COMPATIBLE PAIRS OF ACTIONS FOR FINITE CYCLIC 2-GROUPS AND THE ASSOCIATED COMPATIBLE ACTION GRAPHS

SAHIMEL AZWAL BIN SULAIMAN

Doctor of Philosophy

UNIVERSITI MALAYSIA PAHANG

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SAHIMEL AZWAL BIN SULAIMAN

Thesis submitted in fulfillment of the requirements
for the award of the degree of
Doctor of Philosophy

Faculty of Industrial Sciences & Technology

UNIVERSITI MALAYSIA PAHANG

JUNE 2018

My humble effort I dedicate to my sweet and loving

Father who could not see this thesis completed

Mother

Siblings and spouses

Two beautiful nieces

Whose affection, love, encouragement and prays of day and night make me able to get such success and honor.

ACKNOWLEDGEMENTS

In the Name of Allah, the Most Gracious the Most Merciful

Alhamdulillah, all praise and thanks to Allah S.W.T. for giving me the opportunity and strength to go through this Doctor of Philosophy (Ph.D.) journey until it complete.

First, I would like to express my gratitude to my supervisor, Dr. Mohd Sham bin Mohamad for his guidance, suggestions, motivations, knowledges and advices. Without his supervision and help, this thesis would not have been possible. Moreover, I would like to thank my co supervisor, Dr. Yuhani binti Yusof for her brilliant comments, guidance and also suggestions during my studies. Her comments and questions were very beneficial in completing of this thesis.

Next, I would like to thank the Ministry of Education of Malaysia (MOE) and Research and Innovation Department of Universiti Malaysia Pahang (UMP) for financial funding of this research through the RDU Vote: 141307 and 1703265. Financial assistance from Institute of Postgraduate Studies of UMP, the Graduate Research Scheme is gratefully acknowledged.

Not forgotten, I would like to thank all of UMP lecturers and staffs especially Assoc. Prof. Dr. Rashid and Dr Roslinazairimah, for their helps and cooperation's during my study.

To my friends especially Dr. Hamzah, Dr. Nor Amirah, Ainur Amira, Zuhri, Syazani, Farid, Farhan, Mohamed, Puteri, Izzatie, Tarmizi, Amalina, Zaleha, Siti Hajjar and Hitfat teammates, you should know that your support and encouragement was worth more than I can express on paper.

Last but not least, I would like to express my gratitude to my parents, for their love and support throughout my life. Thank you both for giving me strength to reach the star and chase my dreams. My siblings, sisters-in-law and two beautiful nieces, deserve my wholehearted thank as well.

Thank you for being such an important part of my Ph.D. journey. This thesis just a beginning for my research journey.

ABSTRAK

Hasil darab tensor tak abelan ditakrifkan untuk sepasang kumpulan yang bertindak antara satu sama lain dengan tindakan tersebut memenuhi syarat keserasian. Pasangan tindakan serasi yang berbeza akan memberikan hasil darab tensor tak abelan yang berbeza. Oleh itu, dalam kajian ini bilangan maksimum hasil darab tensor tak abelan yang berbeza antara dua kumpulan didapati dengan menentukan jumlah bilangan tindakan pasangan serasi dan hanya memfokus kepada kumpulan kitaran berperingkat kuasa-2. Kajian ini bermula dengan menentukan jumlah sebenar pasangan tindakan serasi dengan menggunakan syarat-syarat perlu dan cukup bagi kumpulan kitaran berperingkat kuasa-2 yang bertindak serasi antara satu sama lain. Dalam mencari bilangan pasangan tindakan serasi bagi kumpulan kitaran berperingkat kuasa-2, graf tindakan serasi diperkenalkan. Kemudian, beberapa ciri bagi graf tindakan serasi dan subgraf bagi graf tindakan serasi diberikan.

ABSTRACT

The nonabelian tensor product is defined for a pair of groups which act on each other provided the actions satisfying the compatibility conditions. Different pairs of compatible actions will give different nonabelian tensor products. Thus, in this research, the maximum number of different nonabelian tensor products between two groups is found by determining the exact number of compatible pairs of actions and focusing only on the finite cyclic 2-groups. This research starts with determination of the exact number of compatible pairs of actions by using the necessary and sufficient conditions of finite cyclic 2-groups to act compatibly on each other. In order to find the number of compatible pairs of actions of finite cyclic 2-groups, the compatible action graph is introduced. Then, some properties of the compatible action graph and a subgraph of the compatible action graph are determined.

