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CUBATURE FORMULA FOR APPROXIMATE CALCULATION OF INTEGRALS OF TWO-DIMENSIONAL IRREGULAR HIGHLY OSCILLATING FUNCTIONS

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The paper presents a new method for the calculation of integrals of two-dimensional irregular highly oscillatory functions for the case when information about functions is given on sets of lines. Estimation of the proposed method is done for the class of differentiable functions.

Keywords: highly oscillating functions, cubature formula, interlineation of functions, two-dimensional functions.

1. Introduction

The methods of digital signal and image processing are widely used in scientific and technical areas nowadays. Research in astronomy, radiology, computed tomography, holography, radars etc. require improvement of existing and development of new mathematical methods, especially for new types of input information. An important case is when an input information about considered function is given as the set of the function' traces on planes or on lines.

The paper proposes a new effective mathematical approach, build on the base of the theory of interlineation and interflatation of functions [1; 2]. The paper demonstrates, how application of the operators of interlineation and interflatation of functions results in the method of numerical integration of highly oscillating functions of many variables.

Oscillatory integrals have various applications, but existing methods for their calculation have known limitations. There is a number of studies, where problems of highly-oscillating functions are discussed for the regular case. The most dated paper was published by Filon in 1928 [3]; let us also mention Luke

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(1954) [4], Flinn (1960) [5], Zamfirescu (1963) [6], Bakhvalov and Vasileva (1968) [7]. Good review and analysis of existing methods were given by A. Iserles [8] and V. Milovanovic [9]. In [10] Eckhoff shown how the traditional Fourier method can be used to develop the numerical high order methods for calculating derivatives and integrals. The Eckhoff's method for univariate and bivariate functions is described in details in [11].

One and two-dimensional methods for the computation of integrals of highly oscillating functions in the irregular case were also discussed in various papers. A collocation procedure for the effective integration of rapidly oscillatory functions is presented in [12]. This method was extended to two-dimensional integration, and numerical results showed its efficiency for rapid irregular oscillations. In [13; 14] the methods for the computation of highly oscillatory integrals (for one and two-dimensional irregular cases) are explored. Outcomes are two families of the methods, one is based on a truncation of the asymptotic series and the other Filon's idea [3]. These papers came with numerical experiments that demonstrate the potential of proposed methods. In [15] the new methods for numerical approximation of the integral of irregular highly oscillatory functions were derived. On the base of the method developed by Levin, Olver proposed a new approach that uses the same type of input information and has the same asymptotic order, as the Filon's method, but without requiring the computation of moments. In [16] a calculation of canonical oscillatory integrals was discussed, where the irregular oscillatory integrals were transformed into canonical ones with respect to the stationary phase points. Two calculation methods for the canonical oscillatory integrals were then proposed: one is the Clenshaw-Curtis method and the other is the improved method of Levin. In [17], an adaptation of the Filon method for the calculation of highly oscillatory integrals with or without stationary points was developed. The main feature of this method is that it optimizes the choice of interpolation points between different oscillatory regimes. In [18] the problems of calculating integrals of an irregular highly oscillatory function in MATLAB were discussed.

Problems of computing rapidly oscillating integrals of differentiable functions using various information operators were considered by V.K. Zadiraka in [19]. In [20] O.M. Lytvyn and O.P. Nechuyviter proposed the formulas for the evaluating two dimensional Fourier coefficients using spline-interlineation. These formulas were constructed for two cases: (1) when input information about a function is given on the set of traces of the function on lines and (2) as a set of values of the function at the points. The main advantages of the proposed methods are the high accuracy of approximation and less amount of the input data needed. This paper discusses a computation of the Fourier coefficients of a function of two variables $f(x,y)$ by classic cubature formula and by cubature formula, which uses piecewise spline interlineation for the case, when information about $f(x,y)$ is given as the set of values of the function at the grid points.

