- = (1, 1)$. In Figure 2.7,

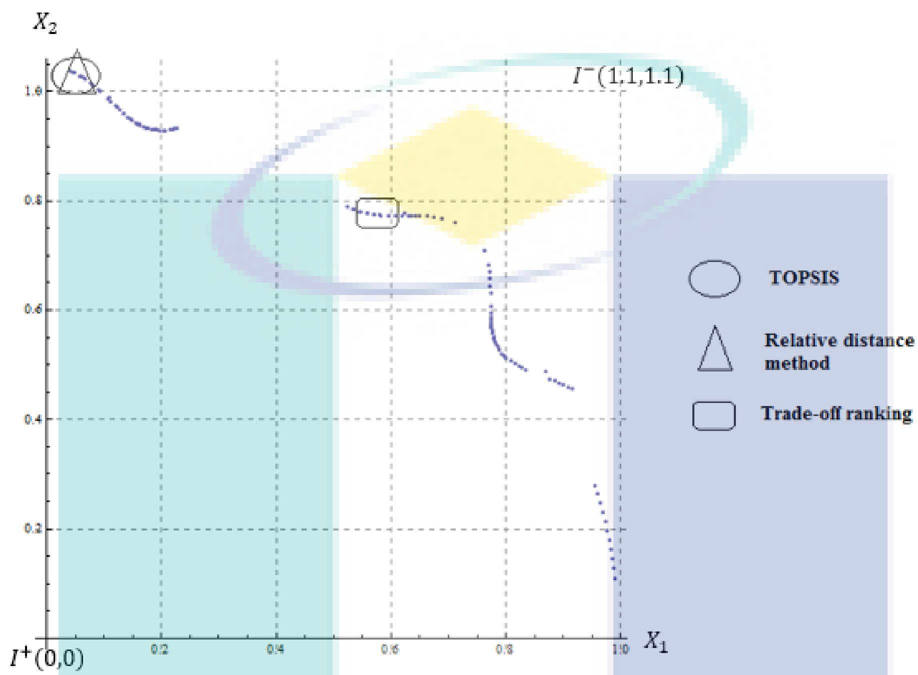

Figure 2.6: Results of the highest ranking for TNK

the preferable alternatives in the trade-off ranking are in the range of F_1 : $[0.3, 0.4]$ and F_2 : $[0.4, 0.45]$. The TOPSIS provides the ranking closest to the trade-off ranking method such that highest ranking solutions are ranging from F_1 : $[0.2, 0.3]$ and F_2 : $[0.45, 0.5]$. It is worth noting that the highest ranked solution in the trade-off approach is only ranked the thirteenth in the TOPSIS method. On the contrary, the first choice in the TOPSIS is ranked the tenth in the trade-off ranking. A similar situation occurs between the TOPSIS and the relative distance method. The most preferable alternative in the relative distance method is ranked the eleventh in the TOPSIS. The weights obtained for the ranking calculation in the relative distance method and the TOPSIS using formula (2.6) are $w_1 = 0.54$ and $w_2 = 0.46$.

ZDT2 problem: this test case is introduced by Zitzler et al. (2000). The

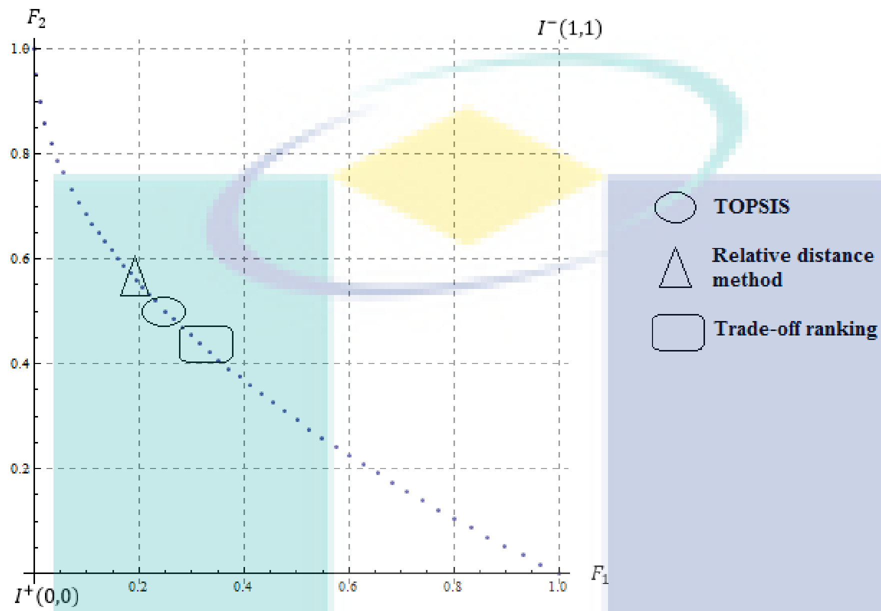

Figure 2.7: Results of the highest ranking for ZDT1

test case has a non-convex Pareto optimal frontier. The results for the best solutions in ZDT2 for each ranking approach are shown in Figure 2.8.

The ideal solution of the test case, for both the TOPSIS and relative distance method, is $I^+ = (0,0)$ and the anti-ideal solution is $I^- = (1,1)$. As shown in Fig. 2.8, a higher ranking alternatives in the trade-off ranking algorithm are situated in the middle of the Pareto set within the range of $F_1 : [0.5, 0.6]$ and $F_2 : [0.7, 0.8]$. The TOPSIS and the relative distance approach have the same ranking alternatives at the top rank. Both methods have the most preferable solutions in the range of $F_1 : [0, 0.1]$ and $F_2 : [0.9, 1]$. The weights obtained for ZDT2 using formula (2.6) are $w_1 = 0.69$ and $w_2 = 0.31$.

ZDT6 problem: this test case is taken from Shukla and Deb (2007). The test case has non-uniform density solutions on a non-convex Pareto optimal

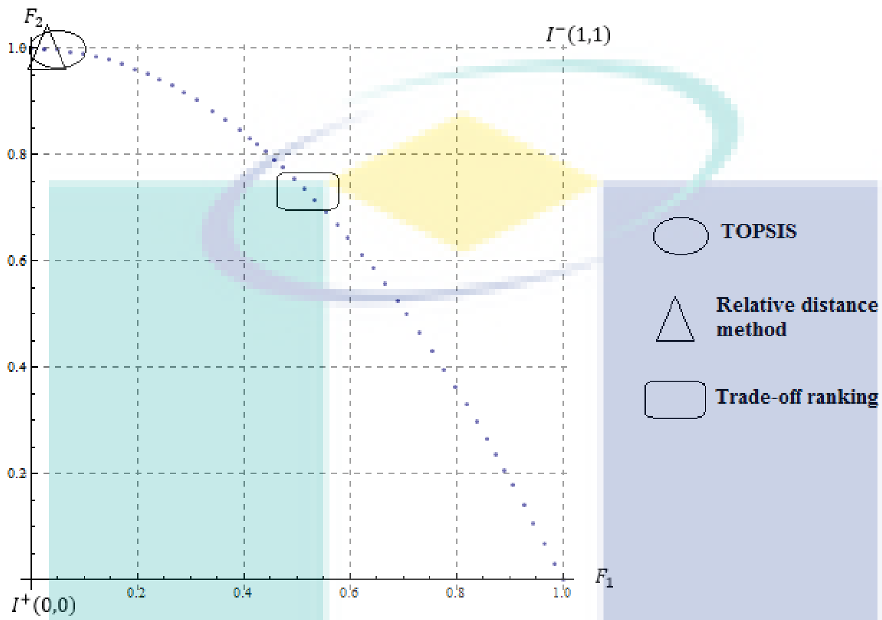

Figure 2.8: Results of the highest ranking for ZDT2

frontier. The results for the most preferable alternatives of ZDT6 problem for each ranking approach are shown in Figure 2.9.

The ideal solution for ZDT6 is $I^+ = (0.38, 0)$ and the anti-ideal solution is $I^- = (1, 0.88)$. They are applied in both the TOPSIS and relative distance method. In Figure 2.9, it can be seen that the top ranked alternatives in the trade-off ranking are in the range between $F_1 : [0.7, 0.75]$ and $F_2 : [0.4, 0.5]$. In this test case, similar to TNK and ZDT2, the preferable alternatives in the TOPSIS and the relative distance method coincide. Both methods give alternatives with greater values for criterion $F_1 : [0.95, 1]$ and smaller values for criterion $F_2 : [0, 0.1]$. The weights for the test case, calculated from formula (2.6) are $w_1 = 0.1$ and $w_2 = 0.9$.

DTLZ5 problem: this three-dimensional test case is introduced by Deb et al. (2005). The test case has only two anchor points despite there are three-

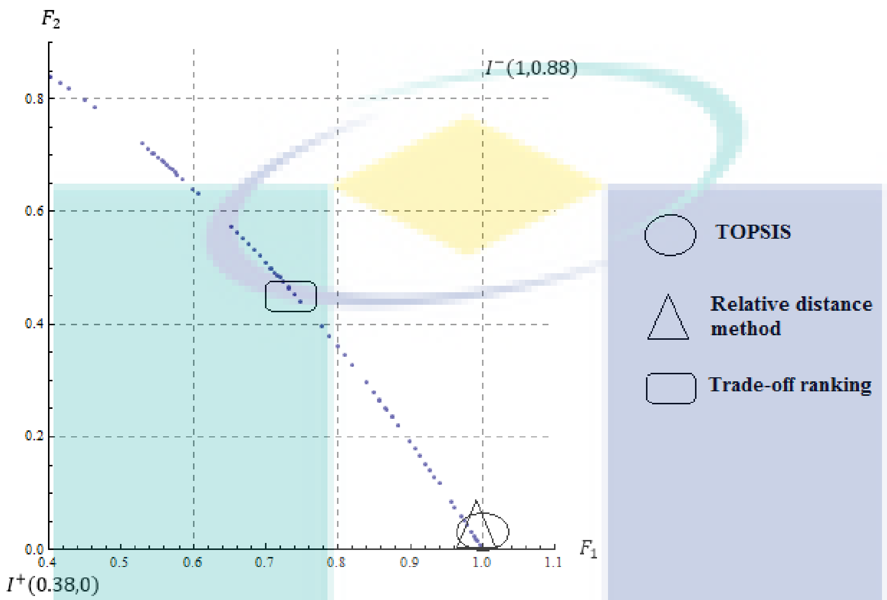

Figure 2.9: Results of the highest ranking for ZDT6

objective functions (Erfani and Utyuzhnikov, 2011). The best alternatives for each ranking method are shown in Figure 2.10.

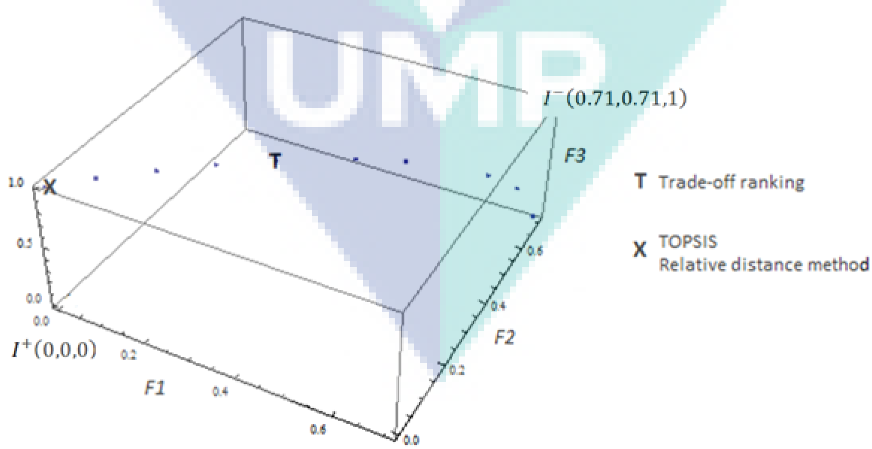

Figure 2.10: Results of the highest ranking for DTLZ5

For this problem, in both the TOPSIS and the relative distance method, the ideal solution is $I^+ = (0, 0, 0)$ and the anti-ideal solution is $I^- = (0.71, 0.71, 1)$. As shown in Figure 2.10, the highest ranking alternative in the trade-off ranking is in the middle of the set of Pareto points. The TOPSIS and the relative distance method have close ranking results with greater values in criterion F_3. The weights obtained for ranking of the relative distance method and the TOPSIS are $w_1 = 0.59$, $w_2 = 0.39$ and $w_3 = 0.02$.

2.5.2 Application Test Case

Consider an example from Jacquet-Lagreze and Siskos (1982) as given in Table 2.3. The data correspond to ten cars evaluated via six criteria: maximum speed, horse power, space of the car, gas consumption in town, gas consumption at 120 km/h and the price.

Table 2.3: Input data for car selection example and the results of ranking

No.	Maximum speed (km/h)	Horse power (CV)	Space (m2)	Gas consumption in town (lt/100 km)	Gas consumption at 120 km/h (lt/100 km)	Price (1000 francs)	Trade-off ranking	Relative distance ranking	TOPSIS
1	173	10	7.88	11.4	10.01	49.5	1	2	4
2	176	11	7.96	12.3	10.48	46.7	2	1	2
3	161	7	5.11	8.6	8.42	35.2	3	3	1
4	148	7	6.15	10.5	9.61	39.15	4	7	5
5	178	13	8.06	14.5	11.05	64.7	5	5	7
6	145	11	8.38	14.3	12.95	55	6	9	10
7	182	11	7.81	12.7	12.26	68.593	7	4	8
8	142	5	5.65	8.2	7.3	32.1	8	8	3
9	180	13	8.47	13.6	10.4	75.7	9	6	9
10	117	3	5.81	7.2	6.75	24.8	10	10	6

In this example, the first three criteria (maximum speed, horse power and space) are maximized while the last three criteria (gas consumption in town, gas consumption at 120 km/h and price) are minimized. Clearly the problem is having a conflicting criteria. The solutions given are the general Pareto solutions. In general, the ideal solution is $I^+ = (182, 13, 8.47, 7.2, 7.3, 24.8)$ while in turn, the anti-ideal solution is $I^- = (117, 3, 5.11, 14.5, 12.95, 75.7)$. However, in this application test case, the criterion value is not in standard form. To standardized the criterion value, formulae (2.2)-(2.5) and formulae (2.9)-(2.10) are used. After the weight calculation using the standardize values and formula (2.6), the weights 0.6346, 0.01, 0.01, 0.01, 0.01 and 0.3254 for each criterion, respectively are obtained (Kao, 2010). The cars are ranked with the trade-off ranking, the TOPSIS and the relative distance ranking method. The results are shown in the last three columns in Table 2.3.

The rankings obtained by the TOPSIS and the relative distance method are different from those in the proposed method, i.e. the trade-off ranking. The trade-off method ranked car no. 1 as the best choice while the relative distance method ranked it as the second and the TOPSIS ranked it as the fourth. In turn, the best choice in the relative distance approach is the second one in both the trade-off ranking and the TOPSIS. Car no. 3 is ranked the highest with the TOPSIS method, while the third with both the trade-off ranking and the relative distance method. To justify the best solution of the trade-off ranking approach in this application problem, refer to the Table 2.4.

Table 2.4: Preference ranking for each criterion in the car example

Car no.	1	2	3	4	5	6	7	8	9	10
Maximum speed	5	4	6	7	3	8	1	9	2	10
Horse power	6	3	7	7	1	3	3	9	1	10
Space	5	4	10	7	3	2	6	9	1	8
Gas consumption in town	5	6	3	4	10	9	7	2	8	1
Gas consumption at 120 km/h	5	7	3	4	8	10	9	2	6	1
Price	6	5	3	4	8	7	9	2	10	1

Notice that each criterion is ranked according to its value from the input data (Table 2.3). For the first three criteria, they are ranked from the highest value to the lowest value since they are the benefits criteria. It is vice versa for the last three criteria, the cost criteria, where they are ranked from the smallest value to the largest one. As can be seen in Table 2.4, car no.1 holds the most balance ranking in all criteria out of the ten cars. It is an evident that the trade-off ranking approach gives the less compromise solution compare to the others in a general Pareto set of solutions.