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LIST OF SYMBOLS

1	identity element
a mod *b*	*a* modulo *b*
a^{-1}	inverse of *a*
$\langle a \rangle$	cyclic subgroup generated by *a*
$[a, b]$	commutator of *a* and *b*
Aut (G)	automorphism group of group *G*
C_n	the cyclic group of order *n*
$\deg^{+}(v)$	the out-degree of a vertex *v*
$\deg^{-}(v)$	the in-degree of a vertex *v*
D_n	the dihedral group of order 2*n*
$E(\Gamma_{G\otimes H})$	the set of edges of the compatible action graph $\Gamma_{G\otimes H}$
(g^k, h^l)	the compatible pair of actions
G	a finite group
$\lvert G \rvert$	the order of *G*
$G\otimes H$	the nonabelian tensor product of the groups *G* and *H*
$G \cong H$	the groups *G* and *H* are isomorphic
${}^{g}h$	action of *g* on *h*
$H \le G$	*H* is a subgroup of *G*
$L \wedge K$	the nonabelian tensor exterior product
$L \square N$	the diagonal ideal
$\mathbb{N}$	the set of natural numbers
Q_n	the quaternion group of order 2*n*
t | s	*t* divides *s*
$V(\Gamma_{G\otimes H})$	the set of vertices of the compatible action graph $\Gamma_{G\otimes H}$
S_n	the symmetric group of degree *n*
SO_n	the special orthogonal group of order *n*
$Z(G)$	the center of *G*
$\mathbb{Z}$	the set of integers
$\in$	element of
$\Gamma_{G\otimes H}$	the compatible action graph of $G\otimes H$
$\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$	the subgraph of the compatible action graph for the groups $C_{\rho^{m-i}}$ and $C_{\rho^{n-i}}$
$\square$	end of proof

CHAPTER 1

INTRODUCTION

1.1 An Overview

This chapter presents an introduction to the whole thesis that contains research background, problem statement, research objectives, research scope and research significance.

1.2 Research Background

The nonabelian tensor product for groups G and H, denoted by $G \otimes H$, started in connection with a generalized Van Kampen Theorem. The structure has its origins in algebraic K-theory and also in homotopy theory. Brown and Loday (1984) introduced the concept of the nonabelian tensor product of groups with compatible actions, which extends the ideas of Whitehead (1950). The nonabelian tensor product is defined for a pair of groups, which acts on each other provided the actions satisfy the compatibility conditions:

$$^{(^{g}h)}g' = {}^{g}(^{h}(^{g^{-1}}g')) \text{ and } ^{(^{h}g)}h' = {}^{h}(^{g}(^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$. If G and H are groups that act compatibly with each other, then $G \otimes H$ is a group generated by $g \otimes h$ with these two relations

$$gg' \otimes h = (^{g}g' \otimes {}^{g}h)(g \otimes h) \text{ and } g \otimes hh' = (g \otimes h)(^{h}g \otimes {}^{h}h')$$

for all $g, g' \in G$ and $h, h' \in H$.

Brown and Loday (1984) studied the finiteness of the nonabelian tensor square $G \otimes G$ for a finite group G. However, the paper by Brown and Loday (1984) was written in Portuguese. Thus, the studies by Brown et al. (1987) became a starting point for investigations of the nonabelian tensor product of a group and was extensively studied by many researchers. Brown et al. (1987) focused on the group theory properties and the determination of the nonabelian tensor squares, denoted by $G \otimes G$.

As a continuity of Brown and Loday′s work, the results for the finiteness of nonabelian tensor product were given by Ellis (1987). Furthermore, Ellis provided the result on the nonabelian tensor product, which is p-power order if G and H are of p-power order. Next, McDermott (1998) investigated the nonabelian tensor product when G and H are a p-group and a q-group respectively, in which p and q are prime numbers. However, this paper is interested in the bound of the order of $G \otimes H$ and some results are given for that case. Meanwhile, the nonabelian tensor product of the cyclic groups of p-power order are investigated by Visscher (1998) and 14 years later, Mohamad (2012) focused on the nonabelian tensor product of the p-power order groups with two-sided actions.