Using interflatation cubature formulas for calculation of 3D Fourier coefficients for the class of differentiable functions were presented in [21]. Information about the function was given by the traces on the system of mutually perpendicular planes. It was proven that the approximation error of the cubature formulas can be evaluated by the estimation of the error of corresponding quadrature formulas. Paper [22] considered the cubature formula for the approximate calculation of triple integrals of rapidly oscillating functions by using Lagrange polynomial interlineation of functions with the optimal choice of the nodal planes for the approximation of the non-oscillating set. The error of the cubature formula was estimated for the class of differentiable functions, defined in a unit cube.

This paper proposes a new effective method for the calculation of two-dimensional integrals from highly oscillating functions for the more general case, when information about the functions is given on the set of lines. It develops a cubature formula for numerical computation of two-dimensional Fourier coefficients for the case, when information about $f(x,y)$ is the set of traces of the function on lines. The paper also compares the results of calculation of Fourier coefficients of the function of two variables by classic cubature formula and by cubature formula using interlineation in the case when information about $f(x,y)$ is a set of values of the function at the grid points.

2. Cubature formula for calculating a two-dimensional integral of the irregular highly oscillating function, for the case when input data is the set of traces of the function on lines

Let us consider $H^{2,r}\left(M,\widetilde{M}\right), r \geq 0$ – the class of functions, which defined in the domain $G=[0,1]^2$ and

$$\left|f^{(r,0)}(x,y)\right| \leq M,\ \left|f^{(0,r)}(x,y)\right| \leq M,\ r \neq 0,\ \left|f^{(r,r)}(x,y)\right| \leq \widetilde{M},\quad r \geq 0,$$

$$f^{(r,0)}(x,y)=\frac{\partial^r f}{\partial x^r},\ f^{(0,r)}(x,y)=\frac{\partial^r f}{\partial y^r},\ f^{(r,r)}(x,y)=\frac{\partial^{2r} f}{\partial x^r \partial y^r},$$

$M, \widetilde{M}$ – are the constants that limit the corresponding partial derivatives.

Definition 2.1. Under the traces of the function $f(x,y)$ on the lines $x_k = k\Delta_1 - \Delta_1/2,\quad y_j = j\Delta_1 - \Delta_1/2,\ k,j=\overline{1,\ell_1}$, $\Delta_1 = 1/\ell_1$ we understand the function of one variable $f(x_k,y),\ 0 \leq y \leq 1$ and $f(x,y_j),\ 0 \leq x \leq 1$.

Definition 2.2. Under the traces of function $g(x,y)$ on the lines $x_p = p\Delta_2 - \Delta_2/2$, $y_s = s\Delta_2 - \Delta_2/2$, $p,s=\overline{1,\ell_2}$, $\Delta_2 = 1/\ell_2$ we understand a function of one variable $g(x_p,y),\ 0\le y\le 1$ or $g(x,y_s),\ 0\le x\le 1$.

A two-dimensional integral of a highly oscillating function is defined as

$$I^2(\omega)=\int_0^1\int_0^1 f(x,y)e^{i\omega g(x,y)}\,dxdy$$

for $f(x,y),\ g(x,y)\in H^{2,1}\left(M,\widetilde{M}\right)$.

Let

$$h1_{0k}(x)=\begin{cases}1, & x\in X1_k,\\ 0, & x\notin X1_k,\end{cases}\quad k=\overline{1,\ell_1},\qquad H1_{0j}(y)=\begin{cases}1, & y\in Y1_j,\\ 0, & y\notin Y1_j,\end{cases}\quad j=\overline{1,\ell_1},$$

$$X1_k=\left[x_{k-1/2},x_{k+1/2}\right],\ Y1_j=\left[y_{j-1/2},y_{j+1/2}\right],$$

$$x_k=k\Delta_1-\Delta_1/2,\qquad x_{k-1/2}=(k-1)\Delta_1,\ \ x_{k+1/2}=k\Delta_1,$$

$$y_j=j\Delta_1-\Delta_1/2,\qquad y_{j-1/2}=(j-1)\Delta_1,\ \ y_{j+1/2}=j\Delta_1,$$