Good decisions come from experience.
Experience comes from making bad decisions.
-Mark Twain, Literary icon.

SECTION THREE

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3 Uncertainty and sensitivity analysis in multi-criteria decision making process

Data in the multi-criteria decision making are often imprecise and changeable. Therefore, it is important to carry out sensitivity analysis test for the multi-criteria decision making problem. This part of the thesis aims to present a sensitivity analysis for some ranking techniques based on the distance measures in multi-criteria decision making. Two types of uncertainties are considered for the sensitivity analysis test. The first type is related to the input data, while the second type is towards the DM preferences (weights). Several test cases are considered to study the performance of each ranking technique in both types of uncertainties.

The ranking techniques considered in this study are TOPSIS, the relative distance and trade-off ranking methods. Recall from Section 2, the TOPSIS and the relative distance method measure a distance from an alternative to the ideal and anti-ideal solutions. In turn, the trade-off ranking calculates a distance of an alternative to the other alternatives. The trade-off ranking method in its original form as illustrated in Section 2.4 has been tested on different test cases and the results were promising by giving the least compromised option as the best solution. However, not every DM opt for the least compromised option. Thus, the trade-off method can be improved considering the extreme options (Figure 2.2) as the solution in a case if the DM opt for that option. In this way, the trade-off ranking modification is suited for every DM preferences. The modification is described in the next subsection.

3.1 Trade-off ranking modification

Recall from Section 2.4, in a conflicting multi-criteria problem, it is not possible to determine an alternative that possess the best value for all the criteria, hence it is wise to seek a compromise solution between those criteria. The formulation of the trade-off ranking method in Section 2.4 ensures the best trade-off solution. Formally, each criterion is assumed to be equally important; however, the weighting may be enforced in the trade-off ranking method formulation.

An even and well distributed set of Pareto solutions is a pre-requisite for the method. Such a set efficiently represents the whole Pareto frontier. Thus, a DM is able to make an efficient decision based on the limited set within a time-constraint situation. Such a set can be obtained from optimization methods such as the Normal Boundary Intersection (NBI) method (Das and Dennis, 1998), the Normal Constraint (NC) method (Messac et al., 2003; Messac and Mattson, 2004) and the Directed Search Domain (DSD) algorithm (Erfani and Utyuzhnikov, 2011; Erfani et al., 2013).

Now, consider q alternatives and m criteria. In this study, we only consider a limited set of q alternatives. Suppose that $q > m$. In the trade-off ranking method, the criteria performance Y_{ij} and the criteria weights w_j are normalised by the formulae:

$$f_{ij} = \frac{Y_{ij} - \min_j Y_{ij}}{\max_j Y_{ij} - \min_j Y_{ij}}, \; i = 1, ..., q; \; j = 1, ..., m, \tag{3.1}$$

$$w'_j = \frac{w_j}{\sum_{j=1}^{n} w_j}, \; j = 1, ..., m. \tag{3.2}$$

The normalization guarantees that the range belongs to [0,1], and eliminates the units of criteria functions. The normalisation step can be ignored if the performance scores vary in the same range and $\sum_{j=1}^{m} w_j = 1$ with $w_j \geq 0, \; j = 1, ..., m$.

Ideally, the set of alternatives includes the anchor points of the multi-objective problem (2.1). The anchor point is a solution for a single-objective problem. Such a solution is called an extreme solution in the trade-off ranking method, i.e. a solution with the best value in at least one criterion (Figure 2.2. We presume that the number of extreme solutions in MCDM problem is equal to m. Practically, a k-th extreme solution, $A_k^*, \; k = 1, ..., m$ is the alternative with the optimal j-th criterion:

$$A_k^* = \{\min_{1 \leq i \leq q} f_{ij}\}, \; j = 1, ..., n \text{ for the cost criteria, or}$$

$$A_k^* = \{\max_{1 \leq i \leq q} f_{ij}\}, \; j = 1, ..., n \text{ for the benefit criteria.} \tag{3.3}$$

The benefit criteria are those to be maximized, such as profit, while the cost criteria are those to be minimized, such as price.

The trade-off ranking modification has two levels of selection. The first level is the trade-off between each alternative and the extreme solutions. In this first level, the ranking measure for the trade-off method is the total distance from an alternative, $A_\alpha = (f_{\alpha 1}, f_{\alpha 2}, ..., f_{\alpha m})^T, \; \alpha = 1, ..., q$, to all the extreme solutions, $A_k^* = (f_{k1}, f_{k2}, ..., f_{km})^T,\ k = 1, ..., m$, given by formulae:

$$
\begin{aligned}
DT1_\alpha = &\sum_{j=1}^{m} w'_j [d_{TOR1}(A_\alpha, A_k^*)], \quad \alpha = 1, ..., q, \\
&\sum_{j=1}^{m} w'_j = 1, \\
&w'_j \geqslant 0,\ j = 1, ..., m,
\end{aligned} \tag{3.4}
$$

where $w'_j,\ j = 1, ..., m$ is the normalized weight of criterion j. The $d_{TOR1}(.,.)$ denotes the distance formula between two points in L_2-metric such as:

$$
d_{TOR1}(A_\alpha, A_k^*) = \left[\sum_{j=1}^{m} (f_{\alpha j} - f_{kj})^2 \right]^{1/2},\ \alpha = 1, ..., q;\ k = 1, ..., m. \tag{3.5}
$$

An alternative with the minimum value of $DT1$, i.e. the degree of first trade-off, is the best alternative with the trade-off ranking method. Such an alternative is the closest option to all the extreme solutions. We choose the extreme solution as the point of reference since it represents the best solution in a single-criterion problem. In a conflicting multi-criteria problem, it is not possible to have a solution that simultaneously satisfies all the criteria. Hence, by obtaining the best solution in most of the criteria, if not all, is considered as a reasonable compromise solution in the conflicting criteria problem.

In the case of the same minimum value of $DT1$, the trade-off ranking formulation is further applied to the second level of selection. In this level, the ranking measures a distance between the alternatives as in Section 2.4 with weights imposed. The general formula for the weighted distance between an alternative $A_\alpha = (f_{\alpha 1}, f_{\alpha 2}, ..., f_{\alpha m})^T,\ \alpha = 1, ..., q$, and an alternative $A_\beta =$

$(f_{\beta 1}, f_{\beta 2}, ..., f_{\beta m})^T,\ \beta = 1, ..., q$, is:

$$d_{TOR2}(A_\alpha, A_\beta) = \left[\sum_{j=1}^{m} w_j'^2 \left(f_{\alpha j} - f_{\beta j}\right)^2\right]^{1/2},$$
$$\sum_{j=1}^{m} w_j' = 1,$$
$$w_j' \geqslant 0,\ j = 1, ..., m. \tag{3.6}$$

Here, $\alpha, \beta = 1, ..., q$, where q is the number of alternatives. The distance formulae $d_{TOR2}(.,.)$ in the trade-off ranking method implies the distance between two alternatives in terms of weighted criteria values.

The sum of distances from one alternative to all the others, i.e. the degree of second trade-off, is calculated as:

$$DT2_\alpha = \sum_{i=1}^{q} \left[d_{TOR2}(A_\alpha, A_i)\right], \alpha = 1, 2, ..., q. \tag{3.7}$$

The value of $DT2_\alpha$ implies the total distances of an alternative α to the other alternatives. A smaller values implies a shorter accumulative distances. Hence, an alternative with a smaller value of $DT2$ has a less compromise with the other alternatives, or a less degree of trade-off for an alternative. Here, the trade-off ranking is determined further by the value of $DT2$ where the least value holds the highest ranking.

Thus, the best solution in the trade-off ranking is the solution with the least compromise in regards to (i) the extreme solutions, and (ii) the other alternatives. As an analogy, consider the price and quality as two conflicting criteria. The best quality item is usually offered for a higher price. In turn, the cheapest item is a trade-off of its quality. Given all the feasible options, including those two extreme cases, the trade-off ranking method is able to give the least trade-off option according to the DM as the best solution. The distance measures in the trade-off ranking modification is illustrated in

Figure 3.1.

The difference in distance measures between the trade-off ranking modification, the TOPSIS and the relative distance method is also illustrated in Figure 3.1.

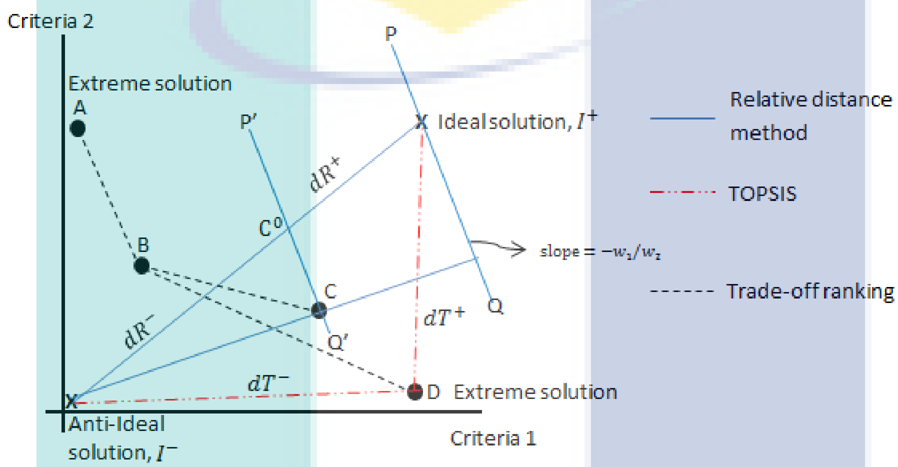

Figure 3.1: Difference in distance measures between the trade-off ranking, TOPSIS and the relative distance method

Figure 3.1 shows the distance measures of the TOPSIS, the relative distance and trade-off ranking methods. Consider four alternatives A, B, C, D, where A and D are the extreme solutions, the ideal and anti-ideal solutions as shown in Figure 3.1. The intervals between alternatives $[B, A]$ and $[B, D]$ contribute to the distance in the first level of trade-off ranking method where the sum of the distances is $DT1_B$ as formula (3.4). The second level of trade-off measures the distance of an alternative to all other alternatives, i.e. $[B, A]$, $[B, C]$ and $[B, D]$, using the weighted distance formula in L_2-metric. The sum of them, $DT2_B$, is the basis for the second level trade-off ranking as in formula (3.7). The intervals $[I^+, D]$ and $[I^-, D]$, represented by dT^+ and dT^- respectively, show the distance in the TOPSIS and are also considered in L_2-metric. For the relative distance method, consider line PQ for calculating a relative measurement of each alternative to the ideal/anti-ideal solutions. The line PQ is determined by constructing a straight line with the slope= $-w_1/w_2$ that passing through the ideal solution I^+. Consider line $P'Q'$ with the same slope passing through the alternative C and suppose the interval $[I^-, I^+]$ intersects the line $P'Q'$ at point C^o as shown in Figure 3.1. Then, C^oI^- is the relative distance of alternative C to the anti-ideal solution, while C^oI^+ is the relative distance of alternative C to the ideal solution. Due to the relative measures, the distance in the relative distance method is calculated using the L_1 distance metric.

3.2 Uncertainty in the input data

In this section, the first uncertainty in the MCDM process, uncertainty in the input data, is considered. This first type of uncertainty can occur due to a variety of reasons, such as imprecise input parameters, lack of data or inaccurate data during the design process. In multi-objective optimization, the input data are used to generate a set of alternatives for MCDM process. Thus, the uncertainty in the input data may generate a different set of alternatives and give a different ranking solution. In this study, the sensitivity of the MCDM methods to the uncertainty in the input data is considered by using the fuzzy set theory (Zadeh, 1965), which allows the multi-objective problem to be formulated in a more flexible way for practical applications. The fuzzy theory was used by Erfani and Utyuzhnikov (2010) to handle the uncertainty in the variables and to develop a robust design of the multi-objective optimization problem by finding a less sensitive solution to the uncertainty of the model. There are several other authors who implemented the fuzzy theory in MCDM methods and application problems (Amiri, 2010; Torlak et al., 2011; Buyukozkan and Cifci, 2012).

By using fuzzy numbers, problem (2.1) can be transformed into a multi-objective fuzzy constrained problem as:

$$\begin{aligned} \text{Minimize } \widetilde{Y} &= \{\widetilde{Y}_1(x), \widetilde{Y}_2(x), ..., \widetilde{Y}_m(x)\}, \\ \text{subject to } \widetilde{g}_k(x) &\leq \widetilde{b}_k, \ , \ k = 1, ..., p, \\ x &\in D, \end{aligned} \tag{3.8}$$

where the tilde denotes that the problem is modelled using fuzzy variables.

The fuzzy problem (3.8) is solved by transforming the fuzzy numbers into a crisp value. To do so, the crisp possibilistic mean value is used (Carlsson and Fuller, 2001). The crisp possibilistic mean value of fuzzy numbers is given by the formula:

$$Mean(A) = a + \frac{c-b}{6}. \tag{3.9}$$

Using formula (3.9) (see Appendix B), the fuzzy problem (3.8) is then converted into a deterministic formulation by substituting the fuzzy variables with their crisp values. Thus, problem (3.8) is reduced to

$$\begin{aligned} \text{Minimize } Y^{mp} &= Y_1^{mp}(x), Y_2^{mp}(x), ..., Y_m^{mp}(x) \\ \text{subject to } g_k^{mp}(x) &\leq b_k^{mp} \ , \ k = 1, ..., p, \\ x &\in D, \end{aligned} \tag{3.10}$$

where the mp notation denotes the possibilistic mean values.

Solution to the deterministic problem (3.10) gives a set of solutions called the Possibilistic Mean Pareto optimal solution according to the following definition:

Definition: (Possibilistic Mean Pareto Optimality) Vector $x^* \in D$ is called the Possibilistic Mean Pareto optimal of problem (3.10) iff there does not exist any $h \in D$ such that $Y_j^{mp}(h) \leq Y_j^{mp}(x^*)$ for any $j = 1, ..., m$ and exists $l \leq m : Y_l^{mp}(h) < Y_l^{mp}(x^*)$.

The set of optimal solutions for problem (3.10) is then treated as Pareto alternatives in MCDM process, where a ranking algorithm is imposed to find the best option.

3.2.1 Robust Set of Alternatives

The DM may prefer a robust (stable) set of solutions, which minimizes the variation of input parameters, rather than an optimal one. In finding for a robust set of solutions, a new function, the measure of robustness, is introduced (Erfani and Utyuzhnikov, 2010), and is defined as follows:

$$R = \frac{1}{nm} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\sigma Y_j}{\sigma x_k}, \tag{3.11}$$

where n is the number of design variables/parameters which vary, and m is the number of objective functions. The denominator σx_k, which denotes the standard deviation of the parameter, is calculated by the variance of fuzzy number (Carlsson and Fuller, 2001) given by (see Appendix B):

$$Var(A) = \frac{(b+c)^2}{24}. \tag{3.12}$$

In turn, the numerator σY_j, which is the standard deviation of the objective function, is calculated using the first order Taylor series (Parkinson et al., 1993) as follows:

$$\sigma^2_{Y_j} = \sum_{j=1}^{m} \left(\frac{\partial Y_j}{\partial p} \right)^2 \sigma^2_p, \tag{3.13}$$

where p is the uncertain parameters of the model. Here, formula (3.12) is also used to calculate the variance of the parameters, σ^2_p.

