NUMERICAL SOLUTION OF BOUNDARY LAYER PROBLEM DUE TO MOVING FLAT PLATE USING RUNGE-KUTTA METHODS

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ABSTRACT

This project is carried out to investigate the flow characteristic of boundary layer over moving plate using numerical solution. The flow characteristic along moving flat plate can be identify using numerical solution of the boundary layer equation. The numerical technique that was used in this project are Runge-Kutta methods. The steady flow, two dimensionless flow, incompressible flow, infinite aluminum flat plate and laminar flow was consider in this project. The programming of the boundary layer equation are developed. The programming was developed using the FORTRAN90 software and compiled by FORTRAN Power Station 4.0. The velocity profile of boundary layer along moving flat plate are investigated. It can be seen that when the flow distance from plate are increases, the velocity in boundary layer will decrease and the wall shear stress will increase. The comparison also been made with Blasius flow and previous study result. The flow characteristic of moving flat plate had been solve numerically using numerical approach.

ABSTRAK

Projek ini dijalankan untuk mengkaji lapisan sempadan punya cirian aliran diatas plat rata yang bergerak menggunakan kaedah berangka. Cirian aliran disepanjang plat yang bergerak boleh dikenali dengan menggunakan kaedah berangka dari persamaan matematik lapisan sempadan. Kaedah berangka yang akan digunakan dalam projek ini ialah kaedah Runge-Kutta. Aliran biasa, aliran dua dimensi, aliran tak termampat , plat aluminum yang panjang akan dipertimbangkan dalam projek ini. Pemprograman persamaan matematik lapisan sempadan dibuat. Dengan menggunakan perisian komputer FORTRAN90 dan FORTAN Power Station 4.0 sebagai penyusun, pemprograman dibuat. Profil halaju lapisan sempadan diatas sepanjang plat bergerak akan dikaji. Didapati apabila jarak aliran dari plat bertambah, halaju sempadan aliran akan berkurang dan tegasan ricih permukaan pula bertambah. Perbandingan dibuat denagan aliran Blasius dan kajian terdahulu. Cirian aliran sepanjang plat bergerak diselesaikan dengan menggunakan kaedah berangka.

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CHAPTER 1

INTRODUCTION

1.1 Project Background

This project is focusing on laminar flow which is the flow of high viscosity with low Reynolds number. The laminar boundary layer behavior on a moving continuous flat surface can be investigated by two methods. One method involves the numerical solution of the boundary layer equation and other one is using integral method which is based on assumed velocity profile that satisfies the appropriate boundary condition. The boundary layer equations represent a significant simplification over the full Navier-Stokes equations in a boundary layer region. The simplification is done by an order-of-magnitude analysis, which is determining terms in the equations are very small relative to the other terms. For simplicity, the boundary layer equations for steady, incompressible, uniform flow over a moving flat plate will be determine. After the equation had been deriving, Runge-Kutta method is important in order to solve the equation. Then the boundary layer equation will be solved to determine the behavior of laminar boundary layer due to moving flat plate.

1.2 Problem Statement

Boundary layer behavior over a moving continuous surface is an important type of flow occurring in several engineering processes. For example, many metallurgical processes involve the cooling of continuous strips or filaments by drawing them through a quiescent fluid. These phenomena are same for the boundary layer along moving flat plate. The boundary layer equations must be used in order to solve boundary layer problem. There are many way in order to solve boundary layer equation such as Taylor's series that the result sometimes is not really accurate. The problem may occur if the calculation that had been made got many error and not accurate. So in order to solve the boundary layer equations, others numerical methods will be used.

1.3 Define Question

What methods that can be used in order to solve boundary layer equation to get an accurate result of boundary layer problem?

1.4 Objective of Project

The main objective of this project is to solve the problem of boundary layer flow due to moving flat plate in viscous flow.

1.5 Scope of Project

These problems are formulated using similarity transformation and solve numerically using the Runge-Kutta methods. The scope will be limited to problem involve two dimensional incompressible flow on a continuous flat surface with a constant velocity.

1.6 Overview of the Project

In order to achieve the objective of this project, there is some guideline need to be understand. Chapter 1 consist of the scope, objective and problem statement of this project. Chapter 2 discuss more about literature review in which how to get the continuity governing equation of boundary layer and a little bit definition of boundary layer flow over moving flat plate and non-moving plate. Runge-Kutta method had been discuss on chapter 3 in which the methods that need to be apply to this project and chapter 4 show the development of boundary layer program using FORTRAN90 software and the discussion on the results.

