NUMERICAL STUDY OF BOUNDARY LAYER DUE TO STATIONARY FLAT PALTE

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ABSTRACT

This thesis presents the numerical study on boundary layer equation due to stationary flat plate. Matlab is the mathematical programming that used to solve the boundary layer equation applied of keller box method. The objective of this project is to solve the boundary layer equation on the stationary flat plate utilizing the Matlab programming software. Matlab source codes are generated base on boundary layer equation. Then it was compiled and analyzes using Matlab. The result for Matlab is compared to the Blasius solution. It is largest deviation is about 0.070% and lowest is 0.005%. The present result compared with the Blasius solution and it can be seen that the both result are good agreement.

ABSTRAK

Tesis ini membentangkan kajian pembelajaran matematik ke atas lapisan sempadan disebabkan oleh plat rata tidak bergerak. Matlab adalah perisian matematik yang digunakan untuk menyelesaikan persamaan lapisan sempadan aplikasi dari cara penggunaan "keller box". Objektif projek ini adalah untuk menyelesaikan persamaan lapisan sempadan ke atas plat rata tidak bergerak dengan menggunakan perisian pengaturcara Matlab. Kod asas Matlab yang dibina adalah berasaskan persamaan lapisan sempadan. Kemudiannya, ia akan di analisis menggunakan perisian Matlab. Keputusan yang dihasilkan dari perisian Matlab dibandingkan dengan keputusan oleh Blasius. Peratus perubahan paling besar dianggarkan sebanyak 0.070% dan yang terendah adalah sebanyak 0.005%. Keputusan yang telah dapat dibandingkan dengan keputusan dari Blasius dan kesimpulan yang dapat dibuat ialah kedua-dua keputusan boleh diterima pakai.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In general, the shear stress on a smooth plane surface is variable over the surface. Hence the total shear force in a given direction is obtained by integrating the component of shear stress in that direction over the total area of the surface. The shear stress on a smooth plane is a direct function of velocity gradient next to the plane and the equation is given by:

$$\tau = \mu \frac{dV}{dy}$$

Therefore, any problem involving shear stress also involves the low pattern in the vicinity of the surface. The layer of fluid near the surface that has undergone a change in velocity because of the shear stress at the surface, which is known as the boundary layer. The purposes of study of the flow pattern in the boundary layer, as well as the associated shear stress at the boundary, are called boundary-layer theory.

Analytical solutions of the steady fluid flows were solved generally two ways. One of these was that of parallel viscous flows and low Reynolds number flows, in which the nonlinear adjective terms were zero and the balance of force was that between the pressure and the viscous force. The second type of solution was that of inviscid flows around bodies of various shapes, in which the balance of force was that between the inertia and pressure force.

Although the equations of motion are nonlinear in this case, the velocity field can be determined by solving the linear Laplace equation. These irrotational solutions predicted pressure force on a streamlined body that agreed surprisingly well with experimental data for flow of fluids with small viscosity. However, these solutions predicted a zero drag force and nonzero tangential velocity at the surface.

1.1.1 History

In 1905, Ludwig Prandtl has made the hypothesized about the boundary layer. He states that for small viscosity, viscous force is negligible everywhere except close to the solid boundaries where the no-slip condition had to be satisfied. The thickness of boundary layer is zero as the viscosity goes to zero. Prandtl has made two contradictory facts. First, he has supported the intuitive idea which is the effect of viscosity are indeed negligible in most of the flow field if v, velocity component in y-direction is small. He able to account for drag by insisting that the no-slip condition must satisfied at the wall, no matter how small the viscosity. Prandtl has showed the equation of motion within the boundary layer can be simplified. The concept of boundary layer has then been generalized. The mathematical techniques involve have been formulized and applied to various other branches of science. Concept of the boundary layer is considered one of the cornerstones in history of fluid mechanics.

1.1.2 Project Background

There are various methods in solving boundary layer equation due to stationary flat plate including Blasius series solution, Karman-Polhausen's method and numerical solution. The boundary layer equation will be solved to determine the behavior of laminar and turbulent boundary layers on a continuous flat surface.

In this project, steady, two dimensional flows and laminar boundary layer is considered to solve the boundary layer problem. For the laminar boundary layer, there are two main method involve. The method involved is numerical solution and the other one is integral method.