Formula (3.11) is minimized to find a robust set of solutions, since minimization of R leads to a smaller value of σY_j, despite having greater value of σx_k. Hence, R is added as a new objective function for multi-objective problem (3.10). Therefore, in searching for a robust set of solutions in multi-objective optimization, problem (3.10) is then converted to

$$\begin{aligned} &\text{Minimize } Y^{mp} = \{Y_1^{mp}(x), Y_2^{mp}(x), ..., Y_m^{mp}(x)\}, \\ &\text{Minimize } R, \\ &\text{subject to } g_k^{mp}(x) \leq b_k^{mp}\ ,\ k = 1, ..., p, \\ &\qquad x \in D. \end{aligned} \tag{3.14}$$

The set of robust solutions is then ranked with the MCDM methods. The rankings for both, Possibilistic Mean Pareto and robust solutions, are demonstrated with a test case in the next subsection.

3.2.2 Test Case: Two-bar Truss Structure

In this subsection, a test case is considered to see the difference in the ranking solutions with the alternatives obtained from the Possibilistic Mean Pareto design (3.10) and the robust design (3.14). The difference reflects the ranking effect of each MCDM method with respect to the uncertainty in the input data.

The test case is introduced by Messac and Ismail-Yahaya (Messac and Ismail-Yahaya, 2002) and is shown in Figure 3.2. The design variables are the diameter of members x_1, and the height of structure x_2. There are two conflicting objective functions to be considered, which are to minimize the total mass of truss members, and to minimize the deflection due to the load $F = (150, 20, 30)$ kN. The parameters of the problem are: member thickness, $t = (2.5, 0.5, 1.5)$ mm, structure width, $w = (750, 100, 50)$ mm, mass density, $\rho = 7.8 \times 10^{-3}$ gr/mm^3 and elastic modulus, $E = 210000$ Nmm^2. The constraints are as follows: the normal stress has to be less than the buckling stress, $1 \leq x_1 \leq 100$ and $10 \leq x_2 \leq 1000$. As can be seen, there are three uncertain variables/parameters in this problem, F, t and w, have been set as the triangular fuzzy numbers.

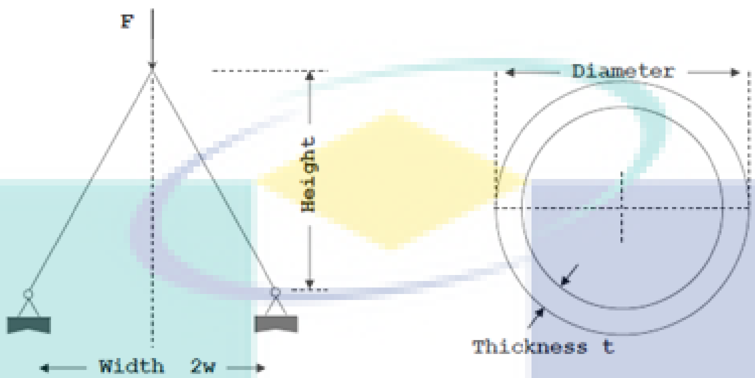

Figure 3.2: Two-bar truss structure

The full formulation of the problem is as follows:

$$\begin{aligned}
\text{Minimize Mass, } F_1 &= 2\pi p t x_1 \sqrt{w^2 + x_2^2}, \\
\text{Minimize Deflection, } F_2 &= \frac{F(w^2 + x_2^2)^{3/2}}{(2\pi t E x_1 x_2)^2}, \\
\text{subject to } s &\le \frac{1}{8}\pi^2 E \frac{t^2 + x_1^2}{w^2 + x_2^2}, \\
& 1 \le x_1 \le 100, \\
& 10 \le x_2 \le 1000, \\
\text{where} & \\
& s = \frac{F}{2\pi t x_1 x_2}\sqrt{w^2 + x_2^2}.
\end{aligned}$$

The triangular fuzzy parameters are substituted by their crisp possibilistic mean values of $F = 151.6$ kN, $t = 2.66$ mm and $w = 741.6$ mm (3.9). Using these deterministic values of F, t and w, problem (3.10) is then solved to obtain the Possibilistic Mean Pareto solutions.

The robust measure is constructed with the variance of 1020.6, 0.4 and

30.61 for F, t and w, respectively, as follows:

$$R = \frac{1}{6}\left(\left(\frac{\sigma F_1}{\sigma F} + \frac{\sigma F_1}{\sigma t} + \frac{\sigma F_1}{\sigma w}\right) + \left(\frac{\sigma F_2}{\sigma F} + \frac{\sigma F_2}{\sigma t} + \frac{\sigma F_2}{\sigma w}\right)\right)$$

.

Both the numerators σF_1 and σF_2 are calculated using formula (3.13) with respect to the uncertain parameters F, t and w.

Both problems (3.10) and (3.14) are solved with the Directed Search Domain (DSD) algorithm (Erfani and Utyuzhnikov, 2011). Erfani and Utyuzhnikov (2010) have tackled the robustness problem in their paper. Their work here is now extended to the ranking problem. For each set of alternatives obtained, the MCDM methods are then used to solve the ranking problem with the assumption of equal weights.

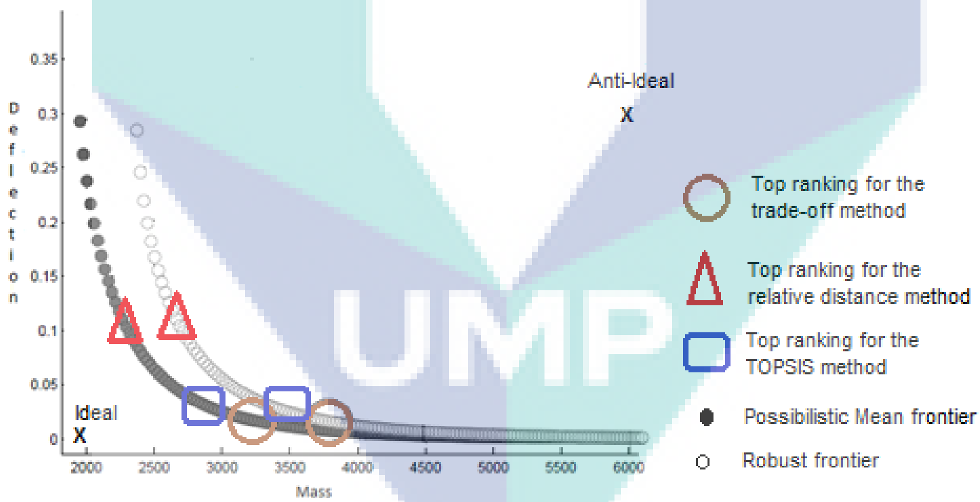

Figure 3.3: Results of the top ranking for Possibilistic Mean and robust Pareto frontier

Figure 3.3 shows the results of the top ranking for Possibilistic Mean and robust Pareto frontiers with three different MCDM methods: trade-off ranking, relative distance method and TOPSIS. For these methods, the top ranking areas are marked by the circle, triangular and rectangular, respectively. As can be seen from Figure 3.3, the top ranking solutions for both the Possibilistic Mean Pareto frontier and the robust one are different for each method. The DM has the option to choose between the Mean Pareto optimal solution and the robust solution. The highest trade-off ranking for the robust frontier is situated in the region, which differs significantly less from the appropriate solutions on the Possibilistic Mean Pareto frontier, than that for any other method. Therefore, it is advisable for the DM to opt for this robust solution, as it is not only robust but also near to the mean Pareto optimal value, with a lower deflection and greater mass. The trade-off ranking method selects the region on the Pareto frontier with the lowest reasonable deflection. Any further decrease of deflection leads to a significant increase of mass.

In the case of the TOPSIS and the relative distance, the top rankings prove to be significantly different for the Possibilistic Mean and robust frontiers. Intuitively, it is preferable to design a truss with a larger mass for obtaining less deflection. This is consistent with the solution provided by the trade-off ranking method in the indicated region of Figure 3.3. By testing this first level of sensitivity analysis, the DM is able to determine a set of robust alternatives in MCDM process. Moreover, the DM is able to gain insight into different solutions provided by the MCDM methods and make the best decision out of all solutions.

3.3 Uncertainty in the Decision Maker's preference

In this section, another type of uncertainty in the MCDM process is considered. In the decision making process, the DM may be uncertain about their preferences. The DM may have ambiguity, for instance, in which criterion they prefer the most or how much the preference is, or they do not know exactly how much they prefer a certain alternative. In the decision making process, the weights assigned to the decision criteria represent the importance of the criteria or the preferences of the DM. The criterion with the highest weight is the most important. The DM can make better decision if he/she is able to determine how sensitive the current ranking of the alternatives to the changes of the weights of the decision criteria. The problem can be tackled via the sensitivity analysis of the criteria weights.

3.3.1 Computational Experiment

A computational experiment is undertaken to study how sensitive each MCDM method to the changes of the weight of each criterion. A traditional approach is used, where random weights under constraints are generated for the sensitivity analysis. The random weights reflect the various techniques of weight evaluations as applied to different MCDM processes (Einhorn and Hogarth, 1975; Hobbs, 1980; Stillwell et al., 1981; Schoemaker and Waid, 1982; Solymosi and Dombi, 1986; Barron and Barrett, 1996; Olson, 2004; Kao, 2010; Toloie-Eshlaghy et al., 2011). Regardless of the techniques used, the obtained weights always have deterministic values. The results of the experiment are shown in some extreme cases of weights using two of the test cases taken from Section 2.5. For a ranking analysis, the trade-off ranking method is compared against the TOPSIS and the relative distance method.

In all the test cases, shapes are used to show the top choices for each method. The rectangular shape is used for the trade-off ranking method; the triangular, for the relative distance approach; and the circle, for the TOPSIS.

In **ZDT2** problem (Zitzler et al., 2000), the optimization problem generates a non-convex Pareto frontier as shown in Figure 3.4. The results for the best solutions for each ranking approach for the selected weights are also shown in the same figure.

The extreme solutions to this problem is (0,1) and (1,0). As seen in Figure 3.4, the preferable choices in the trade-off ranking method vary as the weight changes. As F_1 is more preferable than F_2, i.e. $w_1 > w_2$, the top choices in the trade-off ranking are skewed to the F_2 area in the graph as the best value

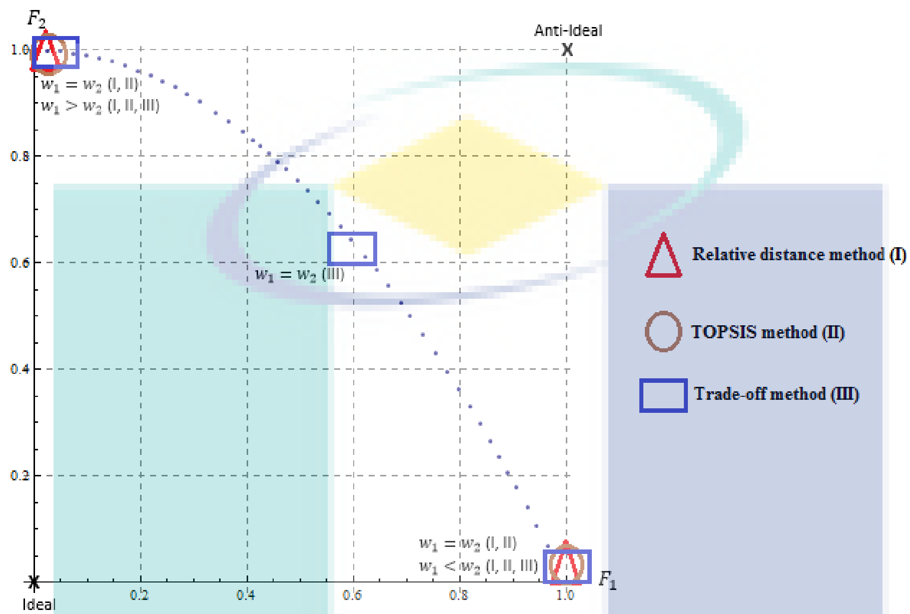

Figure 3.4: Top ranking for each method with each weight case for ZDT2

of criterion F_1 (minimum) is situated in the area. In turn, if the DM opt for criterion F_2 than F_1, the preferable choices are on the F_1 area. As the weights are equal ($w_1 = 0.5, w_2 = 0.5$), the top solutions are at the center of the graph, implying a solution with the least compromise among others in the two conflicting criteria problem. In the equal weights case, the second level of trade-off is imposed as the first level calculation revealed the same minimum value of $DT1$. The same ranking situation occurs in the relative distance and the TOPSIS methods. They are only differ for the case of equal weights between criteria.

Similar to the trade-off ranking method, as criterion F_1 is more preferable to criterion F_2, the top choices for the relative distance and the TOPSIS methods are situated on the left-hand side of the graph. Otherwise, the top ranking alternatives are situated on the right-hand side region.

For the equal weights between criterion F_1 and criterion F_2 ($w_1 = 0.5, w_2 = 0.5$), the top choices for the relative distance method reveal two extreme cases, in which both solutions (0,1) and (1,0) give the same minimum value for the ranking. There is a huge difference in choosing (0,1) as the opposite to (1,0), which implies that it is necessary to choose one criteria while completely ignoring the other. Such ranking solutions occur because the problem has the same extreme values for both criteria, which are $F_1 = 1$ and $F_2 = 1$. Thus, the rankings depend on the weights, i.e. the DM preferences. Therefore, when the weights are equal, the two best solutions occur. In the case of having two or more of the best solutions, the TOPSIS and the relative distance methods do not have an extra algorithm to tackle this kind of problem.

The three dimensional test case, **DTLZ5**, is introduced by Deb et.al. (2005). In this case, the same alternative, (0,0,1), is chosen twice as the extreme solutions since it has the minimum value in two criteria, F_1 and F_2. The third extreme solution, implies an alternative with the minimum value in criteria F_3, is (0.71,0.71,0). The best alternatives for each ranking method with several chosen weight cases are shown in Figure 3.5. The chosen weights are given in Table 3.1.

Table 3.1: Weight cases for DTLZ5

Weight case	a	b	c	d	e	f
Weights	$w_1 = 1$ $w_2 = 0$ $w_3 = 0$	$w_1 = 0$ $w_2 = 1$ $w_3 = 0$	$w_1 = 0$ $w_2 = 0$ $w_3 = 1$	$w_1 = 1/3$ $w_2 = 1/3$ $w_3 = 1/3$	$w_1 = 0.25$ $w_2 = 0.25$ $w_3 = 0.5$	$w_1 = 0.3$ $w_2 = 0.3$ $w_3 = 0.4$

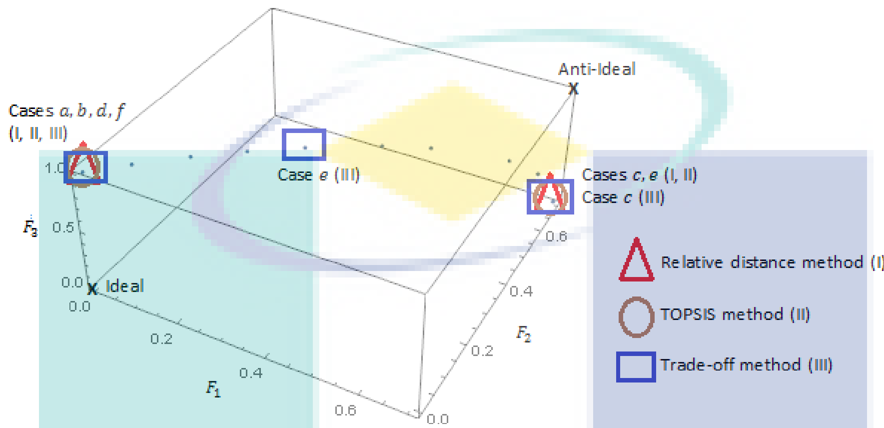

Figure 3.5: Top ranking for each method with each weight case for DTLZ5

In Table 3.1, the weight cases a, b and c represent the importance in only one criterion, either F_1, F_2 or F_3, at a time. In turn, the case d represents an equally importance criteria.