CHAPTER 2

LITERITURE REVIEW

2.1 Introduction

The boundary layer is a very thin layer of air flowing over the surface of object or plate where viscous forces are important [1]. In other word, the boundary layer is the region where velocity gradients are large enough to produce significant viscous stresses and significant dissipation of mechanical energy.At the top of the boundary layer, the molecules move at the same speed of stream flow as the molecule outside the boundary layer. This speed is called the free-stream velocity in which there are no significant velocity gradients and viscous stresses are negligible.The actual speed at which the molecules move depends upon the shape of the object, the viscosity or the stickiness of the air and its compressibility [2].

The boundary layer flow is very important for many problems in aerodynamics, including the skin friction drag. The theory describes boundary layer effects was first presented by Ludwig Prandtl in the early 1900s. Prandtl was the first to realize that the forces experienced by a wing increased from the layer very near the wing's surface to the region far from the surface. The fluid flow sometimes can be laminar and sometimes turbulent. The flow of high viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluids motion that typically occurs at high velocities is typically turbulent. The flow of low viscosity such as air at high velocities is typically turbulent [3].The transition from laminar to turbulent depends on Reynolds number. Boundary layer thickness is usually defined as the distance away from the wall at which the velocity component parallel to the wall is 99 percent of fluid speed outside the boundary layer where $u = 0.99U$, where U is the free stream velocity.

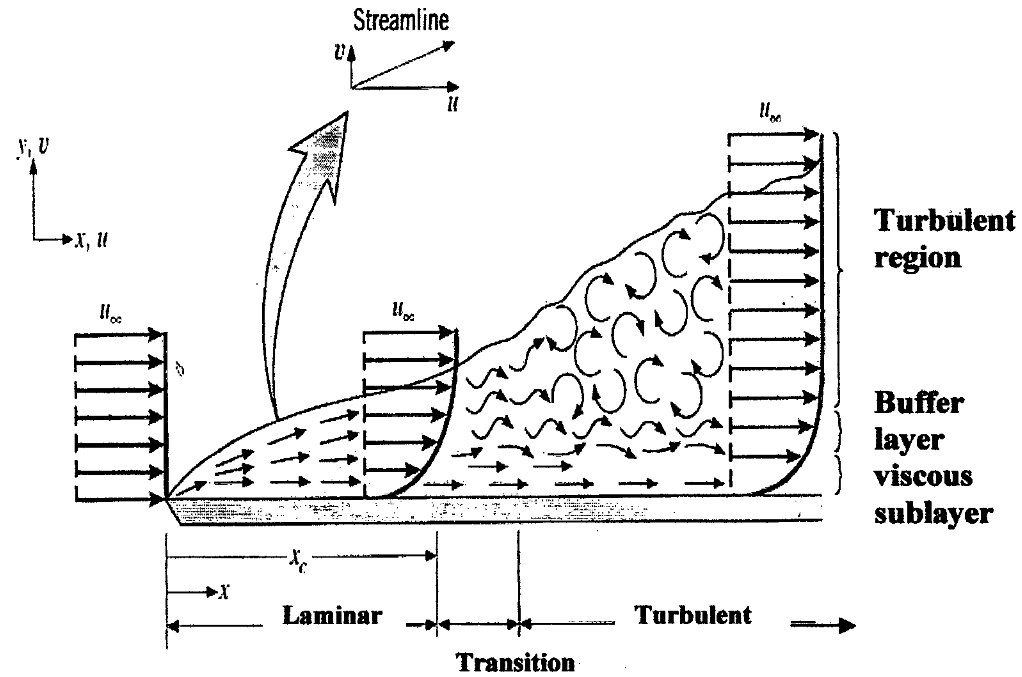

Figure 2.1: Velocity boundary layer developments on a flat plate

2.2 Two Dimensional Incompressible Flow

Fluid flow can be either in one, two or three dimensional. A typical fluid flow is in three dimensional [$\vec{V}(x, y, z)$]. But in order to analyze fluids flow in easier way, to use either one or two dimensional are recommended. Incompressible flow is applied to any situation where changes in the density of a particle are negligible. To recognize whether the flow is in incompressible flow or not are by identifying it Mach number. The Mach number for incompressible flow should be low (Ma<0.3)[1]. Mach number is the expression of the flow speed where Mach number is equal to speed of flow divides by speed of sound.