$$k,j=\overline{1,\ell_1},\ \Delta_1=1/\ell_1,$$

$$h2_{0p}(x)=\begin{cases}1, & x\in X2_p,\\ 0, & x\notin X2_p,\end{cases}\quad p=\overline{1,\ell_2},\qquad H2_{0j}(y)=\begin{cases}1, & y\in Y2_s,\\ 0, & y\notin Y2_s,\end{cases}\quad s=\overline{1,\ell_2},$$

$$X2_p=\left[x_{p-1/2},x_{p+1/2}\right],\quad Y2_s=\left[y_{s-1/2},y_{s+1/2}\right],$$

$$x_p=p\Delta_2-\Delta_2/2,\qquad y_s=s\Delta_2-\Delta_2/2,\ \ p,s=\overline{1,\ell_2},\ \Delta_2=1/\ell_2.$$

Let us define two operators of the interlineation. The first is

$$Jf(x,y)=\sum_{k=1}^{\ell_1} f(x_k,y)h1_{0k}(x)+\sum_{j=1}^{\ell_1} f(x,y_j)H1_{0j}(y)-$$
$$\sum_{k=1}^{\ell_1}\sum_{j=1}^{\ell_1} f(x_k,y_j)h1_{0k}(x)H1_{0j}(y) \tag{1}$$

and the second is given by

$$Og(x,y)=\sum_{p=1}^{\ell_2} g(x_p,y)h2_{0p}(x)+\sum_{s=1}^{\ell_2} g(x,y_s)H2_{0s}(y)-$$
$$\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2} g(x_p,y_s)h2_{0p}(x)H2_{0s}(y). \tag{2}$$

The following cubature formula

$$\Phi^2(\omega) = \int_0^1 \int_0^1 Jf(x,y)\, \mathrm{e}^{i\omega Og(\mathrm{x,y})}\, dxdy \qquad (3)$$

is proposed for the numerical calculation of

$$I^2(\omega) = \int_0^1 \int_0^1 f(x,y)\, \mathrm{e}^{i\omega g(\mathrm{x,y})}\, dxdy\,. \qquad (4)$$

Theorem 2.1. Let us suppose that $f(x,y),\ g(\mathrm{x,y}) \in H^{2,1}\left(M,\widetilde{M}\right)$. Let the functions $f(x,y),\ g(\mathrm{x,y})$ be defined by the $N = 2\ell_1 + 2\ell_2$ traces $f\left(x_k, y\right)$, $f\left(x, y_j\right),\ k,j = \overline{1,\ell_1}$ and $\mathrm{g}(x_p, y),\ \mathrm{g}(x, y_s),\ p,s = \overline{1,\ell_2}$, on the systems of perpendicular lines in the domain $G = [0,1]^2$. It is true that

$$\rho\left(I^2(\omega), \Phi^2(\omega)\right) \triangleq$$

$$= \left| \int_0^1 \int_0^1 f(x,y)\, \mathrm{e}^{i\omega g(\mathrm{x,y})}\, dxdy - \int_0^1 \int_0^1 Jf(x,y)\, \mathrm{e}^{i\omega Og(\mathrm{x,y})}\, dxdy \right| \le$$

$$\frac{\widetilde{M}}{16}\frac{1}{\ell_1^{\,2}} + \widetilde{M} \min\left(2; \frac{\widetilde{M}\omega}{16}\frac{1}{\ell_2^{\,2}}\right).$$

Proof. It is important to note that

$$f(x,y) - Jf\left(x,y\right) = \int_{x_k}^{x} \int_{y_j}^{y} f^{(1,1)}\left(\xi,\eta\right) d\xi d\eta$$

and

$$\mathrm{g}(x,y) - Of\left(x,y\right) = \int_{x_s}^{x} \int_{y_p}^{y} g^{(1,1)}\left(\xi,\eta\right) d\xi d\eta\,.$$