In Figure 3.5, the rectangular shapes represent the top choices for the specified weight cases in the trade-off ranking method. The triangular shapes imply the top rankings for the relative distance approach, and the circles denote the top ranking for the TOPSIS. From the results shown, it follows that the top ranking in the trade-off ranking method, the TOPSIS and the relative distance approach are the extreme solutions (0,0,1) and (0.71,0.71,0) as the weight changes. In this problem, the trade-off ranking method gives the same top solution as the TOPSIS and the relative distance method. In the cases of a, b and c, the top solution in the trade-off ranking method is an alternative with the best value in each important criterion, respectively. For example, (0,0,1) is the best solution for the cases a ($w_1 = 1$) and b ($w_2 = 1$) as it has the minimum values at $F_1 = 0$ and $F_2 = 0$.

Apart from other similar ranking results with the TOPSIS and the relative distance approach, the trade-off ranking gives the middle point as the best solution in a specific weights case e, which are $w_1 = 0.25, w_2 = 0.25$ and $w_3 =$

0.5. In the case e, the extreme solutions are said to be equally preferred by the DM as the weights w_1 and w_2 represent the importance of the same extreme solution (0,0,1). In this weight case, the trade-off ranking algorithm is imposed further to the second level, i.e. $DT2$, and hence the middle solution. The result of this weight case is different for the other two methods, in which they give (0.71,0.71,0) as the best solution. Supposedly, in the relative distance method and the TOPSIS, it is an alternative that closest to the ideal (0,0,0) and farthest from the anti-ideal (0.71,0.71,1) solutions.

For the equal weights case d, the three MCDM methods give the same top solution (0,0,1). The equal weights represent equally important criteria among the three F_1, F_2 and F_3. Again, in the trade-off ranking method, the weights $w_1 = 1/3$ and $w_2 = 1/3$ represent the importance of the same extreme solution (0,0,1) since it is the alternative with the best value in criteria F_1 as well as F_2. Therefore, whenever the preferences towards both criteria F_1 and F_2 exceed the preference towards the third criteria F_3, the extreme solution (0,0,1) is the top solution with the trade-off ranking method. As an example, in the weight case d, the extreme solution (0,0,1) is preferred 2/3 times more than the extreme solution (0.71,0.71,0). The same ranking occurs in the weight case f where the importance of extreme solution (0,0,1) is 0.6 ($w_1 + w_2$) compared to the extreme solution (0.71,0.71.0) which is only 0.4 (w_3).

3.3.2 Objective Weights via the Trade-off Ranking Method

There are two types of weights in a MCDM problem, the subjective and objective weights. The subjective weight is the weight determined by the DM since the human judgements may be affected by their past experience, intuition, biased and etc. On the other hand, the objective weight is calculated from the data. By obtaining the weight via a data calculation, we minimize the uncertainty in the DM preference, giving a set of weights that would optimize the ranking calculation. In this section, we show how to determine the objective weights using the trade-off ranking method.

The weights are determined by minimizing formula (3.4) such as:

$$
\begin{aligned}
\text{Minimize } & \sum_{i=1}^{q}\left[\sum_{j=1}^{m} w_j[d_{TOR1}(A_i,A_k)]\right], i=1,...,q; k=1,...,m,\\
\text{subject to } & \sum_{j=1}^{m} w_j = 1,\\
& w_j \geqslant 0,\ \ j=1,...,m,\\
& \text{where}\\
& d_{TOR1}(A_i,A_k) = \left[\sum_{j=1}^{m}(f_{ij}-f_{kj})^2\right]^{1/2},\\
& f_{ij} = \frac{Y_{ij} - \min\limits_{j} Y_{ij}}{\max\limits_{j} Y_{ij} - \min\limits_{j} Y_{ij}}.
\end{aligned} \tag{3.15}
$$

The result of optimization problem (3.15) is a set of weights that would minimize the total distances of the alternatives and the extreme solutions. Therefore, if the data are skewed to one of the extreme solutions, the result is a set of optimal weights that would give such an extreme solution as the best option. As examples of the optimization results, consider the two trivial data sets as shown in Figure 3.6.

Figure 3.6 shows two graphs, 3.6(a) and 3.6(b), as examples of the objective weights calculation using the trade-off ranking method for problem (3.15). In Figure 3.6(a), the Pareto solutions are skewed to the extreme solution F_2, while the data in Figure 3.6(b) show otherwise. By solving optimization problem (3.15) using the set of alternatives in Figure 3.6(a), we obtain the optimal weights of $w_1 = 1$ and $w_2 = 0$. The optimal weights give the extreme solution (0,1) as the best option out of three Pareto alternatives in Figure 3.6(a) with the trade-off ranking method. As can be seen, the extreme solution (0,1) is situated in the F_2 area, which is the area where the

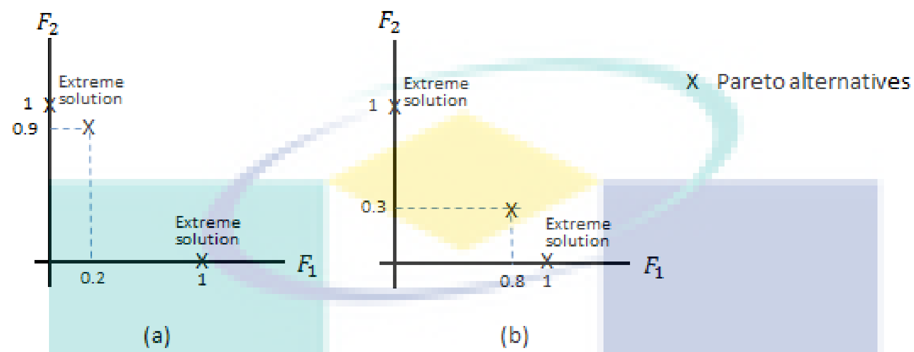

Figure 3.6: Examples for the objective weights calculation

data in Figure 3.6(a) are skewed.

On the other hand, data in Figure 3.6(b) give $w_1 = 0$ and $w_2 = 1$ as the optimal weights. Such weights give the extreme solution (1,0) as the best option in Figure 3.6(b). The results are consistent with the data in Figure 3.6(b) which are scattered towards the F_1 area. Thus, the optimization result for problem (3.15) depends on the distribution of the data set on the Pareto frontier. Therefore, if the data (i.e. the alternatives) are evenly distributed, the optimization problem (3.15) may gives finitely many sets of optimal weights as the result. In such a case, formulae (3.6) and (3.7) should be employed for the ranking calculation using the equal weights.

As mentioned before, imposing the objective weights may minimize the uncertainty in the DM preferences. However, when the DM preferences are available, the subjective weights should be considered instead of the objective weights.

3.4 Critical criterion analysis in multi-criteria decision making

In this section, we seek for a critical criterion in a multi-criteria decision making problem via analysing the sensitivity of ranking method to various criteria weights. Prior to this section, the DM may already defined the best solution according to their preferences (criteria weights) with the trade-off ranking, the TOPSIS and the relative distance method. Now, let say the DM want to change those preferences. The change may or may not affect the current best solution. The critical criterion is determined by analysing the smallest change in the preferences value that affects the current solution. Again, the trade-off ranking, the relative distance method and the TOPSIS, are used for comparison in the analysis.

3.4.1 Methodology

The methodology used for the analysis is given in this section. Let $\eth_1$ denotes the change in the current weight w_1 associated with criterion C_1. Thus, a new weight for criterion C_1 is $w_1^* = w_1 + \eth_1$ where $\eth_1 \geq -w_1$ since $w_j^* \geq 0 \quad (j = 1, , m)$. Note that the weights w_j are normalised such that $\sum_{j=1}^{m} w_j = 1$. Hence, the new normalised weights, $w_j^{'} \quad (j = 1, .., m)$ for the case of the weight change in criterion C_1 are then given by the formulae:

$$
\begin{aligned}
w_1^{'} &= \frac{w_1^*}{w_1^* + w_2 + \quad + w_m} \\
w_j^{'} &= \frac{w_j}{w_1^* + w_2 + \quad + w_m} \quad \text{for } j \neq 1.
\end{aligned}
\tag{3.16}
$$

The new ranking is calculated by substituting the new normalised weights, which are formulae (3.16) into formulae (2.7) and (2.8) for the relative distance method, formulae (2.11) and (2.12) for the TOPSIS and formulae (3.4) or (3.7) for the trade-off ranking. The weight change analysis is done for each criterion $C_j \quad (j = 1, , m)$ with any possible value $\eth_j \quad (j = 1, , m)$. Formula (3.16) are replaced by a different value of j according to the new weight in each analysis case.

The critical criterion is determined after the whole analysis has been completed. The critical criterion is defined by referring to Triantaphyllou

& Sánchez (1997). Triantaphyllou & Sánchez introduced four definitions of critical criterion based on absolute term, relative term, top ranking and any ranking. In this study, the critical criterion is defined as a criterion with the smallest changes in the current weight, which affects the current ranking. The smallest change is determined by the utilization of absolute value.

3.4.2 Critical criterion analysis

Consider the data shown in Table 3.2 taken from Triantaphyllou & Sánchez (1997). The data consist of four alternatives and four criteria with associated weights. It can be seen that the criterion C_1 has the largest weight, while C_4 has the smallest.

Table 3.2: Data of the problem

	C_1	C_2	C_3	C_4
	$w_1 = 0.3277$	$w_2 = 0.3058$	$w_3 = 0.2876$	$w_4 = 0.0790$
A_1	0.3088	0.2897	0.3867	0.1922
A_2	0.2163	0.3458	0.1755	0.6288
A_3	0.4509	0.2473	0.1194	0.0575
A_4	0.0240	0.1172	0.3184	0.1215

Table 3.3: Current ranking for each method

Alternative	Trade-off ranking	TOPSIS	Relative distance method
A_1	1	4	4
A_2	2	2	3
A_3	3	3	2
A_4	4	1	1

Table 3.3 presents the current ranking for each decision making method calculated using the respective ranking formulae. The current ranking for the trade-off method is different compared to the other two methods; the TOPSIS and the relative distance ranking. The best alternative in the trade-off method (without modification) is ranked the lowest in the other two

methods. In turn, their best solution is ranked the worst in the trade-off approach.

The analysis on changes in each of the weights (w_1, w_2, w_3, w_4) separately may give insight in determining the critical criterion. The results of the analysis are given in Tables 3.4-3.7.

Table 3.4: New ranking for each method with weight change in w_1
I = Trade-off ranking; II = TOPSIS; III = Relative distance

δ_1	-0.3			-0.2			-0.1			0.2			0.4			0.6		
$\begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix}$	$\begin{pmatrix} 0.04 \\ 0.44 \\ 0.41 \\ 0.11 \end{pmatrix}$			$\begin{pmatrix} 0.16 \\ 0.38 \\ 0.36 \\ 0.10 \end{pmatrix}$			$\begin{pmatrix} 0.25 \\ 0.34 \\ 0.32 \\ 0.09 \end{pmatrix}$			$\begin{pmatrix} 0.44 \\ 0.25 \\ 0.24 \\ 0.07 \end{pmatrix}$			$\begin{pmatrix} 0.52 \\ 0.22 \\ 0.21 \\ 0.06 \end{pmatrix}$			$\begin{pmatrix} 0.58 \\ 0.19 \\ 0.18 \\ 0.05 \end{pmatrix}$		
Method	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*
A_1	2	4	4	1	4	4	1	4	4	1	3	4	1	3	3	1	3	3
A_2	4	3	3	2	3	3	2	3	3	2	2	2	2	2	2	2	2	2
A_3	1	1	1	3	2	2	3	2	2	3	4	3	3	4	4	3	4	4
A_4	3	2	2	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1

Table 3.4 shows the result of rankings for the weight changes in w_1 for criterion C_1 analysis. As can be seen, the current ranking for the trade-off ranking method starts to change with $\eth_1 = -0.3$. The DM may analyse the change in the range of $\eth_1 = [-0.3, -0.2]$ to determine the smallest change in w_1 that affects the current trade-off ranking.

The current ranking by the TOPSIS does not retain with any of the changes in w_1. However, the best solution for the TOPSIS (alternative A_4) is to remain the first ranking until $\eth_1 = -0.3$ where it starts downgrading into the second choice in contrast to alternative A_3. A drastic change in the TOPSIS ranking for alternative A_3 starts to occur in the range of $\eth_1 = [-0.1, 0.2]$ as it changes from the worst option to the second one. Further analysis in the specific range is required if the DM wants to determine the exact value of $\eth_1$ at which the change occurs.

The current ranking for the relative distance approach retains with $\eth_1 =$

$[-0.2, -0.1]$. However, at $\eth_1 = -0.3$, the current ranking changes with the switch on the first and second rankings. The first current option (alternative A_4) swaps with the second current option (alternative A_3). With $\eth_1 = 0.2$, the same shift occurs with the second and third rankings, while the best and worst alternatives are retained. For $\eth_1 = 0.4$ and $\eth_1 = 0.6$, all the second, third and fourth current rankings are changed while the first one remains. The DM may want to analyse the changes in the whole ranking or may be interested in looking into the changes of the best ranking only.

Table 3.5: New ranking for each method with weight change in w_2
I = Trade-off ranking; II = TOPSIS; III = Relative distance

$\eth_2$	-0.3			-0.2			-0.1			0.2			0.4			0.6		
$\begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix}$	$\begin{pmatrix} 0.47 \\ 0.01 \\ 0.41 \\ 0.11 \end{pmatrix}$			$\begin{pmatrix} 0.41 \\ 0.13 \\ 0.36 \\ 0.10 \end{pmatrix}$			$\begin{pmatrix} 0.36 \\ 0.23 \\ 0.32 \\ 0.09 \end{pmatrix}$			$\begin{pmatrix} 0.27 \\ 0.42 \\ 0.24 \\ 0.07 \end{pmatrix}$			$\begin{pmatrix} 0.23 \\ 0.50 \\ 0.21 \\ 0.06 \end{pmatrix}$			$\begin{pmatrix} 0.20 \\ 0.57 \\ 0.18 \\ 0.05 \end{pmatrix}$		
Methods	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*
A_1	2	4	4	2	4	4	2	4	4	1	4	4	1	4	4	1	4	4
A_2	1	2	2	1	2	2	1	2	3	2	2	3	2	3	3	3	3	3
A_3	4	3	3	3	3	3	3	3	2	3	3	2	3	2	2	2	2	2
A_4	3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1

Table 3.5 displays the new rankings of the weight changes in w_2 for criterion C_2 analysis. Current ranking for the trade-off ranking method is preserved in the range of $\eth_2 = [0.2, 0.4]$. If $\eth_2 = 0.6$, the second and third choices of the current trade-off rankings swap their places. However, the alternative A_1 is remained as the first choice. Similarly, the current ranking is started to change with $\eth_2 = -0.1$, where alternative A_1 is no more the best option, and is replaced by alternative A_2. In the analysis conducted, the change of $\eth_2 = -0.1$ is regarded as the smallest change in w_2 that affects the ranking for the trade-off ranking method.

For the analysis with the TOPSIS method, the current ranking is observed to change from $\eth_2 = 0.4$, where the changes only occur in the second and third options. Meanwhile, the best and worst rankings retain as the same alternatives. The same ranking changes also occur in the analysis with the relative distance approach. The first and last choices are retained throughout the changes. However, the ranking places of second and third options are started to change at $\eth_2 = -0.2$.