At the end of this project, the boundary layer problem will be solved using integral method, experimental and numerical solution also known as keller-box method.

1.2 Objective

The main objective in this project is to solve and compared the boundary layer on the stationary flat plate between Matlab programming and theoretical Blasius solution applying the keller box method.

1.3 Scope of Project

Scope of the project contain the method which is to be used in solving boundary layer problem with considering only steady and two dimensional flow in the xy – plane in Cartesian coordinates. The methodology involve in solving boundary layer problem includes:

i) Derive boundary layer equation

ii) Numerical solution of boundary layer problem

1.4 Project Flow Chart

Solve the boundary layer due to stationary flat plate using Keller-Box method. The project flowchart is shown in figure 1.1.

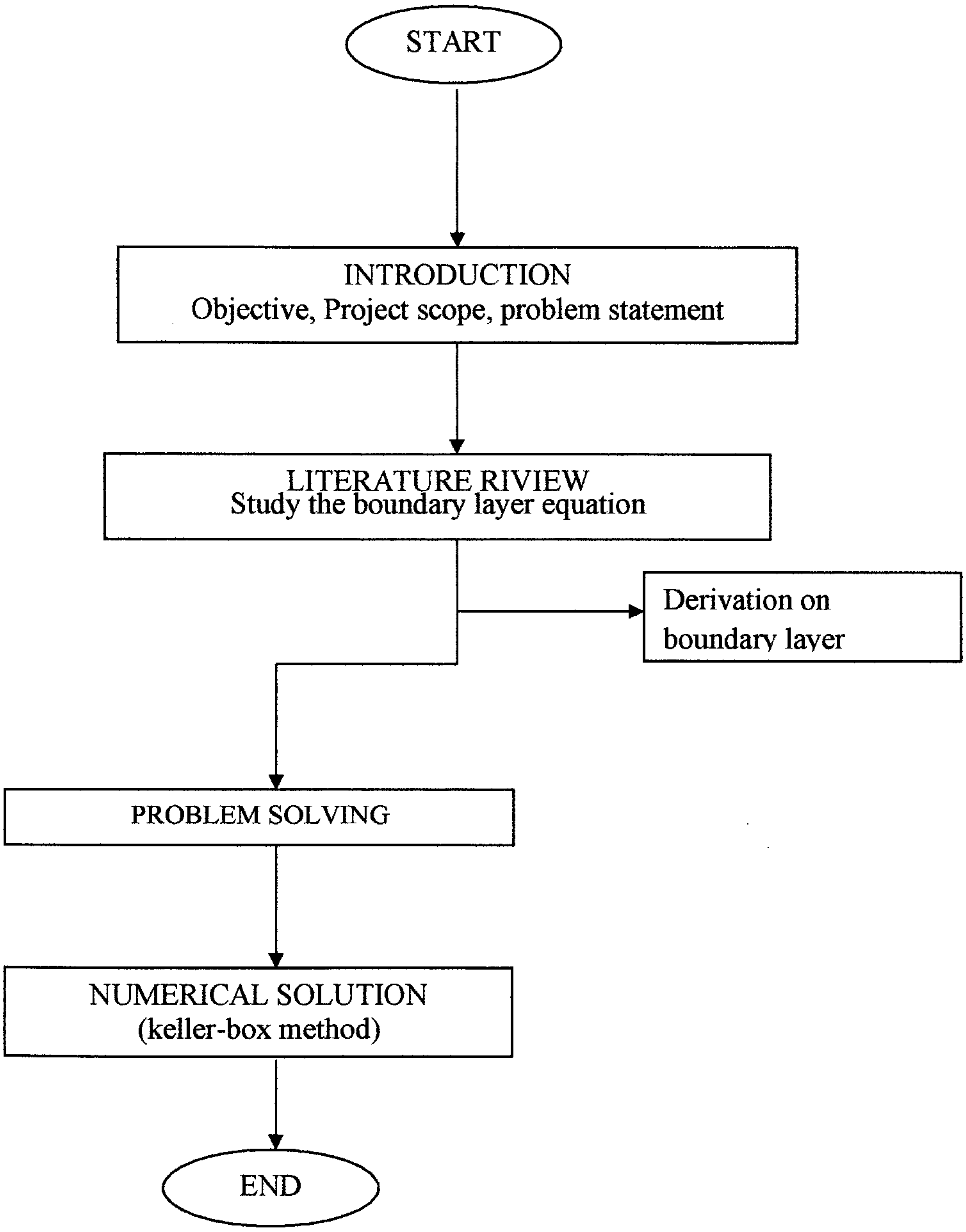

Figure 1.1: Project flowchart

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Consider a fluid flows past the leading edge of a stationary flat surface aligned with the flow direction. According to the laws of perfect fluid flow, the surface should not influence the flow in any way. Besides, the velocity should be U_∞ everywhere in the flowing fluid. However, the no-slip condition requires that right at the surface there is no relative motion between the fluid and the surface. Strong velocity gradients appear in the region near the surface. In viscous flow, there should exit a velocity gradient in the y direction extending out to infinity. In that case Prandtl suggested that the flow could be conceptually divided into two parts. In the region close to the solid surface, the effects of viscosity are too large to be ignored. However, this is a fairly small region where outside this region the effects of viscosity are small and can be neglected.

Boundary layer is the region where velocity gradients are large enough to produce significant viscous stresses and significant dissipation of mechanical energy. The region outside the boundary layer is called the free stream (or undisturbed stream or potential flow regime), in which there are no significant velocity gradients and viscous stresses are negligible. At any level in the boundary layer, the viscous stresses tend to decrease the velocity of the flow on the high speed side of the layer and increase the velocity on the low speed side.

At the edge of boundary layer, therefore, viscous action will tend to slow the free stream fluid and proceed downstream more and more of the free-stream flow is affected by friction. Therefore the thickness of the boundary layer grows with distance downstream.