The integral $I^2(\omega)$ can be rewritten as

$$I^2(\omega) = \int_0^1 \int_0^1 Jf(x,y)\, \mathrm{e}^{i\omega Og(\mathrm{x,y})}\, dxdy +$$

$$+\int_0^1 \int_0^1 \left[f(x,y) - Jf(x,y)\right] \mathrm{e}^{i\omega Og(\mathrm{x,y})}\, dxdy + \int_0^1 \int_0^1 f(x,y)\left[\mathrm{e}^{i\omega g(\mathrm{x,y})} - \mathrm{e}^{i\omega Og(\mathrm{x,y})}\right] dxdy\,.$$

Hence, it is sufficient to show that

$$\rho\left(I^2(\omega),\Phi^2(\omega)\right)$$

$$\le \int_0^1\int_0^1 \left|f(x,y)-Jf(x,y)\right|dxdy+\int_0^1\int_0^1 \left|f(x,y)\right|\left|\mathrm{e}^{i\omega g(\mathrm{x,y})}-\mathrm{e}^{i\omega Og(\mathrm{x,y})}\right|dxdy\,.$$

Let us use the fact that

$$\mathrm{e}^{i\omega g(\mathrm{x,y})}-\mathrm{e}^{i\omega Og(\mathrm{x,y})}=2i\sin\frac{\omega\,\mathrm{g(x,y)}-\omega O\,\mathrm{g(x,y)}}{2}\mathrm{e}^{i\frac{\omega}{2}\left(g(\mathrm{x,y})+Og(x,y)\right)}.$$

Therefore

$$\rho\left(I^2(\omega),\Phi^2(\omega)\right)\le\int_0^1\int_0^1\left|f(x,y)-Jf(x,y)\right|dxdy+$$

$$+\int_0^1\int_0^1\left|f(x,y)\right|\left|2i\sin\frac{\omega\,\mathrm{g(x,y)}-\omega O\,\mathrm{g(x,y)}}{2}\mathrm{e}^{i\frac{\omega}{2}\left(g(\mathrm{x,y})+Og(x,y)\right)}\right|dxdy\le$$

$$\le\sum_{k=1}^{\ell_1}\sum_{j=1}^{\ell_1}\int_{x_{k-\frac12}}^{x_{k+\frac12}}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\left|\int_{x_k}^{x}\int_{y_j}^{y}f^{(1,1)}(\xi,\eta)\,d\xi d\eta\right|dxdy+$$

$$2\widetilde{M}\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2}\int_{x_{p-\frac12}}^{x_{p+\frac12}}\int_{y_{s-\frac12}}^{y_{s+\frac12}}\left|\sin\frac{\omega(g(x,y)-Og(x,y))}{2}\right|dxdy\le$$

$$\le\widetilde{M}\sum_{k=1}^{\ell_1}\sum_{j=1}^{\ell_1}\int_{x_{k-\frac12}}^{x_{k+\frac12}}\left|x-x_k\right|dx\int_{y_{j-\frac12}}^{y_{j+\frac12}}\left|y-y_j\right|dy+$$

$$2\widetilde{M}\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2}\int_{x_{p-\frac12}}^{x_{p+\frac12}}\int_{y_{s-\frac12}}^{y_{s+\frac12}}\min\left(1;\frac{\omega\left|g(x,y)-Og(x,y)\right|}{2}\right)dxdy\le$$

$$\le \widetilde{M}\sum_{k=1}^{\ell_1}\sum_{j=1}^{\ell_1}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\left|x-x_k\right|dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left|y-y_j\right|dy+$$

$$2\widetilde{M}\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2}\int_{x_{p-\frac{1}{2}}}^{x_{p+\frac{1}{2}}}\int_{y_{s-\frac{1}{2}}}^{y_{s+\frac{1}{2}}}\min\left(1;\frac{\omega}{2}\left|\int_{x_p}^{x}\int_{y_s}^{y}g^{(1,1)}(\xi,\eta)d\xi d\eta\right|\right)dxdy\le$$

$$\le \widetilde{M}\ell_1^{\,2}\frac{\Delta_1^{\,2}}{4}\frac{\Delta_1^{\,2}}{4}+$$