There were some researchers that were interested in finding the compatible pair of actions for the finite cyclic 2-groups. First, Visscher (1998) provided the characterizations on the compatibility conditions for the finite cyclic 2-groups, such that the actions act on each other in compatible ways. Mohamad (2012) gave some necessary and sufficient conditions for the finite cyclic 2-groups to act compatibly on each other, with the order of the actions included as one of the conditions. Sulaiman et al. (2015) studied on some compatible pairs of the nontrivial actions of order two and four for some of the finite cyclic 2-groups and presented the compatible pair of nontrivial actions for the cyclic groups when the two groups are the same and the actions are in the same order as in Sulaiman et al. (2016).

1.3 Problem Statement

Let G and H be two groups, which act on each other and act on themselves by conjugation. Different pairs of the compatible actions will give a different nonabelian tensor product. Most of the researchers calculated the nonabelian tensor product for trivial actions by letting the action to have order one. However, only some researchers worked on the nontrivial actions where the actions have an order greater than one and calculated the nonabelian tensor product under this condition. McDermott (1998) focused on the quaternion and dihedral groups and gave a different nonabelian tensor product for a different compatible pair of non-trivial actions. Other than that, Visscher (1998) discussed the nonabelian tensor product for the finite cyclic 2-groups where the actions have order two for both and gave eight different cases. According to the definition of the compatible actions, the compatible pair of actions is required in order for the nonabelian tensor product to be computed. Therefore, the exact number for the compatible pairs of actions between the two groups is found where the number will give the maximum number of the different nonabelian tensor products between G and H.

1.4 Research Objectives

The objectives of this study are:

i. to determine the number of compatible pairs of actions between two finite cyclic 2-groups.

ii. to introduce the compatible action graph of the finite cyclic 2-groups and their properties.

iii. to determine the properties of a subgraph of compatible action graph for the finite cyclic 2-group

iv. to validate the results in (i) by using the computer algebra system Groups, Algorithm and Programming (GAP) to compute the compatible pairs of actions of the finite cyclic 2-groups.

1.5 Research Scope

This research is concentrated on the compatible pairs of actions and the groups considered are limited to the finite cyclic 2-groups only.

1.6 Research Significance

This research will be significant in terms of:

i. New Findings/Knowledge

The major contribution of this thesis is in determining the number of compatible pair of actions, which focuses on the finite cyclic 2-groups. Furthermore, a new graph, namely the compatible action graph is introduced.

ii. Specific or Potential Application

In this research, the GAP algorithm that has been built can be used to determine the total number of the compatible pairs of actions easily and more quickly. In addition, this algorithm can be modified to find the number of compatible pairs of actions for other types of groups such as the dihedral and quaternion groups.

1.7 Thesis Organization

Throughout this thesis, all groups are assumed to be finite unless stated otherwise. The first chapter contains research background, problem statement, research objectives, research scope, and research significance.

Chapter 2 focuses on the literature review of the work done by various researchers regarding the nonabelian tensor product and a view of graph theory.

Some definitions and preparatory results on the automorphism groups, compatibility conditions, graph theory and GAP algorithm are provided in Chapter 3. By using the GAP algorithm, the compatible conditions and number of compatible pairs of actions were determined. All results in this chapter are used in proving the new results in subsequent chapters.

The number of compatible pairs of actions for the cyclic 2-groups are investigated in Chapter 4. By using the necessary and sufficient conditions for two cyclic 2-groups to act compatibly with each other, the number of compatible pairs of actions between two cyclic 2-groups is determined. Some examples for the number of compatible pairs of actions between the two cyclic 2-groups are also provided.

Next, the compatible action graph of the cyclic 2-groups is introduced. Some results on the properties of the compatible action graph of the cyclic groups of 2-power order are presented in Chapter 5.

Chapter 6 focuses on the subgraph of the compatible action graph. Some results on properties of the subgraph of compatible action graph are also highlighted.

Lastly, Chapter 7 presents the summary of this research and some suggestions for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter presents details of the literature review on the nonabelian tensor product of groups, compatible pairs of actions and some background on graph theory.