The character of the flow field is a function of the shape of the body. Flows past relatively simple geometric shapes such as sphere or circular cylinder are expected to have less complex flow fields than flows past a complex shape such as an airplane or a tree. The character of the flow should depend on the various dimensionless parameters involved. For typical external flows the most important of these parameters are the Reynolds number, Mach number and Froude number. For the present, it is considered how the external flow and its associated lift and drag vary as a function of Reynolds number. Recall that the Reynolds number represents the ratio of inertial effects to viscous effects.

In the absence of all viscous effects ($\mu = 0$) the Reynolds number is infinite, thus it can be said that flows with large Reynolds number (Re > 100) are dominated by inertial effects. On the other hand, in the absence of all inertial effects (negligible mass or $\rho = 0$), the Reynolds number is zero, and can be said that the flows with small Reynolds number (Re < 1) are dominated by viscous effects. Generally, flow past the objects can be illustrated by considering flows past two objects - one a flat plate parallel to the upstream velocity and the other a circular cylinder. The Reynolds number is given as:

$$Re = \frac{\rho U_{avg} l}{\mu}$$

where,

Re = Reynolds number

ρ = Fluid density

l = Characteristic of Length

U_{avg} = Average velocity

μ = Fluid viscosity

2.2 Conservation of Linear Momentum

To generate the Navier-Stokes equation, must know the Cauchy's equation first. The general expression for the conservation of linear momentum as applied to a control volume. The equation can be expressed as Equation (2.1) [2]

$$\Sigma\vec{F} = \int_{CV} \rho\vec{g}\, dV + \int_{CS} \sigma_{ij} \,.\, \vec{n}\ dA = \int_{CV} \frac{\partial}{\partial t}\left(\rho\vec{V}\right) dV + \int_{CS} \left(\rho\vec{V}\right) d\vec{V} \cdot \vec{n}\ dA \tag{2.1}$$

where, σ_{ij} = stress tensor

$\vec{V}$ = Absolute velocity

A = Surface area

$\vec{n}$ = Unit normal vector

$\vec{g}$ = Gravitational acceleration and its magnitude

ρ = Density

Components of σ_{ij} on the positive faces of an infinitesimal rectangular control volume are shown in Figure 2.1.

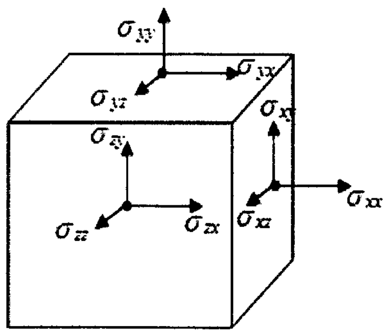

Figure 2.1: Infinitesimal rectangular control volume

The Figure 2.1 shows the positive components of the stress tensor in Cartesian coordinates on positive (right, top and front) faces of an infinitesimal rectangular control volume. The dots indicate the center of each face. Positive components of the negative (left, bottom and back) faces are in the opposite direction of those shown.