$$2\widetilde{M}\min\left(\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2}\int_{x_{p-\frac{1}{2}}}^{x_{p+\frac{1}{2}}}\int_{y_{s-\frac{1}{2}}}^{y_{s+\frac{1}{2}}}dxdy,\frac{\widetilde{M}\omega}{2}\sum_{p=1}^{\ell_2}\sum_{s=1}^{\ell_2}\int_{x_{p-\frac{1}{2}}}^{x_{p+\frac{1}{2}}}\int_{y_{s-\frac{1}{2}}}^{y_{s+\frac{1}{2}}}\left|x-x_p\right|\left|y-y_s\right|dxdy\right)=$$

$$=\frac{\widetilde{M}}{16}\Delta_1^{\,2}+2\widetilde{M}\min\left(\ell_2^{\,2}\Delta_2^{\,2},\frac{\widetilde{M}\omega}{2}\ell_2^{\,2}\frac{\Delta_2^{\,2}}{4}\frac{\Delta_2^{\,2}}{4}\right)=$$

$$=\frac{\widetilde{M}}{16}\Delta_1^{\,2}+\widetilde{M}\min\left(2;\frac{\widetilde{M}\omega}{16}\Delta_2^{\,2}\right)=\frac{\widetilde{M}}{16}\frac{1}{\ell_1^{\,2}}+\widetilde{M}\min\left(2;\frac{\widetilde{M}\omega}{16}\frac{1}{\ell_2^{\,2}}\right).$$

3. Cubature formulas for calculating two-dimensional Fourier coefficients for various types of data

Let us suppose that in the formula

$$I_s^2(\omega)=\int_0^1\int_0^1 f(x,y)\sin\left(\omega g(\mathrm{x},\mathrm{y})\right)dxdy$$

we have $\omega g(\mathrm{x},\mathrm{y})=2\pi mx+2\pi ny$ and $\ell_1=\ell_2=\ell$. Then the cubature formula

$$\Phi_1^2\left(m,n\right)=\sum_{k=1}^{\ell}\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mxdx\int_0^1 f(x_k,y)\sin 2\pi nydy+$$

$$+\sum_{j=1}^{\ell}\int_0^1 f(x,y_j)\sin 2\pi mxdx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi nydy-$$

$$-\sum_{k=1}^{\ell}\sum_{j=1}^{\ell} f(x_k,y_j)\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mx dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi ny dy,$$

$$x_k = k\Delta-\frac{\Delta}{2},\ y_j = j\Delta-\frac{\Delta}{2},\ k,j=\overline{1,\ell},\ \Delta=\frac{1}{\ell}$$

is used for numerical calculation of two-dimensional Fourier coefficients

$$I_1^2(m,n)=\int_0^1\int_0^1 f(x,y)\sin 2\pi mx \sin 2\pi ny dx dy$$

in the case when information about $f(x,y)$ is the set of traces of the function on lines [23].

Theorem 3.1. [23, p. 335] Suppose that $f(x,y)\in H^{2,1}\left(M,\widetilde{M}\right)$. Let the function $f(x,y)$ be defined by the $N=2\ell$ traces $f\left(x_k,y\right)$, $f\left(x,y_j\right)$, $k,j=\overline{1,\ell}$ on the system of perpendicular lines in the domain $G=\left[0,1\right]^2$. It is true that

$$\rho\left(I_1^2(m,n),\Phi_1^2(m,n)\right)\le\frac{\widetilde{M}}{16\ell^2}.$$

In [23] there were proposed the cubature formulas for calculating two-dimensional Fourier coefficients for the case, when information about $f(x,y)$ is a set of values of the function at the grid points

$$\widetilde{\Phi}_1^2(m,n)=\sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^2} f(x_k,\tilde{y}_{\tilde{j}})\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mx dx\int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}}\sin 2\pi ny dy+$$

$$+\sum_{j=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^2} f(\tilde{x}_{\tilde{k}},y_j)\int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}}\sin 2\pi mx dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi ny dy-$$

$$-\sum_{k=1}^{\ell}\sum_{j=1}^{\ell} f(x_k,y_j)\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}}\sin 2\pi mx dx\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\sin 2\pi ny dy\ ,$$