2.2 The Nonabelian Tensor Product

The notion of nonabelian tensor product of groups was introduced by Brown and Loday (1984) and it is an extended idea of Whitehead (1950). The nonabelian tensor product of a groups was defined for a pair of groups G and H, provided the groups act on each other in compatible ways and satisfy the compatibility conditions. The paper by Brown and Loday (1987) motivated many researchers to investigate the group theoretical aspects of nonabelian tensor products extensively.

Brown and Loday (1987) studied the group theoretical properties, especially on computing the nonabelian tensor square, $G \otimes G$ when G is a group. In their study, Brown and Loday (1987) proved the finiteness of the nonabelian tensor square for a finite group G. Furthermore, the computation of the nonabelian tensor squares for groups of order up to 30 was given by using GAP. Later, Ellis (1987) extended the results for the nonabelian tensor product without any analytical proof. Furthermore, the nonabelian tensor product is of p-power order if G and H are of p-power order were presented.

Next, Bacon and Kappe (1993) determined the nonabelian tensor square of 2-generator p-groups of nilpotency class 2, where p is an odd prime. Moreover, the fact that nonabelian tensor square is abelian when G is a nilpotent group of class 2 was proven. In other studies, Ellis and Leonard (1995) developed an algorithm for computation of the nonabelian tensor product and Schur multiplier of a finite group. Furthermore, Ellis and Leonard (1995) developed a method for determining the nonabelian tensor products for all pairs of normal subgroups G and H of orders up to 14 and gave the nonabelian tensor square and Schur multiplier of the Burnside groups, namely $B(2,4)$ and $B(3,3)$ of order 2^{12} and 3^7, respectively.

McDermott (1998) developed an algorithm for computing the nonabelian tensor product of groups and implemented the algorithm using GAP. The order of the nonabelian tensor product for all normal subgroups G and H of the quaternion group of order 32 was presented. Also, the nonabelian tensor product of Q_n and D_n of order 8 was determined and two cases were considered namely the actions act compatibly on each other and the actions do not act compatibly with each other. Next, Ellis and McDermott (1998) improved Rocco′s bound (Rocco, 1991) and extended it to the case of the nonabelian tensor product of prime power groups G and H. The results showed that when G has order p^n and d is the minimal number of generators of G, then the order of $G \otimes G$ does not exceed p^{dn}.

Extended from Ellis and McDermott′s work, Visscher (1998) investigated the nonabelian tensor product of the cyclic p-groups namely p is an odd prime and $p = 2$. Furthermore, Visscher determined the characterisation of the compatibility condition and classification of all nonabelian tensor products of the cyclic p-groups. Moreover, the bounds on the nilpotency class and solvability length of $G \otimes H$ were presented.

Nakaoka (2000) studied the nonabelian tensor product of solvable groups. As a result, Nakaoka (2000) discussed a group construction regarding the nonabelian tensor products as a second proof of the result by Ellis (1987). Furthermore, a group $\eta(G, H)$ is defined as follows:

$$\left\langle G, H \middle| \left[g, h^{\phi}\right]^{g_1} \left[g^{g_1}, (h^{g_1})^{\phi}\right], \left[g, h^{\phi}\right]^{h_1^{\phi}} = \left[g^{h_1}, (h^{h_1})^{\phi}\right], \forall g, g_1 \in G, \forall h, h_1 \in H \right\rangle.$$

where G and H are groups acting compatibly on each other and an extra copy of H, isomorphic through $\phi : H \to H^{\phi}, h \mapsto h^{\phi}$ for all $h \in H$.

Nakaoka and Rocco (2001) presented the nonabelian tensor products for two nilpotent groups when the groups act on each other. Some examples when at least one of the actions were non-nilpotent were presented. Moreover, the nonabelian tensor square for a finite cyclic group was provided. Morse (2005) focused on the polycyclic groups for the nonabelian tensor square. The nonabelian tensor square of the finite group G and the presentation for the finite group denoted by $\nu(G)$ was defined. In addition, the subgroup $[G, G^{\varphi}]$ that is isomorphic to the nonabelian tensor square has been computed. Next, Moravec (2007) studied the nonabelian tensor product for the polycyclic groups. The results showed that $M \otimes N$ is polycyclic for two polycyclic groups that act compatibly with each other. In addition, the generating set for the nonabelian tensor product for the normal polycyclic subgroups has been provided.

