FIFTH-STAGE STOCHASTIC RUNGE-KUTTA METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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DOCTOR OF PHILOSOPHY

UNIVERSITI MALAYSIA PAHANG

SUPERVISOR'S DECLARATION

We hereby declare that We have checked this thesis and in our opinion, this thesis is adequate in terms of scope and quality for the award of the degree of Doctor of Philosophy.

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STUDENT'S DECLARATION

I hereby declare that the work in this thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at Universiti Malaysia Pahang or any other institutions.

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Thesis submitted in fulfilment of the requirements
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In dedication to:

My Parents,
Encik Ariffin Ahmad and Puan Noor Faizah Mohammed Yunus,
Strong and gentle souls, who have taught me to trust in Allah and to earn an honest living.

My Beloved Husband,
Dr. Abdul Rahman Mohd Kasim,
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My Siblings,
Nasrul, Adibah, Arifah and Adilah,
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LIST OF SYMBOLS

h	Step size
r	Time delay
t	Time
T	Terminal time
X	Random variable or stochastic process
R	Remainder
$\mathbb{R}$	Real number
$\mathbb{T}$	Rooted Tree
$\mathscr{B}$	*Borel* sets
$\mathscr{F}$	σ–field or σ–algebra
$\mathscr{R}$	Stability function
$\emptyset$	Empty set
Ω	Sample space
σ	Diffusion coefficient
$\subseteq$	Subset or is included in
$\subset$	Subset
$\in$	Element
$\cap$	Intersection
$\cup$	Union
$(\cdot)^T$	Transpose
f_α	Partial derivative with respect to α
f'	First derivative
f''	Second derivative
$f^{(k)}$	k–th derivative
$\int \circ dW(t)$	Stratonovich Integral
x_{max}	Maximum cell concentration
μ_{max}	Maximum specific growth rate
YE1	Control medium
YE2	Medium of yeast and NH_4Cl
YE3	Medium of yeast and NH_4NO_3
$\mathrm{E}(X)$	Expected value of X
$W(t)$	Wiener process
$\mathrm{Var}(X)$	Variance of X

LIST OF ABBREVIATIONS

MS	Mean–square
RK	Runge–Kutta
ODEs	Ordinary differential equations
SDEs	Stochastic differential equations
SRK	Stochastic Runge–Kutta
SRK2	2–stage stochastic Runge–Kutta
SRK4	4–stage stochastic Runge–Kutta
SRK2.0	Specific stochastic Runge–Kutta
RMSE	Root mean square error