$$x_k = k\Delta-\frac{\Delta}{2},\ y_j = j\Delta-\frac{\Delta}{2},\ \ k,j=\overline{1,\ell},\ \ \Delta=\frac{1}{\ell},$$

$$\tilde{x}_{\tilde{k}} = \tilde{k}\Delta_1 - \frac{\Delta_1}{2},\ \tilde{y}_{\tilde{j}} = \tilde{j}\Delta_1 - \frac{\Delta_1}{2},\ \tilde{k},\tilde{j} = \overline{1,\ell^2},\quad \Delta_1 = \frac{1}{\ell^2}.$$

The advantages of proposed formula are the high accuracy of approximation and the possibility to decrease the size of input data about function, need for its computation.

Theorem 3.2. [24, p. 45] Suppose that $f(x,y) \in H^{2,1}\left(M,\widetilde{M}\right)$. Let the function $f(x,y)$ be defined by $f(x_k,\tilde{y}_{\tilde{j}})$, $f(\tilde{x}_{\tilde{k}},y_j)$, $k,j=\overline{1,\ell}$, $\tilde{k},\tilde{j}=\overline{1,\ell^2}$ knots in the domain $G=[0,1]^2$. It is true that $\rho\left(I_1^2(m,n),\widetilde{\Phi}_1^2(m,n)\right) \le \frac{M}{2\ell^2} + \frac{\widetilde{M}}{16\ell^2} = O\left(\frac{1}{\ell^2}\right)$.

4. Results and discussion

Let us prove the theorem 2.1. Let us calculate

$$I_s^2(\omega) = \int_0^1\int_0^1 f(x,y)\sin\left(\omega g(\mathrm{x},\mathrm{y})\right)dxdy,$$

by the cubature formula

$$\Phi_s^2(\omega) = \int_0^1\int_0^1 Jf(x,y)\sin\left(\omega Og(\mathrm{x},\mathrm{y})\right)dxdy,$$

where Jf, Og are given by (1) and (2) correspondingly.

When $f\left(x,y\right)=\sin(x+y)$, $g\left(x,y\right)=\cos(x+y)$, $\omega=2\pi$ and $\omega=5\pi$. For $\omega=2\pi$, $\omega=5\pi$ exact values were calculated in MathCad system of version 15.0:

$$I_s^2(2\pi)=0.062699216073162,\ I_s^2(5\pi)=0.022780463640219.$$

Let us denote the computing error $\varepsilon_{ex} = \left|I_s^2(\omega)-\Phi_s^2(\omega)\right|$. It is clear that $\varepsilon_{ex}=\varepsilon_{ex}\left(\ell_1,\ell_2\right)$. Let's show that $\varepsilon_{ex}\le\varepsilon_{th}$, where $\varepsilon_{th} = \frac{\widetilde{M}}{16}\frac{1}{{\ell_1}^2} + \widetilde{M}\min\left(2;\frac{\widetilde{M}\omega}{16}\frac{1}{{\ell_2}^2}\right)$ for various ℓ_1,ℓ_2. In our case, for $f\left(x,y\right)=\sin(x+y)$, $g\left(x,y\right)=\cos(x+y)$ we have $M=\widetilde{M}=1$ and $\varepsilon_{th} = \frac{1}{16{\ell_1}^2} + \min\left(2;\frac{\omega}{16{\ell_2}^2}\right)$.

Table 1 presents the results of computing $I_s^2(\omega)$ by cubature formula $\Phi_s^2(\omega)$ for $\omega = 2\pi$, $\omega = 5\pi$ and values of ε_{th} and ε_{ex} when ℓ_1, ℓ_2 are changed. The numerical results show that $\varepsilon_{ex} \le \varepsilon_{th}$.