Later, Blyth and Morse (2009) studied the theory of computation of the nonabelian tensor squares for polycyclic groups. The results provided the computations and the basis of an algorithm for computing the nonabelian tensor squares for any polycyclic group. Next, Moravec (2009) discussed the nonabelian tensor squares for powerful p-groups. Then, some fundamental properties of the nonabelian tensor square, which focused on powerful p-groups were provided. Moreover, the bounds for the order of $G \otimes G$ for a given p-group G were given.

Thomas (2010) proved the Ellis's results (Ellis, 1987) for the nonabelian tensor product for the two finite groups with an algebraic proof. Furthermore, the nonabelian tensor product of two p-groups was presented. Blyth et al. (2010) investigated the nonabelian tensor squares for the class of groups G from the derived subgroup which focused on the classes of free solvable and free nilpotent groups of finite rank and some classes of finite p-groups. Furthermore, Russo (2010) determined the nonabelian tensor products of two Chernikov groups and the nonabelian tensor products of two solvable minimax groups.

Meanwhile, Hassim et al. (2010) focused on the nonabelian tensor squares and homological functors of all 2-Engel groups of orders up to 16. Furthermore, a GAP algorithm to compute nonabelian tensor squares was developed. Thus, the result obtained was verified using the GAP software. Next, Salemkar et al. (2010) investigated the nonabelian tensor product of two Lie algebras K and L, denoted by $L \otimes K$. Some common properties of the Lie algebras and tensor product, such as the bound on the nilpotency class and solvability length of $L \otimes K$ were presented. Moreover, the bound for the dimension of $L \otimes K$ when K and L are finite-dimensional nilpotent Lie algebras and ideals of a single Lie algebra were given. Moravec (2010) focused on the powerful action and $M \otimes N$ for the powerful p-groups M and N. Some results on the properties of powerful actions of powerful p-groups were derived. Furthermore, the nonabelian tensor product with powerful p-groups M and N acting on each other in compatible ways were determined. In addition, the bounds for parameters, such as order, exponent and rank of $M \otimes N$ were estimated.

The nonabelian tensor squares for symplectic groups and projective symplectic groups were discussed by Rashid et al. (2011a). The commutator subgroups and Schur multiplier for these groups and also special linear groups and projective special linear groups were computed. Next, Erfanian et al. (2011) discussed the structure of the nonabelian tensor square for the polycyclic groups with a trivial centre. In addition, the Hirsch length and conditions on the Schur multiplier were provided. Rashid et al. (2011b) discussed on the group G that is capable when it is isomorphic to the central factor group $H/Z(H)$ for some group H. Furthermore, the nonabelian tensor square and its capability of groups of order p^2q, where p and q are prime were computed. In addition, the capability of G when $Z(G) = 1$ or $p < q$ and $G^{sb} = \mathbb{Z}_p \times \mathbb{Z}_p$ were proven.

Extended from Visscher's work, Mohamad (2012) investigated the concept of the nonabelian tensor product of the cyclic p-groups, namely $p = 2$ and p is an odd prime. The characterisation of automorphisms and new necessary and sufficient conditions for the actions that are compatible on each other for the cyclic p-groups where the order of the actions included as one condition have been determined. In addition, the nonabelian tensor product of some cyclic p-groups was proven to be cyclic.

Moreover, Mohamad et al. (2012) has obtained the nonabelian tensor product of the finite cyclic groups of order p^2 when the actions have order p. Next, Jafari (2012) provided the improvement for a bound on the order of $G \otimes G$ which is $p^{(n-e)d+m}$ when G/G' has exponent p^e. Furthermore, Rashid et al. (2012) focused on determining the Schur multiplier and the nonabelian tensor square for special orthogonal groups denoted by $SO_n(F_q)$ and spin groups denoted by $Spin_n(F_q)$ where F_q is a field with q elements.