2.3 Boundary Layer Flow over Moving Surface

Process such as fabrication of adhesive tapes and production of plastic sheet is the important process that involves laminar flow of Non-Newtonian fluids such as polymer solution, blood, and liquid plastic above moving plate or belt. This type of flow sometimes referred to as Sakiadis flow[4]. The Sakiadis flow has opposite boundary layer profile from Blasius flow. By referring Figures 2.2 and 2.3 below, it can be seen the differences between these two types of flow. In Sakiadis flow, there was a slip condition. The meaning of slip condition is the fluid past the solid without contact. The velocity on the body at plate is equal to the flow velocity.

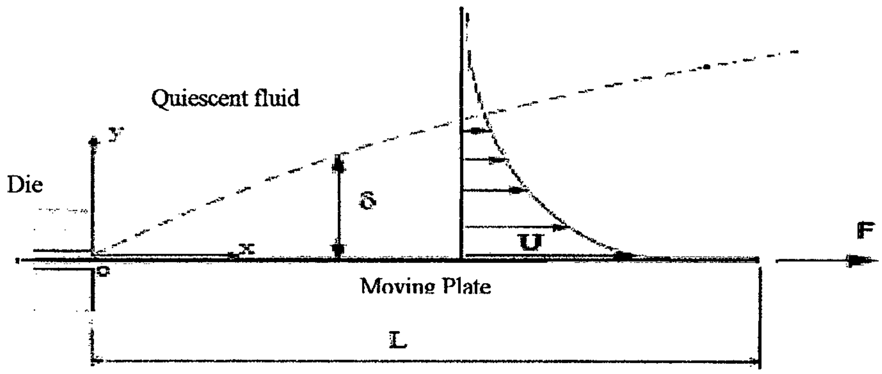

Figure 2.2: The boundary layer profile on moving flat plate (Sakiadis flow)

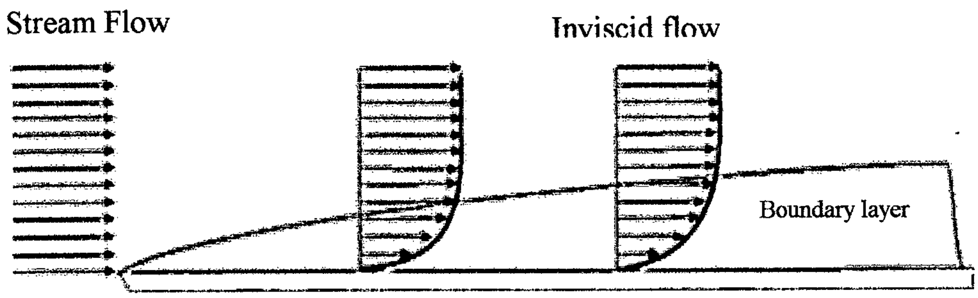

Figure 2.3: The boundary layer profile on a static flat plate (Blasius flow)

Since the effect of viscosity is to resist fluid motion, the velocity close to the solid surface continuously decreases towards upstream. In previous study, Sakiadis flow has been used to predict the force required to pull a plate steadily through a quiescent incompressible Newtonian fluid [4]. Newtonian fluid is a fluid for which the rate of deformation is proportional to the sheer stress such as water, air, gasoline and oils. Relying on boundary layer theory, Sakiadis flow succeeded in predicting this force with a high degree of accuracy.

2.4 Theoretical Background

The derivation on boundary layer equations must include the derivation on continuity equation, Cauchy's equation and Navier-stokes. Before starting the derivation of the equation, identify all terms that related in order to derive all the equations that involve.

2.4.1 Derivation on Continuity Equation

The derivation of continuity equation start from the conservation of mass principle which is the principle can be expressed as the net mass transfer to or from control volume during a time interval is equal to the net change increase or decrease in the total mass within the control volume(CV) during Δt where Δt is time interval.

(Total mass entering the CV during Δt)- (total mass leaving the CV during Δt) = (net change in mass within the CV during Δt)

$$m_{in} - m_{out} = \Delta m_{cv}$$

The rate form can be written as Equation (2.1)

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{cv}}{dt} \qquad (2.1)$$

where $\dot{m}_{in}$ and $\dot{m}_{out}$ are the total rates of mass flow into and out of the control volume, and $\frac{dm_{cv}}{dt}$ is the rate of change of mass within the volume boundaries. The mass within the control volume is $dm = \rho dV$ where ρ are density and dV are the differential control volume. The total mass within the control volume at any instant time t are:

$$m_{cv} = \int_{cv} \rho dv$$

So the rate of change of mass within the control volume:

$$\frac{dm_{cv}}{dt} = \int_{cv} \rho dv \qquad (2.2)$$

The net flow rate into or out of control volume through the entire control surface is obtained by integrating $\delta \dot{m}$ over the entire surface, Differential mass flowrate:

$$\delta m = \rho\left(\vec{V_n}\cdot\vec{n}\right)dA \tag{2.3}$$

V_n=normal velocity, dA=the flow area

Net mass flow rate

$$m_{net} = \int_{cs}\delta m = \int_{cs}\rho V_n dA = \int_{CS}\rho(\vec{V}\cdot\vec{n})dA \tag{2.4}$$