The way to derive the differential form of conservation of momentum is applied the divergence theorem. Divergence theorem can be written as Equation (2.2)

$$\int_v \vec{\nabla} \cdot G_{ij}\, dV = \oint_A G_{ij} \cdot \vec{n}\, dA \tag{2.2}$$

where G_{ij} is second-order tensor. Let replace G_{ij} in the extended divergence theorem with the quantity $(\rho\vec{V})\vec{V}$ which is a second-order tensor and the last term in Equation (2.1) become,

$$\int_{CS} (\rho\vec{V})\vec{V} \cdot \vec{n}\ dA = \int_{cv} \vec{\nabla} \cdot (\rho \vec{V} . \vec{V})\ dV \tag{2.3}$$

That $\vec{V}.\vec{V}$ is a vector product called the outer product of the velocity vector with itself. Now let replace G_{ij} by stress tensor σ_{ij} to the second term on the left-hand side of Equation (2.1) and it becomes

$$\int_{CS} \sigma_{ij} \cdot \vec{n}\ dA = \int_{CV} \vec{\nabla} \cdot\ \sigma_{ij}\, dV \tag{2.4}$$

By applying the equation (2.3) and (2.4) into the equation (2.1), the two surface integrals become volume integrals. Combine and rearranging

$$\int_{cv} \left[\frac{\partial}{\partial t}\left(\rho\vec{V}\right) + \vec{\nabla} \cdot \left(\rho\vec{V}\,\vec{V}\right) - \rho\vec{g}\ -\ \vec{\nabla} \cdot \sigma_{ij} \right] dV = 0 \tag{2.5}$$

Equation (2.5) must hold any control volume regardless of its size and shape. This is possible only if the integrand is identically zero. Now the equation for conservation of linear momentum knows as Cauchy's equation.

Cauchy's equation can be expressed as Equation (2.6)

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \vec{\nabla} \cdot (\rho\vec{V} \cdot \vec{V}) = \rho\vec{g} + \vec{\nabla} \cdot \sigma_{ij} \tag{2.6}$$

The Cauchy's equation is valid for compressible as well as incompressible flow since there is no assumption about incompressibility. Equation (2.6) is a vector equation and represents three scalar equations, one for each coordinate axis in three-dimensional problem.

However, the Cauchy's equation is not very useful because the stress tensor σ_{ij} contains nine components where six are independent (because of symmetry). Plus density and three velocity component, there are six additional unknowns for a total of ten unknowns. In Cartesian coordinates the unknowns are $\rho, u, v, w, \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}$ and σ_{zz}. The constitutive equation is going to discuss because it is enable to write the component of stress tensor in terms of the velocity field and pressure field.

The first thing is, separate the pressure stress and the viscous stresses. When the fluid at rest, the only stress acting is hydrostatic pressure P, and the pressure is acting inward and normal to surface. For fluid at rest the stress tensor reduces to:
Fluid at rest, the stress tensor can be written as Equation (2.7)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} \tag{2.7}$$

When the fluid is moving, pressure is still acts inwardly normal, but the viscous stress may also exist. For moving fluids as,

Moving fluids, the stress tensor can be expressed as Equation (2.8)

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{2.8}$$

where τ_{ij} is called the viscous stress tensor or the deviatoric stress tensor. Replace the six unknown components of σ_{ij} with six unknown component of τ_{ij} and another unknown which is pressure, P. In incompressible fluid, P, is no longer define as thermodynamic pressure. Instead, P is define as mechanical pressure,

Mechanical pressure can be written as Equation (2.9)

$$P_m = -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) \tag{2.9}$$

where P_m is mechanical pressure

2.3 Derivation Of Navier-Stokes Equation

Assume incompressible flow (ρ = constant), nearly isothermal flow where local change in temperature is small: this eliminates the need for a differential energy equation. By the assumption, fluid properties such as dynamic viscosity μ and kinematic viscosity, v are constant as well. The viscous stress tensor reduces to

$$\tau_{ij} = 2\mu\,\varepsilon_{ij} \tag{2.10}$$

where ε_{ij} is the strain rate tensor. In Cartesian coordinates, the nine components of the viscous stress tensor are listed, six of which are independent due to symmetry. The equation (2.7) becomes:

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix} \tag{2.11}$$

and Equation (2.8) becomes:

$$\sigma ij = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & 2\mu\dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) & 2\mu\dfrac{\partial w}{\partial z} \end{pmatrix} \tag{2.12}$$

Substitute Equation (2.12) into the Cartesian components of Cauchy's equation. Consider x-component first. So the equation becomes:

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \tag{2.13}$$

Note that as long as velocity components are smooth functions of x, y, and z the order of differentiation is irrelevant. The first part of the last term in Equation (2.13) becomes:

$$\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right) = \mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right) \tag{2.14}$$

After rearrangement of the viscous terms in equation (2.13) becomes:

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\frac{\partial w}{\partial z} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$= -\frac{\partial P}{\partial x} + \rho g_x + \mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{2.15}$$

Term in parentheses is zero because of the continuity equation for incompressible flow. The last three terms as Laplacian of velocity component u in Cartesian coordinates.

Thus, the momentum equation in x, y and z component is:

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + \mu\nabla^2 u \tag{2.16}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \rho g_y + \mu\nabla^2 v \tag{2.17}$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial P}{\partial z}+\rho g_z+\mu\nabla^2 w \tag{2.18}$$

Finally the combination of the three components into one vector equation becomes Navier-Stokes equation for incompressible flow with constant viscosity. Incompressible Navier-Stokes equation can be written as Equation

$$\rho\frac{D\vec{V}}{Dt}=-\vec{\nabla}P+\rho\vec{g}+\mu\nabla^2\vec{V} \tag{2.19}$$

2.4 Boundary Layer Equation

The Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, are a set of equations that describe the motion of fluid substances such as liquids and gases. These equations establish that changes in momentum in infinitesimal volumes of fluid are simply the product of changes in pressure and dissipative viscous forces (similar to friction) acting inside the fluid. These viscous forces originate in molecular interactions and dictate how viscous a fluid is. Thus, the Navier-Stokes equations are a dynamical statement of the balance of forces acting at any given region of the fluid.

For the first stage, the Navier stokes equation.

x - axis component

$$\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)=-\frac{dP}{dx}+\rho g+\mu\left(\frac{d^2u}{d_x^{\,2}}+\frac{d^2u}{d_y^{\,2}}+\frac{d^2u}{d_z^{\,2}}\right) \tag{2.20}$$

y - axis component

$$\rho\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)=-\frac{dP}{dx}+\rho g+\mu\left(\frac{d^2v}{d_x^{\,2}}+\frac{d^2v}{d_y^{\,2}}+\frac{d^2v}{d_z^{\,2}}\right) \tag{2.21}$$

z - axis component

$$\rho\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right)=-\frac{dP}{dx}+\rho g+\mu\left(\frac{d^2w}{d_x^2}+\frac{d^2w}{d_y^2}+\frac{d^2w}{d_z^2}\right) \qquad (2.22)$$

where u, v and w are the x, y and z components of velocity. The equation have rearranged so the acceleration terms are on the left side and the force terms are on the right.

Then, the Navier-stokes equation is reducing with considering steady, two – dimensional (which is only x and y component) laminar flows and negligible gravitational effects to:

$$u\frac{du}{dx}+v\frac{du}{dy}=-\frac{1}{\rho}\frac{dP}{dx}+v\left(\frac{d^2u}{d_x^2}+\frac{d^2u}{d_y^2}\right) \qquad (2.23)$$

and

$$u\frac{dv}{dx}+v\frac{dv}{dy}=-\frac{1}{\rho}\frac{dP}{dy}+v\left(\frac{d^2v}{d_x^2}+\frac{d^2v}{d_y^2}\right) \qquad (2.24)$$

Equation (2.23) and (2.24) are known as continuity equation. The differential equation for conservation of mass is valid for steady or unsteady flow, and compressible or incompressible fluids.