Rashid et al. (2013) investigated the nonabelian tensor square and its capability for the groups of order $8q$, where q is an odd prime. The capability of the group using the the Schur multiplier of the groups of order $8q$ was determined. The results showed that G is capable when $Z(G) = 1$ only. Fauzi et al. (2014) presented the nonabelian tensor squares of a Biebierbach group of dimension five with a dihedral point group of order eight denoted by $B_l(5)$ was investigated. Thus, the nonabelian tensor square of the first Biebierbach group of dimension five with the dihedral point group of order eight generated by ten elements was determined.

Niroomand et al. (2015) focused on the decomposition of the nonabelian tensor product of Lie algebras denoted by $L \otimes N$ using the diagonal ideal when $N = L$. Next, Donadze et al. (2015) investigated the closure and the finiteness properties for the nonabelian tensor product of groups. Some classes that are closed under the formation of the nonabelian tensor product, such as solvable by finite, nilpotent by finite, polycyclic by finite, nilpotent of nilpotency class n and supersolvable groups were presented. In addition, some necessary and sufficient conditions for the nonabelian tensor products for the finitely generated groups were provided. Recently, Jafari (2016) categorised finite p-groups by the order of their nonabelian tensor squares.

In this research, our focus is on the compatible pair of actions for the cyclic groups. Visscher (1998) is the pioneer who studied the nonabelian tensor product of cyclic 2-groups, which gave the characterization of the compatibility conditions. However, Visscher (1998) has covered the cases for one-sided actions and when both actions have order two only. Next, Mohamad (2012) extended Visschers work (Visscher, 1998) to cover the case for the nonabelian tensor products of the cyclic 2-groups with two-sided actions. The necessary and sufficient conditions for the two cyclic 2-groups to act compatibly with each other, which is the order of the actions included as one of the conditions were provided. Later, Sulaiman et al. (2015) studied on the compatible pairs for the cyclic groups with non-trivial actions, but only considered the case of the compatible pairs of some cyclic 2-groups with non-trivial actions. Also, Sulaiman et al. (2016) investigated the compatible pair of non-trivial actions for two same cyclic 2-groups and the order of actions are the same. Next, Shahoodh et al. (2016) studied the compatible pairs of actions for the cyclic groups of 3-power order.

There are researchers that investigated the compatible pair of actions for the cyclic groups of p-power order with non-trivial actions, but none of them presented the exact number of compatible pairs of actions between two groups. Therefore, this research leads to determine the number of compatible pairs of actions between two finite cyclic 2-groups.

2.3 A View of Graph Theory

There were many researchers interested in the study of algebraic structure, especially in the properties of graphs. Most of the researchers try to investigate the interplay between group theory and graph theory. The paper by Abdollahi et al. (2006) used the associate graph called the non-commuting graph of G denoted by Γ_G where G is a nonabelian group and $Z(G)$ is the center of G. Some properties of the non-commuting graph were determined, such as the connectivity of Γ_G, Hamiltonian, planarity when G is isomorphic with the groups S_3, D_8 or Q_8. Also, the group properties of the two nonabelian groups that are isomorphic with two similar non-commuting graph were presented.

Then, Darafsheh (2009) investigated the non-commuting graph for two groups G and H, where G is the nonabelian finite group and H is the finite nonabelian simple group of Lie type, such as $A_n(q), B_n(q), C_n(q), D_n(q), F_4(q)$ and $G_2(q)$. In addition, $\Gamma_G \cong \Gamma_H$ implies $G \cong H$ was presented. Jahandideh et al. (2015) studied the conditions on the edges and vertices of a non-commuting graphs. As a result, some properties of the non-commuting graph, such as the number of edges denoted by $|E(\Gamma_G)|$, the degrees of the vertices of the non-commuting graph and also the number of conjugacy classes of a finite group was provided.

In connection between group theory and graph theory, Mansoori et al. (2016) defined the non-coprime graph associated to the group G denoted by $\prod_G$ where vertex set is $G\setminus\{e\}$, e is the identity element of G and two distinct vertices are adjacent whenever their orders are relatively non-coprime. The general properties of the non-coprime graph, such as diameter, girth, connectivity, Hamiltonian, independence number, domination number and also planarity were presented.

There are many researchers studied the specific graph on groups and determine the properties of the graph for the group, as shown by Abdollahi et al. (2006), Jahandideh et al. (2015) and Mansoori et al. (2016). In this research, our interest is to introduce the compatible action graph of the finite cyclic 2-groups and their properties.