Table 3.6: New ranking for each method with weight change in w_3
I = Trade-off ranking; II = TOPSIS; III = Relative distance

δ_3	-0.2			-0.1			0.2			0.4			0.6			0.8		
$\begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix}$	$\begin{pmatrix} 0.41 \\ 0.38 \\ 0.11 \\ 0.10 \end{pmatrix}$			$\begin{pmatrix} 0.36 \\ 0.34 \\ 0.21 \\ 0.09 \end{pmatrix}$			$\begin{pmatrix} 0.27 \\ 0.25 \\ 0.41 \\ 0.07 \end{pmatrix}$			$\begin{pmatrix} 0.23 \\ 0.22 \\ 0.49 \\ 0.06 \end{pmatrix}$			$\begin{pmatrix} 0.21 \\ 0.19 \\ 0.55 \\ 0.05 \end{pmatrix}$			$\begin{pmatrix} 0.18 \\ 0.17 \\ 0.61 \\ 0.04 \end{pmatrix}$		
Methods	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*
A_1	1	3	3	1	3	4	2	4	4	3	4	4	3	4	4	3	4	4
A_2	2	2	4	2	2	3	1	2	3	1	1	3	1	1	3	1	2	3
A_3	3	4	2	3	4	2	4	3	2	4	2	2	4	2	1	4	1	1
A_4	4	1	1	4	1	1	3	1	1	2	3	1	2	3	2	2	3	2

Table 3.6 demonstrates the result of the rankings for the weight changes in w_3 for criterion C_3. Current ranking for the trade-off method are changing at $\eth_3 = 0.2$ onwards where alternative A_2 becomes the first choice instead of alternative A_1. Further analysis in the range of $\eth_3 = [-0.1, 0.2]$ can be done to find the exact value where the changes start to occur. In this analysis, the value of $\eth_3 = 0.2$ is the smallest change in w_3 that affects the current ranking.

The current ranking for the TOPSIS retains with $\eth_3 = 0.2$. As $\eth_3 = -0.1$, the third and the fourth rankings have shifted toward each other, while the highest ranking retains. In the relative distance method, the current ranking is changed at $\eth_3 = -0.2$. Again, only the third and fourth options are swapped. The highest ranking in the relative distance approach (alternative A_3) is downgraded into the second choice with $\eth_3 = 0.6$.

Table 3.7 presents the new rankings of weight w_4 changes for criterion C_4.

Table 3.7: New ranking for each method with weight change in w_4
I = Trade-off ranking; II = TOPSIS; III = Relative distance

δ_4	0.1			0.2			0.4			0.6			0.8			1.0		
$\begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \end{pmatrix}$	$\begin{pmatrix} 0.30 \\ 0.28 \\ 0.26 \\ 0.16 \end{pmatrix}$			$\begin{pmatrix} 0.27 \\ 0.25 \\ 0.24 \\ 0.23 \end{pmatrix}$			$\begin{pmatrix} 0.23 \\ 0.22 \\ 0.21 \\ 0.34 \end{pmatrix}$			$\begin{pmatrix} 0.20 \\ 0.19 \\ 0.18 \\ 0.42 \end{pmatrix}$			$\begin{pmatrix} 0.18 \\ 0.17 \\ 0.16 \\ 0.49 \end{pmatrix}$			$\begin{pmatrix} 0.16 \\ 0.15 \\ 0.14 \\ 0.54 \end{pmatrix}$		
Methods	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*	*I*	*II*	*III*
A_1	1	4	3	1	3	3	1	3	3	1	3	3	1	3	3	1	3	3
A_2	2	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
A_3	3	2	2	2	2	2	3	2	2	3	2	2	3	2	2	3	2	2
A_4	4	1	1	3	1	1	2	1	1	2	1	1	2	1	1	2	1	1

The current ranking for the trade-off method is noticed to change at $\eth_4 = 0.2$. However, the best alternative (alternative A_1) is retained throughout the changes. The affected rankings are the second, third and fourth ranks only. The top ranking for the TOPSIS (alternative A_4) is also retained throughout the changes. However, the full current ranking is not preserved in any of these changes. Even with the smallest change of $\eth_4 = 0.1$, the second and third ranks are reversed. The same ranking situation occurs in the relative distance approach. There are no changes in the weight that preserve the current ranking. Nevertheless, from the change of $\eth_4 = 0.1$ onwards, the full ranking of the relative distance method remains the same with alternative A_4 as the best option and alternative A_2 as the worst one.

From the results of the analysis in Tables 3.4-3.7, the critical criterion in the trade-off ranking method is the criterion C_2 where the smallest change of $\eth_2 = 0.1$ affects the current full ranking. For the TOPSIS, the changes of the weights in criteria C_1, C_3 and C_4 obtain the same critical value at $\eth_1 = \eth_3 = \eth_4 = 0.1$. After further analysis, the critical criterion in the TOPSIS is criterion C_4 with the smallest change of $\eth_4 = 0.07$. Criterion C_2 is considered as the most non-critical criterion in the TOPSIS method. In the relative distance approach, criterion C_4 is the critical criterion with the smallest change of $\eth_4 = 0.1$ affecting the current full ranking. It is interesting to note that criterion C_1 has the largest weight in the current data. However, it is not the critical criterion in the trade-off ranking, the TOPSIS or the relative distance method.

Making good decisions
is a crucial skill
at every level.
-Peter Drucker, The founder of modern management.

SECTION FOUR

Parts of this section have been submitted for publication to the Journal of Soft Computing.

4 Fuzzy multi-criteria decision making method

The aim of this section is to present a trade-off ranking method in a fuzzy multi-criteria decision making environment. The triangular fuzzy numbers are used to represent the imprecise numerical quantities in the criteria values of each alternative, and in the weight of each criterion. The fuzzy trade-off ranking method is developed to solve the fuzzy multi-criteria decision making problem with conflicting criteria. Recall from Section 2.4 and Section 3.1, the trade-off ranking method tackles this type of multi-criteria problem by giving the least-compromised solution as the best option. The proposed method for the fuzzy decision making problems is compared against two other existing fuzzy decision making approaches, fuzzy VIKOR and fuzzy TOPSIS.

The fuzzy trade-off ranking method is proposed to tackle the MCDM problem in a conflicting fuzzy environment. This type of a problem has led to the development of the trade-off ranking method, in order to give the least-compromised solution as the best option. In comparison with the other MCDM methods, the trade-off ranking method is based on the overall evaluation score of an alternative with respect to the other alternatives, by taking into account the position of each alternative. This strategy is essentially different from VIKOR and TOPSIS; for example, the TOPSIS approach is based on an individual evaluation score, in which an alternative is calculated based on its distance to the ideal and anti-ideal solutions. This type of evaluation is obviously the best suited to non-conflicting multi-criteria problems, where the closest alternative is also the farthest from the anti-ideal solution. However, in a conflicting multi-criteria problem, such an assumption cannot always be realized. This drawback in TOPSIS is addressed in a few articles; (Opricovic and Tzeng, 2004; Wang et al., 2009; Kao, 2010). Kao (2010) has suggested the use of L_1-norm distance formula as opposed to the L_2-norm in the conventional TOPSIS method to overcome the inconsistency problem, while, Wang et al. (2009) proposed a merge of the fuzzy TOPSIS with the fuzzy Analytical Hierarchy Process and the metric distance method to surpass the problem. Nevertheless, TOPSIS is widely accepted and used due to its simplicity.

4.1 Multi-criteria decision making and fuzzy numbers

A conventional MCDM problem can be expressed in a matrix form as

$$
\begin{array}{c|ccccc}
 & & & \text{Criterion} & & \\
 & Y_1 & Y_2 & Y_3 & \ldots & Y_m \\
\text{Weight} & w_1 & w_2 & w_3 & \ldots & w_m \\
\hline
\text{Alternative} & & & & & \\
A_1 & Y_{11} & Y_{12} & Y_{13} & \ldots & Y_{1m} \\
A_2 & Y_{21} & Y_{22} & Y_{23} & \ldots & Y_{2m} \\
A_3 & Y_{31} & Y_{32} & Y_{33} & \ldots & Y_{3m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
A_q & Y_{q1} & Y_{q2} & Y_{q3} & \ldots & Y_{qm}
\end{array},
$$

where the performance of criterion j in alternative i is represented by Y_{ij} and the weight of each criterion is denoted by w_j, for $i = 1, .., q, j = 1, .., m$. Here, m is the number of criteria, and q is the number of alternatives.

Traditionally, MCDM solutions assume all Y_{ij} and w_j values are crisp numbers, but in reality, the values can be crisp, fuzzy or linguistic. Consider an example where two candidates are considered for an engineer position. The criteria considered are creativity (Y_1), communication skill (Y_2) and years of experience (Y_3). The rating for the first two criteria, Y_1 and Y_2, are represented by linguistic terms such as "very good", "average", "poor", and so on. The rating for criteria Y_3 can be some integer numbers. Furthermore, for group decision making that has K number of decision makers (DMs), the preferences towards each criterion may be different for every DM. In turn, each DM has his/her own uncertainty on the importance of each criterion. Thus, this MCDM problem contains a mixture of fuzzy, linguistic and crisp data sets.

To tackle such a problem, the weights of the criteria, $\widetilde{w}_j, j = 1, ..., m$, and the performance of the alternative, $\widetilde{Y}_{ij}, i = 1, ..., q, j = 1, ..., m$, for the fuzzy MCDM problems are considered as linguistic variables, expressed in positive triangular fuzzy numbers, shown in Tables 4.1 and 4.2 respectively (Chen, 2000). In turn, the membership function of linguistic variables in the alternative performance is presented in Figure 4.1.

Table 4.1: Fuzzy numbers for the importance weight of each criterion

Very low (VL)	(0, 0, 0.1)
Low (L)	(0, 0.1, 0.3)
Medium low (ML)	(0.1, 0.3, 0.5)
Medium (M)	(0.3, 0.5, 0.7)
Medium high (MH)	(0.5, 0.7, 0.9)
High (H)	(0.7, 0.9, 1.0)
Very high (VH)	(0.9, 1.0, 1.0)

Figure 4.1 shows the membership functions for the data stated in Table 4.2. As can be seen from Figure 4.1, the intervals to represent the linguistic variables are chosen in order to have a uniform representation from 0 to 10 in the triangular fuzzy numbers. The intervals are not unique and can have other representations (Zadeh, 1975a,b; Chen, 2000; Wang et al., 2009; Sodhi and T V, 2012).

Table 4.2: Linguistic variables for the alternative performance

Very poor (VP)	(0, 0, 1)
Poor (P)	(0, 1, 3)
Medium poor (MP)	(1, 3, 5)
Fair (F)	(3, 5, 7)
Medium good (MG)	(5, 7, 9)
Good (G)	(7, 9, 10)
Very good (VG)	(9, 10, 10)

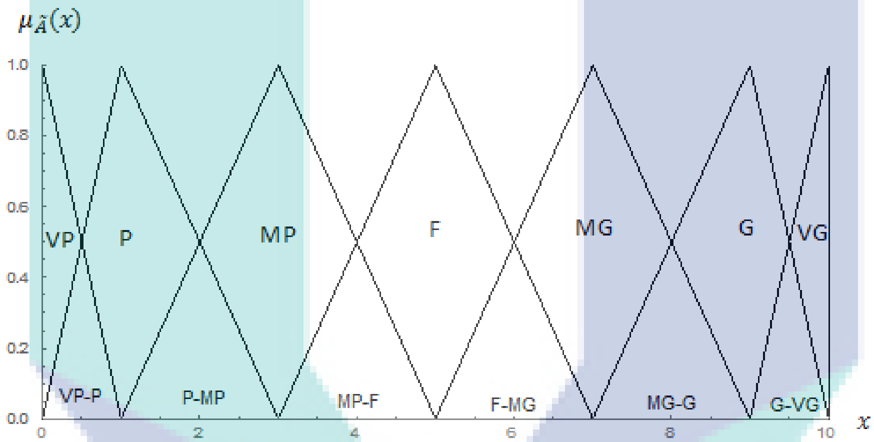

Figure 4.1: Membership functions of the linguistic variables

Thus, a fuzzy MCDM problem can be expressed in a matrix form as

$$\begin{array}{c|ccccc}
 & & & \text{Criterion} & & \\
 & \widetilde{Y}_1 & \widetilde{Y}_2 & \widetilde{Y}_3 & ... & \widetilde{Y}_m \\
\text{Weight} & \widetilde{w}_1 & \widetilde{w}_2 & \widetilde{w}_3 & ... & \widetilde{w}_m \\
\hline
\text{Alternative} & & & & & \\
\widetilde{A}_1 & \widetilde{Y}_{11} & \widetilde{Y}_{12} & \widetilde{Y}_{13} & ... & \widetilde{Y}_{1m} \\
\widetilde{A}_2 & \widetilde{Y}_{21} & \widetilde{Y}_{22} & \widetilde{Y}_{23} & ... & \widetilde{Y}_{2m} \\
\widetilde{A}_3 & \widetilde{Y}_{31} & \widetilde{Y}_{32} & \widetilde{Y}_{33} & ... & \widetilde{Y}_{3m} \\
: & : & : & : & : & : \\
\widetilde{A}_q & \widetilde{Y}_{q1} & \widetilde{Y}_{q2} & \widetilde{Y}_{q3} & ... & \widetilde{Y}_{qm}
\end{array},$$

where the performance of criterion j in alternative i for $i = 1, .., q, j = 1, .., m$ is now evaluated by the triangular fuzzy number $\widetilde{Y}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ and the weight of each criterion is represented by the triangular fuzzy number $\widetilde{w}_j = (w_{j1}, w_{j2}, w_{j3})$. Again, m is the number of criteria, and q is the number of alternatives.

For group decision making, consider a decision group that has K DMs. Each DM is required to rate the performance of the alternatives, and the weights of the criteria using the linguistic variables as in Tables 4.1 and 4.2. The final values for the alternative performance with respect to each criterion and the weight of each criterion are considered as the average values from the rating scores, given by the formulae:

$$\widetilde{Y}_{ij} = \frac{1}{K}\left[\widetilde{Y}_{ij}^1 \oplus \widetilde{Y}_{ij}^2 \oplus ... \oplus \widetilde{Y}_{ij}^K\right], \tag{4.1}$$

$$\widetilde{w}_j = \frac{1}{K}\left[\widetilde{w}_j^1 \oplus \widetilde{w}_j^2 \oplus ... \oplus \widetilde{w}_j^K\right]. \tag{4.2}$$

Here, $\widetilde{Y}_{ij}^K$ and $\widetilde{w}_j^K$ are the fuzzy performances of the alternatives and the fuzzy weight of each criterion, evaluated by the K-th decision maker (Chen, 2000). The operator $\oplus$, an addition of fuzzy numbers, is described further in the next section.

4.1.1 Arithmetic operations on triangular fuzzy numbers

In this section, several basic definitions and notations of fuzzy sets are briefly introduced. These definitions and notations are used in this section for each fuzzy MCDM method.