Table 1

Computation of $I_s^2(\omega)$ by cubature formula $\Phi_s^2(\omega)$

ω	ℓ_1	ℓ_2	$\Phi_s^2(\omega)$	ε_{ex}	ε_{th}
2π	4	4	0.062432583948326	$2.6 \cdot 10^{-4}$	$2.8 \cdot 10^{-2}$
2π	7	7	0.062683978467995	$1.5 \cdot 10^{-5}$	$9.2 \cdot 10^{-3}$
5π	6	6	0.022786668787906	$6.2 \cdot 10^{-6}$	$2.9 \cdot 10^{-2}$
5π	10	4	0.022808425368659	$2.7 \cdot 10^{-5}$	$6.1 \cdot 10^{-2}$
5π	10	10	0.02277048162594	$9.9 \cdot 10^{-6}$	$1.04 \cdot 10^{-2}$

The second example compares cubature formula $\widetilde{\Phi}_1^2(m,n)$ with the well-known formula [23, p. 254].

$$\widehat{\Phi}_1^2(m,n) = \sum_{k=1}^{L}\sum_{j=1}^{L} f(x_k, y_j) \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \sin 2\pi mx dx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \sin 2\pi ny dy.$$

It is necessary to note [23, p. 255] that $\rho\left(I_1^2(m,n), \widehat{\Phi}_1^2(m,n) \right) \le O\left(\frac{1}{L} \right)$.

To have the same order of approximation as in theorem 3.2 let us suppose $L = \ell^2$. In this case $\rho\left(I_1^2(m,n), \widehat{\Phi}_1^2(m,n) \right) \le O\left(\frac{1}{\ell^2} \right)$. It is easy to see that formula

$$\widetilde{\Phi}_1^2(m,n) = \sum_{k=1}^{\ell}\sum_{\tilde{j}=1}^{\ell^2} f(x_k, \tilde{y}_{\tilde{j}}) \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \sin 2\pi mx dx \int_{\tilde{y}_{\tilde{j}-\frac{1}{2}}}^{\tilde{y}_{\tilde{j}+\frac{1}{2}}} \sin 2\pi ny dy +$$

$$+\sum_{j=1}^{\ell}\sum_{\tilde{k}=1}^{\ell^2} f(\tilde{x}_{\tilde{k}}, y_j) \int_{\tilde{x}_{\tilde{k}-\frac{1}{2}}}^{\tilde{x}_{\tilde{k}+\frac{1}{2}}} \sin 2\pi mx dx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \sin 2\pi ny dy -$$

$$-\sum_{k=1}^{\ell}\sum_{j=1}^{\ell} f(x_k, y_j) \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \sin 2\pi mx dx \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \sin 2\pi ny dy \ ,$$

$$x_k = k\Delta - \frac{\Delta}{2},\ y_j = j\Delta - \frac{\Delta}{2},\ \ k, j = \overline{1, \ell},\quad \Delta = \frac{1}{\ell},$$

$$\tilde{x}_{\tilde{k}} = \tilde{k}\Delta_1 - \frac{\Delta_1}{2},\ \tilde{y}_{\tilde{j}} = \tilde{j}\Delta_1 - \frac{\Delta_1}{2},\ \tilde{k}, \tilde{j} = \overline{1, \ell^2},\quad \Delta_1 = \frac{1}{\ell^2}$$

uses less number of numeric data - values of the function at the grid points.

Next example shows the comparative analysis of two formulas $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{\Phi}_1^2(m,n)$ for the function $f(x, y) = \sin(x + y)$. Let us analyze the difference between such characteristics:

- the number of values of the function at the points Q for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{Q}$ for $\widehat{\Phi}_1^2(m,n)$;
- the time spent T for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{T}$ for $\widehat{\Phi}_1^2(m,n)$;
- the size of the memory P for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{P}$ for $\widehat{\Phi}_1^2(m,n)$ for data computing.

Using Wolfram Mathematica 8 was calculated

$$I_1^2(m,n) = \int_0^1 \int_0^1 f(x, y) \sin 2\pi mx \sin 2\pi ny dx dy$$

by formula $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{\Phi}_1^2(m,n)$. We denote $\varepsilon_1 = \varepsilon_1(m,n) = \left| I_1^2(m,n) - \widetilde{\Phi}_1^2(m,n) \right|$ and $\varepsilon_2 = \varepsilon_2(m,n) = \left| I_1^2(m,n) - \widehat{\Phi}_1^2(m,n) \right|$.