2.4 Conclusion

In this chapter, literature on the compatible actions, the nonabelian tensor product of groups and graph theory were presented. From the literature, our motivation is on determining a different compatible pair of actions, which can give a different nonabelian tensor product even for the same groups. Furthermore, our focus is on the determination of the properties of the compatible action graph of the cyclic 2-groups.

CHAPTER 3

PRELIMINARY RESULTS

3.1 Introduction

In Chapter 3, some definitions and preliminary results on the automorphism groups, compatibility conditions and graph theory are given. By using GAP, the number of compatible pairs of actions are determined. All results in this chapter will be used in the following chapters.

3.2 Automorphism Groups

Let G and H be finite cyclic groups generated by $g \in G$ and $h \in H$, respectively. The automorphism group of G, denoted by Aut(G), is the set consisting of all isomorphisms $\sigma : G \rightarrow G$ such that $\sigma : g \mapsto g^t$ where t is an integer with gcd(t, $|g|$)=1. The automorphism group of C_{2^n} is given below.

Theorem 3.1 (Dummit and Foote, 2004)
Let G be a cyclic group of order 2^n, $n \geq 3$. Then, Aut(G)$\cong C_2 \times C_{2^{n-2}}$ and $|\text{Aut}(G)| = \varphi(2^n) = 2^{n-1}$.

In view of Theorem 3.1, $\varphi(2^n)$ is the Euler's $\varphi-$function that represents the number of positive integers not greater than 2^n but relatively prime to 2^n. Therefore, $|\text{Aut}(C_{2^n})| = \varphi(2^n) = 2^{n-1}$.

Now, let $G = \langle g \rangle$ be a finite cyclic group of order n. Then the order of any power of g is stated in the next proposition.

Proposition 3.1 (Burton, 2005)
Let G be group and $g \in G$ with $|g| < \infty$. Then $|g^k| = \dfrac{|g|}{\gcd(k, |g|)}$ for any $k \in \mathbb{N}$.

Mohamad (2012) characterised the automorphisms of all cyclic 2-groups. Starting with the following proposition, every automorphism is stated in terms of these generators.

Proposition 3.2 (Mohamad, 2012)
Let $G = \langle g \rangle$ with $|G| = 2^n$, $n \geq 3$. Then $\rho : g \longrightarrow g^5$ is an automorphism of order 2^{n-2}.

Then, the automorphism of a given order in $\langle \rho \rangle \leq \mathrm{Aut}(C_{2^n})$ is given as follows.

Proposition 3.3 (Mohamad, 2012)
Let ρ^j be an automorphism of order 2^s of $\langle g \rangle \cong C_{2^n}$, where $s = 1, 2, \ldots, n-2$ and $\rho^j : g \longrightarrow g^{5j}$. Then $\gcd(j, 2^{n-2}) = 2^{n-s-2}$. Furthermore, there are 2^{s-1} automorphisms of order 2^s in $\langle \rho \rangle$.

In the next theorem, all automorphisms of cyclic 2-groups are given as follows.

Theorem 3.2 (Mohamad, 2012)
Let $G = \langle g \rangle \cong C_{2^n}$, $n \geq 3$. Then Aut(G)=$\langle \tau \rangle \times \langle \rho \rangle$, where $\tau(g) = g^{-1}$ and $\rho(g) = g^5$ and every $\sigma \in \mathrm{Aut}(G)$ can be represented as $\sigma = \tau^i \rho^j$ with $i = 0, 1$ and $j = 0, 1, \ldots, 2^{n-2} - 1$ and $\tau^i \rho^j(g) = g^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$.

The order of actions is included as one of the conditions in the necessary and sufficient conditions for actions that act compatibly with each other. Hence, the number of automorphisms of a specific order for a cyclic 2-groups given in the following proposition.