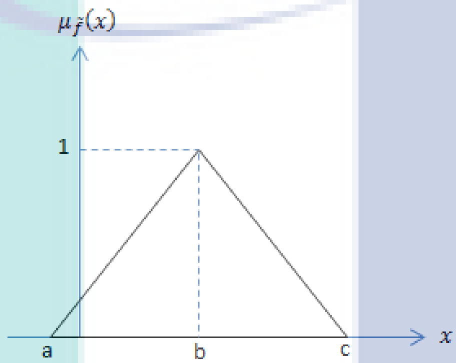

Figure 4.2: Triangular fuzzy number $\widetilde{f} = (a, b, c)$

Figure 4.2 shows a triangular fuzzy number $\widetilde{f} = (a, b, c)$, where a, b, c are real numbers. The interval $[a, c]$ reflects the fuzziness of the evaluation data b, where a closer interval means a lower degree of fuzziness. The membership function $\mu_{\widetilde{f}}(x)$ is defined as:

$$\mu_{\widetilde{f}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & Otherwise \end{cases}$$

With given a real number r, some arithmetic operations on the fuzzy numbers are defined as follows:

Addition:

$$\sum_{l=1}^{k} {}_{\oplus}\widetilde{f}_l = \left(\sum_{l=1}^{k} a_l, \sum_{l=1}^{k} b_l, \sum_{l=1}^{k} c_l \right) \tag{4.3}$$

Scalar addition:

$$\widetilde{f} \oplus r = (a + r, b + r, c + r) \tag{4.4}$$

Subtraction:

$$\widetilde{f}_1 \ominus \widetilde{f}_2 = (a_1 - c_2, b_1 - b_2, c_1 - a_2) \tag{4.5}$$

Multiplication:

$$\widetilde{f}_1 \otimes \widetilde{f}_2 = (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2) \tag{4.6}$$

Scalar multiplication:

$$r \times \widetilde{f} = (r \times a, r \times b, r \times c) \tag{4.7}$$

Scalar division:

$$\widetilde{f}/r = (a/r, b/r, c/r), r > 0 \tag{4.8}$$

Operator MAX:

$$\underset{l}{\mathrm{MAX}}\ \widetilde{f}_l = (\max_l\ a_l, \max_l\ b_l, \max_l\ c_l) \tag{4.9}$$

Operator MIN:

$$\underset{l}{\mathrm{MIN}}\ \widetilde{f}_l = (\min_l\ a_l, \min_l\ b_l, \min_l\ c_l) \tag{4.10}$$

Defuzzification:

$$\mathrm{Crisp}(\widetilde{f}) = \frac{a + 2b + c}{4} \tag{4.11}$$

Distance of two fuzzy numbers:

$$d(\widetilde{f}_1, \widetilde{f}_2) = \sqrt{\frac{1}{3}\left[(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2\right]} \tag{4.12}$$

The distance between two triangular fuzzy numbers, formula (4.12), is also known as a vertex method (Chen, 2000). In turn, the defuzzification, formula (4.11), is known as the second weighted average formula (Opricovic, 2011). As mentioned in Section 1.5, the defuzzification process turns the triangular fuzzy numbers into a crisp value. Such a process is the simplest way of tackling the fuzzy MCDM problem.

The definitions and properties of the above operations, (4.3)-(4.11), are discussed further in several articles (see Klir and Yuan, 1995; Giachetti and Young, 1997; Chen, 2000; Chiu and Wang, 2002; Opricovic, 2011).

4.2 Trade-off ranking method with defuzzification

As mentioned in Section 2, the trade-off ranking method is developed to solve the MCDM problem with conflicting criteria - such a problem gives a set of Pareto solutions. Eventually, the DM has to choose only one solution out of many. Therefore, an evenly-distributed Pareto set is important in the trade-off ranking method. The evenness property gives a sufficient set of solutions that represent the whole Pareto solutions for the DM to make a decision in a limited time.

In a fuzzy MCDM problem, the simplest way of solving the problem is by defuzzification, in which the fuzziness is dissolved at an early stage of the decision making process. The defuzzification process turns the fuzzy numbers into a crisp value.

Thus, the first task in solving the fuzzy MCDM problem is to defuzzify the alternative performance $\widetilde{Y}_{ij} = (a_{ij}, b_{ij}, c_{ij}), i = 1, .., q, j = 1, .., m$ and the weight of each criterion $\widetilde{w}_j = (w_{j1}, w_{j2}, w_{j3}), j = 1, .., m$ using formula (4.11). Each defuzzification is then denoted as Y_{ij} and w_j, respectively.

After the defuzzification process, the ranking of the alternatives is then calculated using a conventional trade-off ranking method as in Section 3.1. To conclude, the steps for the trade-off ranking algorithm are given as follows:

1. Normalization of Y_{ij} and w_j where

$$f_{ij} = \frac{Y_{ij} - \min\limits_{j} Y_{ij}}{\max\limits_{j} Y_{ij} - \min\limits_{j} Y_{ij}}, \; i = 1, ...q, \; j = 1, ..., m,$$

$$w'_j = \frac{w_j}{\sum_{j=1}^{n} w_j}, \; j = 1, ..., m.$$

2. Determination of the extreme solutions, A_k^*, $k = 1, ..., m$ i.e. solutions with the best value in at least one criterion. Thus, a k-th extreme solution is the alternative with the optimal j-th criterion, such as:

$$A_k^* = \{\min_{1 \le i \le q} f_{ij}\}, \; j = 1, ..., m \text{ for the cost criteria, or}$$

$$A_k^* = \{\max_{1 \le i \le q} f_{ij}\}, \; j = 1, ..., m \text{ for the benefit criteria.}$$

3. Calculation of the first level of ranking measures, distance of each alternative to the extreme solutions. Alternative with the least value in distance holds the highest ranking.

$$d_{TOR1}(A_k^*, A_\alpha) = \left[\sum_{j=1}^{m} (f_{kj}, f_{\alpha j})^2\right]^{1/2}, \; \alpha = 1, ..., q, \; k = 1, ..., m.$$

$$DT1_{A_\alpha} = \sum_{j=1}^{m} [w'_j \times d_{TOR1}(A_k^*, A_\alpha)], \; \alpha = 1, ..., q, \; k = 1, ...m.$$

4. Calculation of the second level of ranking measures, distance between the alternatives, if needed. Alternative with the minimum value in

distance holds the highest ranking.

$$d_{TOR2}(A_\alpha, A_\beta) = \left[\sum_{j=1}^{m} w_j'^2 (f_{\alpha j} - f_{\beta j})^2 \right]^{1/2}, \quad \alpha, \beta = 1, ..., q,$$

$$DT2_{A_\alpha} = \sum_{i=1}^{q} \left[d_{TOR2}(A_\alpha, A_i) \right], \; \alpha = 1, ..., q.$$

4.3 Fuzzy trade-off ranking

Another way to solve the fuzzy MCDM problem is using the fuzzy MCDM method, where the fuzzy numbers are processed until the end of the algorithms. In this way, the fuzzy information is preserved and the final solution is more accurate. In a fuzzy MCDM method, the distance formula between fuzzy numbers (4.12) is used for the ranking determination. An algorithm for the fuzzy trade-off ranking (FTOR) method is presented in the following steps:

1. Normalization of the performance of criterion j in alternative i, $\widetilde{Y}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, by formulae:

$$\widetilde{f}_{ij} = \frac{\widetilde{Y}_{ij} \ominus \widetilde{a}}{\max\limits_{j} c_{ij} - \min\limits_{j} a_{ij}}, i = 1, ..., q, \; j = 1, ..., m,$$

   where $\widetilde{a} = (\min\limits_{j} a_{ij}, \min\limits_{j} a_{ij}, \min\limits_{j} a_{ij}), i = 1, ..., q, \; j = 1, ..., m$. The operator $\ominus$ is the subtraction of the fuzzy numbers such as formulae (4.5). The result of the normalized performance $\widetilde{f}_{ij}$ is denoted as a triangular fuzzy number $\widetilde{f}_{ij} = (f_{ij1}, f_{ij2}, f_{ij3})$.

2. Determination of the extreme solutions, $\widetilde{A}_k^*, \; k = 1, ..., m$, by formula:

$$\widetilde{A}_k^* = \{\max_{1 \le i \le q} f_{ij3}\}, \text{ for the benefit criteria,}$$

$$\widetilde{A}_k^* = \{\min_{1 \le i \le q} f_{ij1}\}, \text{ for the cost criteria, } j = 1, ..., m.$$

3. Calculation of the distance of an alternative α to an extreme solution $\widetilde{A}_k^*$, denoted as $d_{FTOR1}(\widetilde{A}_k^*, \widetilde{A}_\alpha)$, as follows:

$$d_{FTOR1}(\widetilde{A}_k^*, \widetilde{A}_\alpha) = \sum_{j=1}^{m}[d(\widetilde{f}_{kj}, \widetilde{f}_{\alpha j})],$$
$$\alpha = 1, ...q, \; k = 1, ..., m. \tag{4.13}$$

The distance between two fuzzy numbers, $d(\cdot, \cdot)$, is calculated using formula (4.12).

4. Calculation of the first level of fuzzy trade-off, which is the trade-off between an alternative with all the extreme solutions, is given by formulae:

$$DFT1_{\widetilde{A}_\alpha} = \sum_{j=1}^{m}[w'_j \times d_{FTOR1}(\widetilde{A}_k^*, \widetilde{A}_\alpha)],$$
$$\alpha = 1, ..., q, \; k = 1, ...m, \tag{4.14}$$

where

$$w'_j = \frac{w_j}{\sum_{j=1}^{n} w_j}, \; j = 1, ..., m,$$
$$w_j = \text{Crisp}(\widetilde{w}_j), j = 1, ..., m.$$

Here, w_j is defuzzified using formulae (4.11). In turn, $\widetilde{w}_j$ is the weight of each criterion in the fuzzy MCDM problem, presented by a triangular fuzzy number $\widetilde{w}_j = (w_{j1}, w_{j2}, w_{j3}), \; j = 1, ..., m$. Similarly to $DT1$, the alternative with the least $DFT1$ is regarded as the best option. Again, in the case of the same $DFT1$ value, the fuzzy trade-off ranking formulation is imposed further, as shown in Step 5 onwards.

5. Calculation of the distance of an alternative to the other alternatives

is determined by

$$d_{FTOR2}(\widetilde{A}_\alpha, \widetilde{A}_\beta) = \sum_{j=1}^{m} \left[d(\widetilde{P}_{\alpha j}, \widetilde{P}_{\beta j}) \right],$$
$$\alpha, \beta = 1, ..., q, \ \alpha \neq \beta \tag{4.15}$$

where $\widetilde{P}_{ij} = \widetilde{w}_j \otimes \widetilde{f}_{ij}$, $i = 1, ..., q, j = 1, ..., m$. The multiplication of two fuzzy numbers is calculated using formula (4.6), while $d(\cdot, \cdot)$ is the distance between two fuzzy numbers determined by formula (4.12). The distance calculation represents the total trade-off in the quantity of each criterion. Hence, the smallest distance value denotes the least trade-off between the two alternatives.

6. Calculation of the second level of fuzzy trade-off, which is the trade-off among the alternatives, is given by

$$DFT2_{\widetilde{A}_\alpha} = \sum_{i=1}^{q} \left[d_{FTOR2}(\widetilde{A}_\alpha, \widetilde{A}_i) \right], \ \alpha = 1, ..., q. \tag{4.16}$$

The degree of fuzzy trade-off $DFT2$ represents the sum of distances between one alternative, and all the other alternatives in a fuzzy problem. The lowest value of $DFT2$ denotes the least value of compromise between the alternatives. Thus, the best alternative in the fuzzy trade-off ranking contains the lowest value of $DFT2$ if $DFT1$ is the same.

In the next two subsections, the existing fuzzy MCDM methods, fuzzy TOPSIS and fuzzy VIKOR, are briefly described. These two fuzzy methods are used for comparison with the proposed fuzzy trade-off ranking for method's validation.

4.4 Fuzzy TOPSIS

The TOPSIS was first developed by Hwang and Yoon (1981). The TOPSIS is based on the concept of having an alternative closest to the ideal solution, and the farthest from the anti-ideal solution, as the best option. The ideal and the anti-ideal solutions are considered as the artificial solutions. In 2000, Chen proposed an extension of the TOPSIS method for fuzzy decision making process (Chen, 2000). Since then, a few researchers have been using the fuzzy TOPSIS method for several applications, including the selection problem (Yurdakul and Iç, 2009; Amiri, 2010; Zouggari and Benyoucef, 2012; Rouyendegh and Saputro, 2014) and the performance evaluation (Sun and Lin, 2009; Sun, 2010; Torlak et al., 2011; Yu et al., 2011).

The first step in the TOPSIS method is to normalize the decision matrix. To avoid the complicated normalization formula used in classical TOPSIS, here linear normalization is used in the fuzzy TOPSIS (Chen, 2000). Therefore, the normalized fuzzy decision matrix $\widetilde{R}$ is given by formulae:

$$\widetilde{R} = [\widetilde{r}_{ij}]_{q\times m}, i = 1, ..., q, j = 1, ..., m$$

where

$$\widetilde{r}_{ij} = \left(\frac{a_{ij}}{c_j^+}, \frac{b_{ij}}{c_j^+}, \frac{c_{ij}}{c_j^+}\right) \text{ and } c_j^+ = \max_i c_{ij}$$

for benefit criteria,

and

$$\widetilde{r}_{ij} = \left(\frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right) \text{ and } a_j^- = \min_i a_{ij}$$

for cost criteria.

Next, the weighted normalized fuzzy decision matrix $\widetilde{V}$ is obtained by multiplying the weights $\widetilde{w}_j$ with $\widetilde{r}_{ij}$ as:

$$\begin{aligned}\widetilde{V} &= [\widetilde{v}_{ij}]_{q\times m} \\ &= [\widetilde{r}_{ij} \otimes \widetilde{w}_j]_{q\times m}, i = 1, ..., q, j = 1, ..., m.\end{aligned}$$

Formulae (4.6) is used for the multiplication of two fuzzy numbers. The Fuzzy Positive Ideal Solution (FPIS), I^+ and the Fuzzy Negative Ideal Solution (FNIS), I^- are then defined as follows:

$$\begin{aligned}I^+ &= (\widetilde{v}_1^+, \widetilde{v}_2^+, ..., \widetilde{v}_m^+), \\ I^- &= (\widetilde{v}_1^-, \widetilde{v}_2^-, ..., \widetilde{v}_m^-),\end{aligned}$$

where $\widetilde{v}_j^+ = (1, 1, 1)$ and $\widetilde{v}_j^- = (0, 0, 0)$ (Chen, 2000).

The distance for each weighted alternative to the FPIS and FNIS is computed by:

$$\begin{aligned}d_i^+ &= \sum_{j=1}^{m} d(\widetilde{v}_{ij}, \widetilde{v}_j^+), \\ d_i^- &= \sum_{j=1}^{m} d(\widetilde{v}_{ij}, \widetilde{v}_j^-), \quad i = 1, ..., q\end{aligned}$$

where $d(\cdot, \cdot)$ is the distance between two fuzzy numbers (4.12).

Finally, the closeness coefficient is calculated, to determine the ranking order of all alternatives, as follows:

$$CC_i = \frac{d_i^-}{d_i^+ + d_i^-}, \quad i = 1, ..., q.$$

The alternative with the highest closeness coefficient represents the best solution.

4.5 Fuzzy VIKOR

The fuzzy VIKOR method was developed to solve fuzzy multi-criteria problem with conflicting and different units criteria (Opricovic, 2011). Several authors have used VIKOR to solve the fuzzy MCDM problem (Wu et al., 2009, 2010). Assume that the alternatives and the weights are evaluated by the triangular fuzzy numbers $\widetilde{Y}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ and $\widetilde{w}_j = (w_{j1}, w_{j2}, w_{j3})$ respectively for $i = 1, ..., q, j = 1, ..., m$. The ranking algorithm for fuzzy VIKOR involves the following steps (Opricovic, 2011):

1. Determination of the ideal $\widetilde{Y}_j^+ = (a_j^+, b_j^+, c_j^+)$ and the anti-ideal $\widetilde{Y}_j^- = (a_j^-, b_j^-, c_j^-)$ for $j = 1, ..., m$, where

   (a) $\widetilde{Y}_j^+ = \underset{i}{\mathrm{MAX}}\ \widetilde{Y}_{ij}$ and $\widetilde{Y}_j^- = \underset{i}{\mathrm{MIN}}\ \widetilde{Y}_{ij}$, if the j-th criteria represents the benefit;

   (b) $\widetilde{Y}_j^+ = \underset{i}{\mathrm{MIN}}\ \widetilde{Y}_{ij}$ and $\widetilde{Y}_j^- = \underset{i}{\mathrm{MAX}}\ \widetilde{Y}_{ij}$, if the j-th criteria represents the cost;

   The MAX and MIN are fuzzy operators as in formula (4.9) and (4.10), respectively.