Substitute Equation (2.4) into the rate form equation $m_{in} - m_{out} = \frac{dm_{cv}}{dt}$ and the equation now can be write as:

$$\int_{cv}\rho dv + m_{net} = \int_{cs}\rho(\vec{V}\cdot\vec{n})dA = 0 \tag{2.5}$$

The divergence theorem is important in purpose to derive the continuity equation. Divergence theorem is a theorem that allow us to transform a volume integral of the divergence of a vector into an area integral over the surface that define the volume that can be write as below:

Divergence theorem $\int_V \vec{\nabla}\cdot\vec{G}\,dv = \oint_A \vec{G}\cdot\vec{n}\,dA$

By using this theorem, the Equation (2.5) can be rewrite as

$$\int_{cv}\frac{\partial p}{\partial t}dv + \int_{cv}\vec{\nabla}\cdot(\rho\vec{v})dv = 0 \tag{2.6}$$

And then simplify the equation

$$\int_{cv}\left[-\frac{\partial p}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) - \right]dv = 0 \tag{2.7}$$

To get the continuity equation, the item in bracket [ ] must equal to zero because the equation must hold any control volume regardless of it size and shape. So the continuity equation are:

$$\frac{\partial p}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{2.8}$$

But in this project, incompressible flow has been used, so the density can be eliminate from the equation because it not a function of time or space, so: $\vec{\nabla} \cdot (\vec{v}) = 0$
Continuity equation for incompressible flow :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.9}$$

2.4.2 Derivation on Cauchy Equation

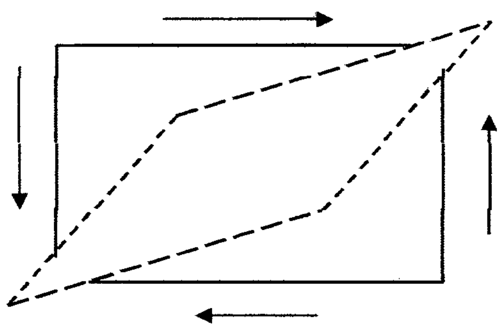

Figure 2.4: Fluid element deformation on shear strain

Shear strain rate can be understood by referring Figure 2.4. In the middle of the rectangle fluid element is the deformation after shear strain. Positive shear strain can be identifying when the initial angle 90 degree of the square fluid element decrease. When the angle 90 degree of the square fluid element increase, it known as negative shear strain [3].

Strain rate tensor in Cartesian coordinates.

$$\varepsilon = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{yx} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{zy} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}) & \frac{1}{2}(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial x}) \\ \frac{1}{2}(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}) & \frac{\partial v}{\partial y} & \frac{1}{2}(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}) \\ \frac{1}{2}(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}) & \frac{1}{2}(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}) & \frac{\partial w}{\partial z} \end{pmatrix}$$

Shear strain rate in Cartesian coordinates

$$\varepsilon_{XY} = \frac{1}{2}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})$$

$$\varepsilon_{zx} = \frac{1}{2}(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})$$

$$\varepsilon_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})$$

The acceleration of fluid element is $\vec{a} = \frac{D\vec{V}}{Dt}$ by definition of the material acceleration. The body force on the control volume in x-direction is

$$\sum F_{x,body} = \sum F_{x,gravity} \cong \rho g_x dxdydz \tag{2.10}$$

The tensors are quantities which have a magnitude and two associated directions. By referring to Figure 2.5, stress that acting has two associated directions which is at the plane on the stress acts and the direction of the of the stress.