Table 2 presents the errors of numerical calculation $I_1^2(m,n)$ by the formula $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{\Phi}_1^2(m,n)$ for $f(x, y) = \sin(x + y)$ and the number of values of the function at the points Q for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{Q}$ for $\widehat{\Phi}_1^2(m,n)$.

Table 2

Errors of numerical calculation $I_1^2(m,n)$ by formulas $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{\Phi}_1^2(m,n)$

m	n	ℓ	ε_1	$Q = 2\ell^3 - \ell^2$	ε_2	$\widehat{Q} = \ell^4$
4	4	10	$1.01 \cdot 10^{-8}$	1900	$1.01 \cdot 10^{-8}$	10000
4	4	25	$2.66 \cdot 10^{-10}$	30625	$2.62 \cdot 10^{-10}$	390625
5	5	25	$1.69 \cdot 10^{-10}$	30625	$1.67 \cdot 10^{-10}$	390625
5	5	35	$4.43 \cdot 10^{-11}$	84525	$4.36 \cdot 10^{-11}$	1500625
5	6	20	$3.43 \cdot 10^{-10}$	15600	$3.40 \cdot 10^{-10}$	160000
5	6	30	$6.83 \cdot 10^{-11}$	53100	$6.73 \cdot 10^{-11}$	810000
5	6	40	$2.16 \cdot 10^{-11}$	126400	$2.12 \cdot 10^{-11}$	2560000

Results in table 2 demonstrate that the errors of numerical calculation of $I_1^2(m,n)$ by the formula $\widetilde{\Phi}_1^2(m,n)$ and formula $\widehat{\Phi}_1^2(m,n)$ have the same order. Table 2 also shows the advantages of the formula $\widetilde{\Phi}_1^2(m,n)$: it needs less number of input data about function for the computation.

Table 3 demonstrates the difference between time spent T and $\widehat{T}$, the size of data in computer memory P and $\widehat{P}$ for the formula $\widetilde{\Phi}_1^2(m,n)$ and for the formula $\widehat{\Phi}_1^2(m,n)$.

Table 3

Values of T, $\widehat{T}$, P, $\widehat{P}$ for $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{\Phi}_1^2(m,n)$

m	n	ℓ	T, c	$\widehat{T}$, c	P, b	$\widehat{P}$, b
4	4	10	0.4	2.0	4010868	40311244
4	4	25	8.4	91.1	39979828	48502900
5	5	25	7.1	84.7	42660708	42863484
5	5	35	21.6	337.9	43036868	43188236
5	6	20	3.3	24.2	44500004	44659660
5	6	30	8.4	165.0	44843860	44991564
5	6	40	13.4	465.5	45150228	45416260

Times spent T for the calculation of $\widetilde{\Phi}_1^2(m,n)$ and $\widehat{T}$ for $\widehat{\Phi}_1^2(m,n)$ and sizes of memory P and $\widehat{P}$ were computed by the functions `Timing` and `MemoryInUse` correspondingly of Wolfram Mathematica `8.` Table 3 shows that a computation of the formula $\widetilde{\Phi}_1^2(m,n)$ and the formula $\widehat{\Phi}_1^2(m,n)$ needs the allocation of the approximately same size of computer memory. At the same time, proposed cubature formula $\widetilde{\Phi}_1^2(m,n)$ for the numerical integration $I_1^2(m,n)$ requires less computation time.

5. Conclusions

The paper develops a cubature formula for approximate calculation of two-dimensional irregular highly oscillatory integrals. A feature of proposed approach is using the sets of traces of a function on lines as the input information. Estimation of the formula for numerical integration has been done for the class of differentiable functions of two variables. Computation of the Fourier coefficients by the classic cubature formula and by the proposed cubature formula was done during numerical experiment. It is shown, that comparatively to classical approaches, based on the operator of interlineation cubature formula uses smaller number of input data and needs less computation time to achieve the same accuracy.

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