Differential equation of conservation can be expressed as Equation (2.25)

$$\left(\frac{d\rho}{dt}\right)+\left(\frac{d(\rho u)}{dx}+\frac{d(\rho v)}{dy}+\frac{d(\rho w)}{dz}\right)=0 \qquad (2.25)$$

In vector notation, the differential equation of conservation can be written as Equation (2.26)

$$d\rho+\nabla\cdot\rho V=0 \qquad (2.26)$$

In this project, consider only incompressible fluids, so the fluids density, ρ, is a constant throughout the flow field.

The equation becomes:

$$\nabla \cdot V = 0 \tag{2.27}$$

or

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \tag{2.28}$$

considering two dimensional, the Equation (2.28) becomes:

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \tag{2.29}$$

Boundary layer is thin with the component of velocity normal to the plate is much smaller than that parallel to the plate and that the rate of change of any parameter across the boundary layer should be greater than along the flow direction[3].

$$\upsilon << u \quad \text{and} \quad \frac{\partial}{\partial x} << \frac{\partial}{\partial y} \tag{2.30}$$

Assume that the flow is primarily parallel to the plate and any fluid property is convected downstream much more quickly than it is diffused across the streamlines. With these assumptions, the equation (2.21), equation (2.22) and equation (2.29) reduce to the following boundary layer equations:

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \tag{2.31}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2.32}$$

2.5 Boundary Layer on A Flat Plate: Blasius Solution

From the boundary layer equation:

$$\frac{du}{dx}+\frac{dv}{dy}=0 \tag{2.33}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial^2 u}{\partial y^2} \tag{2.34}$$

Which is subject to: $y=0$: $u=v=0$, $x>0$

$y \rightarrow \infty$: $u \rightarrow U$

$x=x_0$: $u(y)$ given, $\mathrm{Re}_{xo} >> 1$

Stream function form is given as Equation (2.35)

$$\varphi=\sqrt{vU_f}\, f(\eta) \tag{2.35}$$

$$\eta=y\sqrt{\frac{U_f}{vx}} \tag{2.36}$$

where, φ = Stream function

v = Fluid kinematic viscosity

U_f = Velocity of continuous solid surface

η = Independent similarity variable

y = Cartesian coordinate from solid surface

x =Cartesian or cylindrical axis coordinate

f = Dependent similarity variable

Rearrange the Equation (2.36) and it becomes:

$$y = \eta\sqrt{\frac{vx}{U}} \tag{2.37}$$

Base on velocity component, where:

$$u = \frac{\partial\varphi}{\partial y} \tag{2.38}$$

$$v = -\frac{\partial\varphi}{\partial x} \tag{2.39}$$

Now substitute the equation (2.35) and equation (2.36) into equation (2.38). The velocity component becomes:

$$u = U\frac{\partial f(\eta)}{\partial\eta} = U\,f\,'(\eta) \tag{2.40}$$

Then the same method is used to the equation (2.39). In this section, equation (2.36) is no need to be replaced. By applying differential concept, the equation now becomes:

$$v = \frac{1}{2}\sqrt{\frac{vU}{x}}\left[\eta\,f\,'(\eta) - f(\eta)\right] \tag{2.41}$$

The other term is equation (2.34) need to be differential. At the last section it comes out as Equation (2.42)

$$\frac{\partial u}{\partial x} = U\frac{\partial f\,'(\eta)}{\partial x}$$

$$= U\,f\,''(\eta)\frac{\partial\eta}{\partial x}$$

$$= -U\frac{f\,''(\eta)\cdot\eta}{2x} \tag{2.42}$$

$$\frac{\partial u}{\partial y} = U \frac{\partial f'(\eta)}{\partial y}$$

$$= U f''(\eta) \frac{\partial \eta}{\partial y}$$

$$= U \sqrt{\frac{U}{vx}} f''(\eta) \tag{2.43}$$

$$\frac{\partial^2 u}{\partial y^2} = U \sqrt{\frac{U}{vx}} f'''(\eta) \left(\frac{\partial \eta}{\partial y} \right)$$

$$= \frac{U^2}{vx} f'''(\eta) \tag{2.44}$$

substitute Equations (2.40 2.41) into boundary layer equation (2.32). Finally, after simplification and rearrangement the equation reduces to:

$$f'''(\eta) + \frac{1}{2} f''(\eta) f(\eta) = 0$$

or

$$2f''' + ff'' = 0 \tag{2.45}$$

which is called Blasius equation with the boundary condition:

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1$$