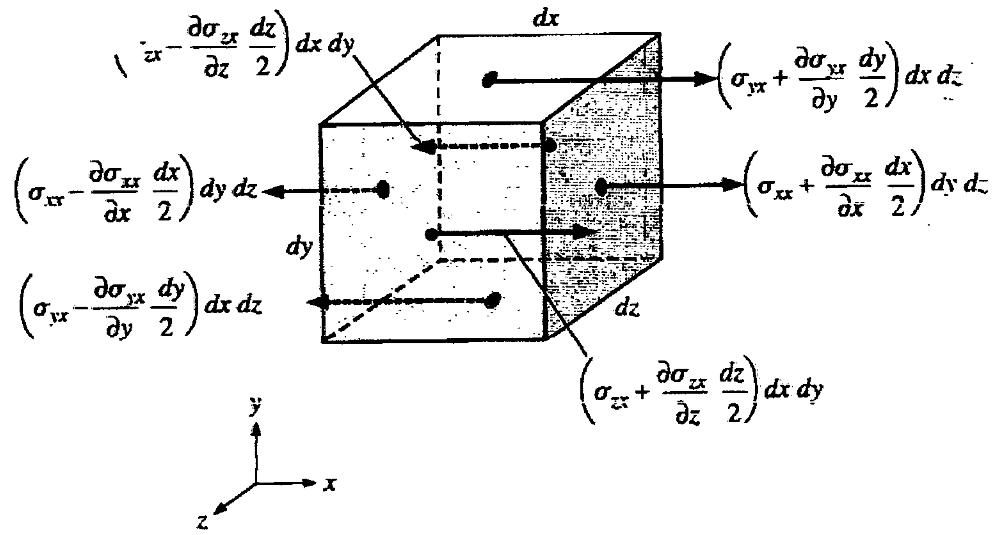

Figure 2.5: Force acting in x-direction due to stress tensor

The net surface force acting on the differential fluid element in x-direction are:

$$\sum F_{x,surface} \cong \left(\frac{\partial}{\partial_{xx}}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{yx} + \frac{\partial}{\partial z}\sigma_{zx} \right) dxdydz \tag{2.11}$$

By Newton's second law applied to a material element of fluid,

$$\Sigma \vec{F} = m\vec{a} = m\frac{D\vec{V}}{Dt} = \rho dxdydz \frac{D\vec{V}}{Dt} \tag{2.12}$$

The total force is equal to the body force Equation (2.10) plus the net surface force Equation (2.11) and the equation will divide by *dxdydz*. So the total force acting on the fluid element are:

$$\rho \frac{D\vec{V}}{Dt} = \rho g + \vec{\nabla} \cdot \sigma ij \qquad (2.13)$$

Equation (2.13) are known as Cauchy's equations. Cauchy's equation in x, y and z component are expressed as Equation (2.14) until (2.16) respectively.

x-component

$$\rho \frac{Du}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \qquad (2.14)$$

y-component

$$\rho \frac{Dv}{Dt} = \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \qquad (2.15)$$

z-component

$$\rho \frac{Dw}{Dt} = \rho g_z + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \qquad (2.16)$$

2.4.3 Derivation of the Navier-Stokes Equation

Viscous stress tensor for an incompressible Newtonian fluid with constant properties $\tau_{ij} = 2\mu \ \varepsilon_{ij}$

$$= \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}) & \mu(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}) \\ \mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}) & 2\mu\frac{\partial v}{\partial y} & \mu(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}) \\ \mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}) & \mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix} \tag{2.17}$$

When a fluid is at rest, the only stress acting at any surface of any fluid element is the local hydrostatic pressure, P. This pressure P always acts inward and normal to the surface.

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} \tag{2.18}$$

When a fluid is moving, pressure still act inwardly normal but viscous stress may also exist.

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{2.19}$$

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}) & \mu(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}) \\ \mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}) & 2\mu\frac{\partial v}{\partial y} & \mu(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}) \\ \mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}) & \mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix} \tag{2.20}$$

By substitutes Equation (2.17) into the Cauchy equation of x-component Equation (2.14) and the equations become:

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \tag{2.21}$$

As long as the velocities are smooth function of the x, y, z, the order of differentiation is irrelevant. So we can write or change the equation as below:

$$\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)=\mu\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right) \tag{2.22}$$

By substitutes this Equation (2.22) into Equation (2.21), the equation now became:

$$\rho\frac{Du}{Dt}=-\frac{\partial \rho}{\partial x}+\rho g_x+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial x}\frac{\partial w}{\partial z}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right] \tag{2.23}$$

The term in $\left[\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]$ is equal to zero because this equation was the continuity equation for incompressible flow. So the equation in x-component can be write as:

$$\rho\frac{Du}{Dt}=-\frac{\partial P}{\partial x}+\rho g_x+\mu\nabla^2 u \tag{2.24}$$

By using the same method of derivation Navier-stokes equation in x-component, derive equation for y and z-component

y-component

$$\rho\frac{Du}{Dt}=-\frac{\partial P}{\partial y}+\rho g_y+\mu\nabla^2 v \tag{2.25}$$

z-component

$$\rho\frac{Du}{Dt}=-\frac{\partial P}{\partial z}+\rho g_z+\mu\nabla^2 w \tag{2.26}$$