2. Compute $\widetilde{S}_i = (S_i^a, S_i^b, S_i^c)$ and $\widetilde{R}_i = (R_i^a, R_i^b, R_i^c)$, $i = 1, ..., q$ by the equations

$$\widetilde{S}_i = \sum_{j=1}^{m}{}_{\oplus}(\widetilde{w}_j \otimes \widetilde{d}_{ij}),$$

$$\widetilde{R}_i = \underset{j}{\mathrm{MAX}}\ (\widetilde{w}_j \otimes \widetilde{d}_{ij}),\ \ j = 1, ..., m,$$

   with

   (a) $\widetilde{d}_{ij} = (\widetilde{Y}_j^+ \ominus \widetilde{Y}_{ij})/(c_j^+ - a_j^-)$, if the j-th criteria represents the benefit;

   (b) $\widetilde{d}_{ij} = (\widetilde{Y}_j^+ \ominus \widetilde{Y}_{ij})/(c_j^- - a_j^+)$, if the j-th criteria represents the cost;

where $\widetilde{d}_{ij}$ is a normalized fuzzy difference, $\widetilde{S}$ is a fuzzy weighted sum as in formula (4.3) and $\widetilde{R}$ is a fuzzy operator MAX (4.9).

3. Compute $\widetilde{Q}_i = (Q_i^a, Q_i^b, Q_i^c)$, $i = 1, ..., q$, by the equation

$$\widetilde{Q}_i = \frac{v(\widetilde{S}_i \ominus \widetilde{S}^+)}{(S^{-c} - S^{+a})} \oplus \frac{(1-v)(\widetilde{R}_i \ominus \widetilde{R}^+)}{(R^{-c} - R^{+a})}$$

   where $\widetilde{S}^+ = \underset{i}{\text{MIN}}\ \widetilde{S}_i$, $S^{-c} = \max_i S_i^c$, $\widetilde{R}^+ = \underset{i}{\text{MIN}}\ \widetilde{R}_i$, $R^{-c} = \max_i R_i^c$ and v is the weight of the maximum group utility, whereas $1 - v$ is the weight of individual regret. Normally, $v = 0.5$ (Opricovic and Tzeng, 2004). For the fuzzy MCDM problem, v is modified as $v = (m+1)/2m$ (Opricovic, 2011).

4. "Core" ranking.
   Rank the alternatives by sorting the values of Q_i^b, $i = 1, ..., q$. A lower value implies a higher ranking. The obtained ranking is denoted by $\{Rank\}_{Q^b}$.

5. Fuzzy ranking.
   The i-th ranking position in $\{Rank\}_{Q^b}$ is confirmed if $\underset{k\in\ell}{\text{MIN}}\ \widetilde{Q}^{(k)} = \widetilde{Q}^{(i)}$, where $\ell = \{i, i+1, ..., q\}$ and $\widetilde{Q}^{(k)}$ is the fuzzy numbers for alternative $A^{(k)}$ at the k-th position in $\{Rank\}_{Q^b}$. Confirmed ordering represents fuzzy ranking $\{Rank\}_{\widetilde{Q}}$.

6. Defuzzification of $\widetilde{S}_i, \widetilde{R}_i, \widetilde{Q}_i, i = 1, ..., q$ to convert the fuzzy numbers into crisp value using formulae (4.11).

7. Defuzzification ranking.
   Rank the alternatives by sorting the crisp values of S, R and Q in Step 6. A lower value implies a higher ranking. The results of the ranking lists are denoted by $\{Rank\}_S$, $\{Rank\}_R$ and $\{Rank\}_Q$ respectively.

8. The best solution ($A^{(1)}$) ranked in $\{Rank\}_Q$ is regarded as the best compromise solution if the following two conditions are satisfied:

   (a) **C1.** Suppose $A^{(1)}$ is the first rank alternative and $A^{(2)}$ is the second rank in $\{Rank\}_Q$, $Adv \geq DQ$ where $DQ = 1/(q-1)$ and $Adv = (Q(A^{(2)}) - Q(A^{(1)}))/(Q(A^{(q)}) - Q(A^{(1)}))$.

   (b) **C2.** The alternative $A^{(1)}$ is also the best solution ranked by S and/or R.

   If one of the conditions is not satisfied, a set of compromise solutions is then proposed compromising the following:

   (a) Alternatives $A^{(1)}$ and $A^{(2)}$ if only condition **C2** is not satisfied, or

   (b) Alternatives $A^{(1)}, A^{(2)}, ..., A^{(M)}$ if condition **C1** is not satisfied; $A^{(M)}$ is determined by the relation $Q(A^{(M)}) - Q(A^{(1)}) < DQ$ for maximum M.

Further reading on the theoretical definitions of S and R values can be made by referring to Opricovic and Tzeng (2004).

4.6 Analysis and comparison

Consider a numerical example of the personnel selection problem, where five benefit criteria are considered in selecting one of three candidates, A_1, A_2 and A_3, for the post of system analysis engineer (Chen, 2000). The criteria considered are stated as follows:

1. Emotional steadiness, Y_1.
2. Oral communication skill, Y_2.
3. Personality, Y_3.
4. Past experience, Y_4.

5. Self-confidence, Y_5.

Here, the fuzzy trade-off ranking, fuzzy TOPSIS and fuzzy VIKOR methods are used to solve the personnel selection problem. Suppose the rating process of each alternative and the weight of each criterion are made by three DMs. The results of the rating evaluations are shown in Tables 4.3 and 4.4. The rating value is described by the linguistic terms expressed in the triangular fuzzy numbers as seen in Tables 4.1 and 4.2 (Section 4.1).

Table 4.3: Criteria weight by decision makers

	DM_1	DM_2	DM_3
Y_1	H	H	H
Y_2	VH	VH	VH
Y_3	VH	H	H
Y_4	VH	VH	VH
Y_5	M	MH	MH

Table 4.4: Alternatives ratings by decision makers

	A_1			A_2			A_3		
	DM_1	DM_2	DM_3	DM_1	DM_2	DM_3	DM_1	DM_2	DM_3
Y_1	MG	G	MG	G	G	MG	VG	G	F
Y_2	VG	VG	VG	MG	MG	MG	G	G	G
Y_3	G	G	G	F	G	G	VG	VG	G
Y_4	G	G	G	VG	VG	VG	VG	G	VG
Y_5	G	G	G	F	F	F	G	G	MG

Formulae (4.1) and (4.2) are applied to each data in Table 4.3 and Table 4.4 respectively, to find the average performance of the alternative and the average weight of each criterion. The fuzzy decision matrix of the problem is then given in Table 4.5.

The defuzzified decision matrix using formula (4.11) is given in Table 4.6.

Table 4.5: The fuzzy decision matrix for the personnel selection problem

	$\widetilde{Y}_1$	$\widetilde{Y}_2$	$\widetilde{Y}_3$	$\widetilde{Y}_4$	$\widetilde{Y}_5$
Weight	(0.7,0.9,1)	(0.9,1,1)	(0.77,0.93,1)	(0.9,1,1)	(0.43,0.63,0.83)
$\widetilde{A}_1$	(5.7,7.7,9.3)	(9,10,10)	(7,9,10)	(7,9,10)	(7,9,10)
$\widetilde{A}_2$	(6.3,8.3,9.7)	(5,7,9)	(5.7,7.7,9)	(9,10,10)	(3,5,7)
$\widetilde{A}_3$	(6.3,8,9)	(7,9,10)	(8.3,9.7,10)	(8.3,9.7,10)	(6.3,8.3,9.7)

Table 4.6: Defuzzified decision matrix

	Y_1	Y_2	Y_3	Y_4	Y_5
Weight	0.875	0.975	0.908	0.975	0.630
A_1	7.60	9.75	8.75	8.75	8.75
A_2	8.15	7.00	7.53	9.75	5.00
A_3	7.83	8.75	9.43	9.42	8.15

This problem aims to maximize all of the criteria. However, the conflicting situation arises since none of the candidates possessed the best properties in all criteria. According to Table 4.6, candidate A_3 is ranked the second in criteria Y_1, Y_2, Y_4 and Y_5, meanwhile, candidate A_3 is ranked the first in criterion Y_3. Furthermore, candidate A_1 is ranked the first in two criteria, which are Y_2 and Y_5, but ranked the third in two other criteria, Y_1 and Y_4. Meanwhile, candidate A_2 is ranked the best in two criteria, Y_1 and Y_4, but the worst in three other criteria, which are Y_2, Y_3 and Y_5.

The normalized defuzzified decision matrix by the trade-off ranking method, Step 1 in the trade-off ranking algorithm (Section 4.2), is given in Table 4.7.

Referring to Table 4.7 and using formula (3.3), the extreme solutions for the trade-off ranking method are determined. As an example, an alternative with the optimal value in criterion Y_5 is the fifth extreme solution for the problem, i.e. $A_5^* = \{0, 1, 0.64, 0, 1\}$ since $\max_{1 \le i \le 3} f_{i5} = 1$.

Table 4.7: Normalized defuzzified decision matrix by the trade-off ranking

	Y_1	Y_2	Y_3	Y_4	Y_5
Weight	0.201	0.223	0.208	0.223	0.144
A_1	0	1	0.64	0	1
A_2	1	0	0	1	0
A_3	0.42	0.64	1	0.68	0.84

After calculating the data in Table 4.7 using formulae (3.4) and (3.5), and the data in Table 4.5 by formula (4.13) and (4.14), the ranking by the trade-off method with defuzzification, and the fuzzy trade-off ranking are given in Table 4.8. As can be observed in Table 4.8, the best candidate ranked by the fuzzy trade-off is candidate A_3. Besides, it is also the best candidate ranked by the pre-defuzzification approach in the trade-off ranking method. Note that, even though candidate A_3 is only ranked the first in one criterion, he/she is not ranked the worst in the other criteria. Thus, this candidate has the most balanced traits, i.e. the least compromise, out of all five criteria compared to A_1 and A_2.

Table 4.8: Ranking by fuzzy trade-off

Ranking	1	2	3
Fuzzy trade-off	A_3	A_1	A_2
Defuzzification	A_3	A_1	A_2

Table 4.9: Results by fuzzy trade-off

	A_1	A_2	A_3
$DFT1$	1.027	1.039	0.984
$DT1$	1.090	1.105	1.030

Table 4.9 shows the results of the fuzzy trade-off $DFT1$ and the defuzzification trade-off $DT1$. The indifference in the ranking by the trade-off

method with defuzzification and the fuzzy trade-off ranking method is due to the small range of fuzziness in the triangular fuzzy numbers, and the small differences in the criteria ratings. The crisp values of the fuzzy numbers given in Table 4.6 are significantly close to their middle values, b_{ij}, presented in Table 4.5. A graphical explanation for this statement is given in Figure 4.3.

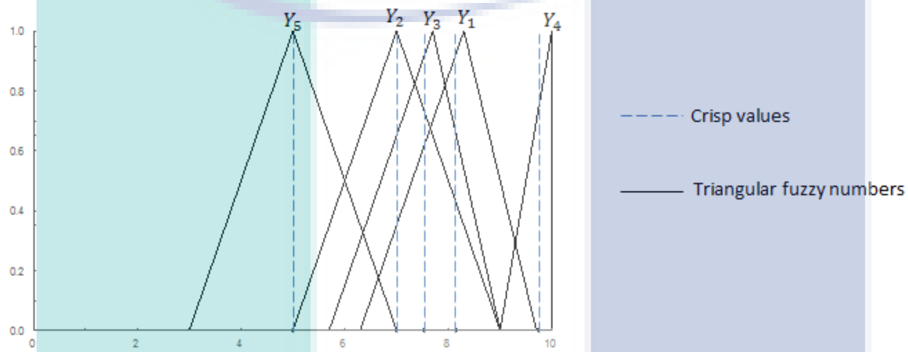

Figure 4.3: Triangular fuzzy numbers and their crisp values of each criterion for alternative A_2

The triangular fuzzy numbers of each criterion for alternative A_2 and their respective crisp values are shown in Figure 4.3. As can be seen, the crisp values are situated close to the middle values of the triangular fuzzy numbers. Hence, there is a small difference in the $DFT1$ and $DT1$ values for each alternative and the indifference in the ranking solutions. In the fuzzy MCDM problem, the final result is a crisp value since the MCDM method must provide a deterministic solution.

Next, consider the ranking by fuzzy VIKOR as given in Table 4.10. The fuzzy ranking $\{Rank\}_{\tilde{Q}}$ in the fuzzy VIKOR method gives a partial ranking, since the first position in $\{Rank\}_{Q^b}$ is not confirmed (Step 5 in Section 4.5.

In the case of the ranking by VIKOR defuzzification, the final decision is the set of compromise solutions $\{A_3,\ A_1,\ A_2\}$ (Step 8 in Section 4.5). Eventually, since there are only three options, the defuzzification ranking by

Table 4.10: Ranking by fuzzy VIKOR

Ordering		1	2	3
$\{Rank\}_{Q^b}$		A_3	A_1	A_2
$\{Rank\}_{\widetilde{Q}}$			A_1	A_2
Defuzzification	$\{Rank\}_Q$	A_3	A_1	A_2
	$\{Rank\}_S$	A_3	A_1	A_2
	$\{Rank\}_R$	A_3	A_1	A_2

VIKOR gives a set of solutions with all three options. The results by the fuzzy VIKOR are given in Table 4.11.

Table 4.11: Results by fuzzy VIKOR

		A_1	A_2	A_3
$\widetilde{S}$	S^a	-1.49	-1.02	-1.42
	S^b	0.62	1.39	0.43
	S^c	3.25	4.01	2.85
	Crisp(S)	0.75	1.44	0.57
$\widetilde{R}$	R^a	-0.18	0	-0.17
	R^b	0.33	0.6	0.2
	R^c	1	1	0.85
	Crisp(R)	0.37	0.55	0.27
$\widetilde{Q}$	Q^a	0	0.11	0.01
	Q^b	0.066	0.24	0
	Q^c	0.095	0.18	0
	Crisp(Q)	0.057	0.193	0.003

Next, the ranking by the fuzzy TOPSIS is given in Table 4.12. This method also identifies candidate A_3 as the best candidate.

As can be seen from Table 4.12, an alternative A_3 is the closest to the ideal solution ($d^+ = 1.45$). However, it is not the farthest from the anti-ideal solution ($d^- = 3.93$). In fact, the alternative farthest from the anti-ideal solution is A_1 ($d^- = 3.95$). It shows that the concept of the TOPSIS method-to have the best solution that is the closest to the ideal and the farthest

Table 4.12: Ranking and results by fuzzy TOPSIS

Ranking	1	2	3
	A_3	A_1	A_2
d^+	1.45	1.48	1.87
d^-	3.93	3.95	3.52
CC	0.731	0.728	0.653

from the anti-ideal solutions - is not realized in this conflicting multi-criteria problem.

Now, suppose the DMs have changed their preference towards each criterion. The new DMs preferences, presented by the linguistic variables, are given in Table 4.13. According to Table 4.1 as well as formula (4.2) and (4.11), the fuzzy and defuzzified weights associated with the new preferences are shown in Table 4.14.

Table 4.13: New criteria weight by decision makers

	DM_1	DM_2	DM_3
Y_1	MH	H	H
Y_2	VL	L	VL
Y_3	ML	ML	ML
Y_4	H	H	VH
Y_5	L	VL	L

Table 4.14: New fuzzy and defuzzified weights

$\widetilde{Y}_j$	$\widetilde{Y}_1$	$\widetilde{Y}_2$	$\widetilde{Y}_3$	$\widetilde{Y}_4$	$\widetilde{Y}_5$
$\widetilde{w}_j$	(0.63,0.83,0.97)	(0,0.03,0.17)	(0.1,0.3,0.5)	(0.77,0.93,1)	(0,0.07,0.23)
w_j	0.82	0.06	0.3	0.91	0.09

As can be seen from Table 4.14, the DMs now prefer criteria Y_1 (emotional steadiness) and criteria Y_4 (past experience) more than the other criteria. The results by each fuzzy MCDM method for the new criteria weights are given in Table 4.15.