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ABSTRAK

Kebanyakan sistem fizikal di sekeliling kita tertakluk kepada faktor-faktor yang tidak terkawal. Oleh itu, persamaan pembezaan stokastik (SDE) diperlukan sebagai model kepada sistem fizikal tersebut. Walau bagaimanapun, penyelesaian tepat kepada model SDE sukar didapati. Di dalam kes ini, kaedah berangka menyediakan kaedah alternatif untuk menyelesaikan sistem tersebut. Pembangunan kaedah berangka untuk SDE masih belum lengkap. Sebaliknya, kaedah berangka untuk persamaan deterministik yang setara dengannya telah dibangunkan dengan pesat. Kekurangan relatif kaedah berangka dalam SDE adalah disebabkan oleh kesukaran untuk menyelesaikan kamiran berganda stokastik berperingkat tinggi. Kamiran stokastik menyediakan maklumat proses Wiener yang kemudiannya menyumbang kepada nilai peringkat bagi kaedah tersebut. Motivasi daripada pembangunan kaedah Runge-Kutta peringkat tinggi dalam menyelesaikan persamaan pembezaan biasa, kajian ini bertujuan untuk membangunkan kaedah stokastik Runge-Kutta peringkat kelima yang baru untuk persamaan pembezaan stokastik dengan peringkat 2.0. Pemerolehan kaedah terbitan bebas ini adalah berasaskan kepada kembangan siri Taylor. Kembangan siri Taylor untuk penyelesaian siri Taylor dan berangka sehingga peringkat penumpuan 2.0 telah diperluaskan. Analisis syarat-syarat peringkat untuk SRK5 telah dilakukan dengan penilaian ke atas ralat pangkasan setempat dalam bentuk min kuasa dua dengan menggunakan MAPLE. Perbezaan antara penyelesaian siri Taylor dan penyelesaian berangka telah dinilai. Untuk menganalisis syarat-syarat peringkat, ralat pangkasan setempat antara kedua-dua penyelesaian telah diminimumkan. Semua persamaan yang timbul dalam analisis syarat-syarat peringkat telah diselesaikan secara serentak dengan menggunakan MATLAB, dan tiga skim SRK5 yang baru dibangunkan telah dibentangkan. Seterusnya, analisis kestabilan min kuasa dua telah dilakukan ke atas skim SRK5 untuk memastikan kecekapan skim berangka yang baru dibangunkan. Fungsi kestabilan bagi setiap skim telah diperoleh dan perubahan pemboleh ubah telah digunakan untuk memplot rantau kestabilan. Rantau kestabilan telah diplot pada satah uv untuk memvisualisasikan sifat kestabilan setiap skim. Di samping itu, eksperimen berangka ringkas telah dilakukan untuk memeriksa sifat kestabilan. Untuk mengesahkan kecekapan skim berangka yang baru dibangunkan, semua skim masing-masing telah digunakan untuk menyelesaikan kedua-dua model stokastik linear dan bukan linear dengan menggunakan C++. Pencapaian skim SRK5 dalam menyelesaikan SDE linear telah diukur dengan membandingkan ralat punca min-kuasa dua dan ralat sejagat yang diperoleh dengan menyelesaikan SDE linear melalui skim SRK5, SRK2.0, SRK4, Milstein dan kaedah Euler-Maruyama. Selain itu, tiga model penapaian yang berbeza telah diselesaikan dengan menggunakan skim SRK5, SRK2.0 dan SRK4. Ralat yang terhasil telah dibandingkan. SRK5 telah dibuktikan sebagai alat yang lebih berkesan untuk penghampiran penyelesaian berangka bagi SDE.

ABSTRACT

Most of the physical systems around us are subjected to uncontrollable factors. Hence, models for these systems are required via stochastic differential equations (SDEs). However, it is often difficult to find analytical solutions of SDEs. In such a case, a numerical method provides an alternative way to solve problems with such systems. The development of numerical methods for SDEs is far from complete. Conversely, numerical methods for their deterministic counterparts are well-developed. The relative paucity of numerical methods in SDEs is due to the complexity of approximating high-order multiple stochastic integrals. A stochastic integral provides information of the Wiener process, which then contributes to the order of the methods. Motivated by the development of high-order Runge-Kutta methods for solving ordinary differential equations (ODEs), this research was aimed to develop a new fifth-stage stochastic Runge-Kutta (SRK5) method for SDEs with a strong order of 2.0. The derivation of this derivative-free method was based on the stochastic Taylor series expansion. The Taylor series expansion for both Taylor series and numerical solutions up to 2.0 order of convergence have been expanded. The analysis of the order conditions for the SRK5 was performed by evaluating the local truncation error in terms of the mean square in MAPLE. The difference between Taylor series solution and numerical solution was evaluated. In order to analyze the order conditions, the local truncation error between both solutions was minimized. All equations arise in order conditions analysis have been solved simultaneously by using MATLAB, and three newly developed SRK5 schemes were presented. A mean-square stability analysis was then performed on the SRK5 scheme in order to ensure the efficiency of the newly-developed numerical scheme. The stability function for each scheme was derived and the change of variables have been applied for stability region plotting purposed. Stability region have been plotted on the *uv*-plane to visualize the stability property of each scheme. In addition, the simple numerical experiments have been performed to check on the stability property. In order to validate the efficiency of the newly develop numerical schemes, all schemes have been used to solve both linear and non-linear stochastic models respectively in C++. The performances of SRK5 schemes in solving linear SDEs have been measured by comparing the root mean-square error and the global error obtained by solving linear SDE via SRK5 schemes, SRK2.0, SRK4, Milstein and Euler-Maruyama methods. Besides, three different models of fermentation process were solved by using SRK5 schemes, SRK2.0 and SRK4. The errors obtained have been compared. The SRK5 is proved to be a more efficient tool for the numerical approximation of solutions to SDEs.

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