Table 4.15: Results by the fuzzy MCDM methods with the new criteria weights

	A_1	A_2	A_3
Fuzzy Trade-off			
$DFT1$	1.700	0.354	1.272
$DT1$	1.799	0.364	1.334
Fuzzy VIKOR			
Crisp(S)	0.662	0.411	0.357
Crisp(R)	0.405	0.276	0.253
Q^a	0.007	0	0.016
Q^b	0.145	0.024	0
Q^c	0.188	0.045	0
Crisp(Q)	0.121	0.023	0.004
Fuzzy TOPSIS			
d^+	3.283	3.215	3.184
d^-	2.074	2.096	2.125
CC	0.387	0.395	0.400

From the results in Table 4.15, the best candidate identified by the fuzzy trade-off ranking method is candidate A_2. Note that, candidate A_2 is ranked the lowest with the previous criteria weights. The difference is related to the DMs preferences. In the previous problem, the DMs preferences towards each criterion are almost equal (Table 4.7). However, in the new weights problem, the DMs prefer criteria Y_1 and Y_4 more than the others. According to Table 4.7, candidate A_2 possesses the best score in both criteria preferred by the DMs, hence, he/she is now regarded as the best choice. In turn, the worst candidate is A_1 since this candidate is ranked the lowest in both criteria Y_1 and Y_4.

Meanwhile, the fuzzy VIKOR method gives a set of compromise solutions $\{A_3, A_2, A_1\}$ as a final ranking result for the new weights case. As for the fuzzy TOPSIS, the best option for the new weights is the same as in the previous weights case, i.e. candidate A_3. The difference now is in the results, such that an alternative that is closer to the ideal solution (d^+) is also farther from the anti-ideal solution (d^-).

Make a decision.
It doesn't have to be a wise decision
or a perfect one.
Just make one.
-Seth Godin, Best selling author.

SECTION FIVE

5 Conclusions

In summary, the study contributes to the knowledge of the subject (multi-criteria decision making) in **developing a new automatic ranking method** that gives the least compromise solution as the best solution. In addition, the **new method comprehends with the DM preferences**, either a risk-taker, a risk-averse or the ones in-between. The study also **introduces an extra algorithm** that tackles a situation in which more than one alternative happens to be the best solution. Moreover, the study **extends the new developed ranking to solve the fuzzy decision-making problem**, in which, a fuzzy trade-off ranking method is developed. The details summary of each main section are given in the next subsection.

5.1 Summary of research findings

I Multi-criteria decision making

A trade-off ranking method has been proposed in this thesis. The method minimizes compromise between the alternatives. The key property of the proposed method is, that in contrast to the other methods, the ranking algorithm is non-local. This means the ranking takes into account the position of each alternative. The trade-off ranking can reflect the position of the Pareto frontier. The application of the method has been tested on different test cases. The obtained results are compared against the rankings provided by the TOPSIS and the relative distance ranking method.

The trade-off ranking method is based on different preference principals than the TOPSIS and the relative distance method. The two latter methods provide solutions which are closer to the ideal solution and farther from the anti-ideal solution. The trade-off ranking method selects least compromise solutions in comparison to the other techniques considered. Meanwhile, it requires the presence of a well-distributed Pareto set. In comparison to the TOPSIS and the relative distance approach, the trade-off ranking might be more adequate for some classes of problems such as a multi-objective problem with conflicting criteria. In any contradiction between objectives, the trade-off ranking method identifies the least compromise solution. As shown in the thesis, the top ranking solutions obtained with the three methods; the trade-off ranking, the relative distance and the TOPSIS; do not always coincide because of the different principals utilized.

II Uncertainty in multi-criteria decision making

The uncertainty analysis of the distance-based ranking techniques, the TOPSIS and the relative distance, including a new MCDM method, the trade-off ranking, is also presented in the thesis. Two types of uncertainties in MCDM process: in the data parameters/variables and in the preferences of the DMs towards each criterion are tested. In the data-uncertainty analysis, the fuzzy numbers have been implemented and a new objective function has been introduced in finding a new robust set of alternatives. In the preference-uncertainty analysis, different weights have been used to represent variation in the preferences of the DM. The effects of the ranking solutions in each analysis have been studied for several test cases.

The first analysis, uncertainty in the data variables, is important for the DM since it provides a robust set of alternatives despite the perturbation in the input data. The DM can also gain insight into different solutions provided by the different MCDM methods. The second analysis, uncertainty in the criteria weights, is valuable because it gives an overall view of the DM preferences. In the second analysis the trade-off ranking method emphasizes on the importance of the criteria. If the DM choose some criteria far greater than the others, the trade-off ranking method is able to give a solution with the best value in the preferred criteria, i.e. one of the extreme solution. Thus, the trade-off ranking method can be acceptable to a different type of DM preferences in the same problem. In comparison to the TOPSIS and the relative distance method, the trade-off ranking method has an extra algorithm to tackle a problem with more than one solution. The extra algorithm in the trade-off ranking method is relevant to the conflicting criteria problem. In such a problem, there is no unique solution. One solution may be better than the other, depending on the criteria weights or the DM preferences. Hence, when the criteria are equally preferred, the extra algorithm gives the least-compromise option as the best solution. Finally, the trade-off ranking method gives a solution that may preferred by the DM from the range of the extreme solutions to the least compromise among alternatives.

The thesis also has presented the sensitivity analysis of weight changes in three multi-criteria decision making methods; the trade-off ranking, TOPSIS and the relative distance approach. The analysis has been done to determine the critical criterion with the smallest changes in the weights that affect the current ranking. The study has showed that, in the analysis with three decision making methods, the critical criterion in each method is not a criterion with the most weight. Thus, it is not an important criterion. Due to that, with the knowledge on the critical criterion, the DM may discard that criterion in order to obtain a more robust ranking due to preference perturbation.

III Fuzzy multi-criteria decision making

A fuzzy trade-off ranking for the fuzzy MCDM problem has been proposed in this thesis. This method has been utilised to find the best solution to the fuzzy conflicting multi-criteria problem. The fuzzy trade-off ranking method is able to capture the solution with the least compromise. The fuzzy trade-off ranking method is also able to comprehend the DM preferences in a fuzzy conflicting MCDM problem. The proposed method has been compared against two other fuzzy MCDM methods, known as fuzzy VIKOR and fuzzy TOPSIS, in a personnel selection problem.

In contrast with the fuzzy trade-off ranking method, the fuzzy TOPSIS method is an individual performance method, where an alternative is only compared against the ideal/anti-ideal solutions, which are the artificial solutions. Such a ranking calculation is the best option for the mutual multi-criteria problem, where an alternative, that is close to the ideal solution, is also far from the anti-ideal solution. However, in a conflicting MCDM problem, such an assumption is not always realized; the best solution which is the closest to the ideal solution may not be the farthest from the anti-ideal solution. Conversely, the fuzzy VIKOR method gives a similar solution to the fuzzy trade-off ranking, since fuzzy VIKOR was also developed to tackle the conflicting MCDM problems. However, in some problems - as shown in the numerical example - the fuzzy VIKOR gives a set of compromise solutions rather than one single solution. In that matter, the DM still has to choose one solution out of the compromise set proposed by the fuzzy VIKOR method.

5.2 Future research implications

There are some other issues which were not fully addressed in the current study. These issues are briefly discussed in this subsection as a motivation for future research.

Decision making under uncertainty Decision making process can provide very little guidance to the DMs beyond offering them some simple decision rules to aid them in their analysis of uncertain situations. There are four basic rules for decision making under uncertainty: the maximax rule, the maximin rule, the minimax rule, and the equal probability rule. The maximax rule looks at the best that could happen under each action. In turn, the maximin rule looks at the worst that could happen under each action. The minimax decision making is based on opportunistic loss, i.e. minimum worst potential regret. The equal probability rule assumes each state of nature is equally likely to occur. It is interesting to see how the proposed method works under each rule.

Decision making under risk Uncertainty and risk are not the same thing. Uncertainty deals with possible outcomes that are unknown, making the probabilities to the outcomes can not be assigned. Whereas, in risk situation, the possible outcomes can be listed and the probabilities to the outcomes can be assigned. The decision rules under conditions of risk are: the expected value rule, the mean-variance rules and the coefficient of variation rule. In the current study, we only consider the uncertainty in decision making problem. Further research towards the behaviour of the trade-off ranking method under risk situations may be explored.

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Appendix A Test problems formulation

TNK

$$
\begin{aligned}
&\text{Minimize } (x_1, x_2)\\
&\quad \text{s.t. } g_1(x) = x_1^2 + x_2^2 - 1 - 0.1\cos\left(16\tan^{-1}(x_1/x_2)\right) \geq 0\\
&\qquad\quad g_2(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5\\
&\qquad\quad 0 \leq x_i \leq \pi \ (i = 1, 2)
\end{aligned}
$$

ZDT1

$$
\begin{aligned}
&\text{Minimize } (F_1(x), F_2(x))\\
&\quad \text{s.t. } 0 \leq x_i \leq 1 \ (i = 1, 2, ..., n)\\
&\quad \text{where}\\
&\quad F_1(x) = x_1,\\
&\quad F_2(x) = g(x)\left(1 - \sqrt{x_1/g(x)}\right),\\
&\quad g(x) = 1 + 9/(n-1)\sum_{i=2}^{n} x_i^2
\end{aligned}
$$

ZDT2

$$
\begin{aligned}
&\text{Minimize } (F_1(x), F_2(x))\\
&\quad \text{s.t. } 0 \leq x_i \leq 1 \ (i = 1, 2, ..., n)\\
&\quad \text{where}\\
&\quad F_1(x) = x_1\\
&\quad F_2(x) = g(x)\left(1 - (x_1/g(x))^2\right)\\
&\quad g(x) = 1 + 9/(n-1)\sum_{i=2}^{n} x_i^2
\end{aligned}
$$

ZDT6

$$
\begin{aligned}
&\text{Minimize } (F_1(x), F_2(x)) \\
&\quad \text{s.t. } 0 \leq x_i \leq 1 \ (i = 1, 2, ..., 10) \\
&\quad \text{where} \\
&\quad F_1(x) = 1 - \exp -4x_1 \sin^6 4\pi x_1, \\
&\quad F_2(x) = g(x)\left(1 - (F_1(x)/g(x))^2\right), \\
&\quad g(x) = 1 + 9\left(\sum_{i=2}^{10} x_i^2/9\right)^{1/4}
\end{aligned}
$$

DTLZ5

$$
\begin{aligned}
&\text{Minimize } (F_1(x), F_2(x), F_3(x)) \\
&\quad \text{s.t } 0 \leq x_i \leq 1 \ \ (i = 1, 2, 3) \\
&\quad \text{where} \\
&\qquad F_1(x) = (1 + g(x_3)) \cos(\theta_1) \cos(\theta_2), \\
&\qquad F_2(x) = (1 + g(x_3)) \cos(\theta_1) \sin(\theta_2), \\
&\qquad F_3(x) = 3\,(1 + g(x_3)) \sin(\theta_1), \\
&\qquad g(x) = (x_3 - 0.5)^2 \\
&\qquad \theta_1 = \frac{\pi}{2}(x_1) \\
&\qquad \theta_2 = \frac{\pi}{4\,(1 + g(x_2))}(1 + 3g(x_3)x_2)
\end{aligned}
$$

Appendix B Mean and variance of fuzzy numbers

Suppose $A = (a, b, c)$ is a triangular fuzzy number where $b > 0$ and $c > 0$ are the left and right-width of the fuzzy number centered at a. Therefore, the α-cut of A is computed by,

$$A_\alpha = [a - (1-\alpha)b, a + (1-\alpha)c] \, \forall \alpha \in [0, 1].$$

The Possibilistic mean value of A is the arithmetic mean of its lower and upper possibilistic mean value (Carlsson and Fuller, 2001), i.e.

$$Mean(A) = \frac{(\underline{M}(A) + \overline{M}(A))}{2},$$

where

$$\underline{M}(A) = 2\int_0^1 \alpha \underline{A} d\alpha, \ \overline{M}(A) = 2\int_0^1 \alpha \overline{A} d\alpha.$$

Here, $\overline{A}$ and $\underline{A}$ are the upper and lower bounds of α-cut of fuzzy number A, respectively. It is easy to prove that the possibilistic mean value of the fuzzy number is given by

$$Mean(A) = a + \frac{c-b}{6}.$$

The variance of a fuzzy number is given by the formulae (Carlsson and Fuller, 2001),

$$Var(A) = \frac{1}{2}\int_0^1 \alpha(\overline{A} - \underline{A})^2 d\alpha.$$

Again, it is easy to prove that the variance of fuzzy number A can be calculated as

$$Var(A) = \frac{(b+c)^2}{24}.$$

Appendix C Data of the test cases

Table 5.1: Data for TNK problem (x_1, x_2)

(0.0417, 1.0384)	(0.7746, 0.5713)	(0.5454, 0.7799)
(0.0613, 1.0267)	(0.7736, 0.5835)	(0.5704, 0.7747)
(0.0796, 1.0123)	(0.7884, 0.5276)	(0.6182, 0.7731)
(0.1120, 0.9813)	(0.7981, 0.5158)	(0.6490, 0.7726)
(0.1339, 0.9601)	(0.7710, 0.6709)	(0.7766, 0.5578)
(0.1556, 0.9430)	(0.8284, 0.4948)	(0.7785, 0.5500)
(0.1691, 0.9356)	(0.9829, 0.1617)	(0.8848, 0.4702)
(0.1826, 0.9310)	(0.9651, 0.2467)	(0.9146, 0.4557)
(0.1961, 0.9291)	(0.2228, 0.9320)	

Table 5.2: Data for ZDT1 problem (F_1, F_2)

(0.2500, 0.5000)	(0.0102, 0.8990)	(0.1106, 0.6674)
(0.2814, 0.4695)	(0.5747, 0.2419)	(0.0858, 0.7071)
(0.2302, 0.5202)	(0.6274, 0.2079)	(0.4330, 0.3420)
(0.3154, 0.4384)	(0.0000, 1.0000)	(0.0548, 0.7659)
(0.1830, 0.5722)	(0.6829, 0.1736)	(0.4775, 0.3090)
(0.2067, 0.5454)	(0.7410, 0.1392)	(0.0327, 0.8192)
(0.1591, 0.6011)	(0.8019, 0.1045)	(0.5248, 0.2756)
(0.3520, 0.4067)	(0.8653, 0.0698)	(0.3911, 0.3746)
(0.1349, 0.6327)	(0.9314, 0.0349)	(1.0000, 0.0000)

Table 5.3: Data for ZDT2 problem (F_1, F_2)

(0.0000, 1.0000)	(0.5342, 0.7146)	(0.4567, 0.7914)
(1.0000, 0.0000)	(0.6226, 0.6124)	(0.9455, 0.1060)
(0.0984, 0.9903)	(0.7071, 0.5000)	(0.3911, 0.8470)
(0.1949, 0.9620)	(0.7986, 0.3622)	
(0.2876, 0.9173)	(0.8746, 0.2351)	

Table 5.4: Data for DTLZ5 problem (F_1, F_2, F_3)

(0.0000, 0.0000, 1.0000)	(0.4217, 0.4217, 0.8027)
(0.0658, 0.0658, 0.9957)	(0.4973, 0.4973, 0.7109)
(0.1460, 0.1460, 0.9785)	(0.6252, 0.6252, 0.4671)
(0.2249, 0.2249, 0.9481)	(0.6755, 0.6755, 0.2958)
(0.3079, 0.3079, 0.9002)	(0.7071, 0.7071, 0.0000)