Proposition 3.4 (Mohamad, 2012)
Let $G = \langle g \rangle \cong C_{2^n}$, $n \geq 3$. Then, Aut$(G) = \langle \tau \rangle \times \langle \rho \rangle$, where $\tau(g) = g^{-1}$ and $\rho(g) = g^5$. Then there exist three automorphisms of order two, namely $\sigma = \tau$, $\rho^{2^{n-3}}$ and $\tau\rho^{2^{n-3}}$ with $\tau(g) = g^{-1}$, $\rho^{2^{n-3}}(g) = g^{5^j}$ and $\tau\rho^{2^{n-3}} = g^{-5^j}$ where $j = 2^{n-3}$. Furthermore, there exist 2^s automorphisms of order 2^s, $s = 2, 3, \ldots, n-2$, namely $\sigma = \rho^j$, $\tau\rho^j$ with $\rho^j(g) = g^{5^j}$ and $\tau\rho^j(g) = g^{-5^j}$ where $\gcd(2^{n-2}, j) = 2^{n-s-2}$.

Based on the presentation of Aut(G) above, there are three automorphisms of order two as given in the following proposition.

Proposition 3.5 (Mohamad, 2012)
Let $G = \langle g \rangle \cong C_{2^n},\ n \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2$. If t is an integer such that $\sigma(g) = g^t$, then

$$t \equiv 2^{n-1} + 1 \pmod{2^n},\ t \equiv 2^{n-1} - 1 \pmod{2^n} \text{ or } t \equiv -1 \pmod{2^n}.$$

All preparatory results on the automorphisms for the cyclic 2-groups have been given in this section. In the next section, all results on the compatible pairs of actions that will be used in this research are given.

3.3 Compatible Actions

In this section, some definitions and previous results on compatible conditions that will be used to compute the total number of compatible pairs of actions are stated. It starts with the definition of an action of group G on group H, which is as follows.

Definition 3.1 : Action (Visscher, 1998)
Let G and H be cyclic groups. An action of G on H is a mapping $\Phi : G \to \text{Aut}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$

for all $g, g' \in G$ and $h \in H$.

Then, the definition of a compatible pair of actions between two groups is given as follows.

Definition 3.2 : Compatible Action (Brown and Loday, 1987)
Let G and H be groups which act on each other. These mutual actions are said to be compatible with each other and with the actions of G and H on themselves by conjugation if

$$^{(^g h)}g' = {}^g({}^h({}^{g^{-1}}g')) \text{ and } {}^{(^h g)}h' = {}^h({}^g({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

Next, the compatibility condition where G and H are abelian is given in the following proposition.

Proposition 3.6 (Visscher, 1998)
Let G and H be groups which act on each other. If G and H are abelian, then the mutual actions are compatible if and only if

$$^{(^g h)}g' = {}^h g' \text{ and } ^{(^h g)}h' = {}^g h'$$

for all $g, g' \in G$ and $h, h' \in H$.

The following proposition gives necessary and sufficient number-theoretic conditions for mutual actions of finite cyclic groups to be compatible.

Proposition 3.7 (Visscher, 1998)
Let $G = \langle x \rangle \cong C_m$ and $H = \langle y \rangle \cong C_n$ be finite cyclic groups. Then there exist mutual actions of G and H on each other such that ${}^y x = x^k$ and ${}^x y = y^l$ for $k, l \in \mathbb{Z}$ if and only if the conditions (i) and (ii) below are satisfied. These actions are compatible if and only if condition (iii) is satisfied as well.

i. $\gcd(k, m) = \gcd(l, n) = 1$

ii. $k^n \equiv 1 \pmod{m}$ and $l^m \equiv 1 \pmod{n}$

iii. $k^{l-1} \equiv 1 \pmod{m}$ and $l^{k-1} \equiv 1 \pmod{n}$

The next proposition shows that the trivial action is always compatible with any other action when G is abelian.

Proposition 3.8 (Visscher, 1998)
Let G and H be groups. Furthermore, let G act trivially on H. If G is abelian, then for any action of H on G the mutual actions are compatible.

According to the presentation of the automorphism group of a cyclic 2-group, Mohamad (2012) presented the necessary and sufficient conditions for a pair of actions that act compatibly with each other with specific order. If one of the actions has order two, then the necessary and sufficient conditions for the other actions to act compatibly with each other are given in the following theorem.

Theorem 3.3 (Mohamad, 2012)
Let $G = \langle x \rangle \cong C_{2^m}$ and $H = \langle y \rangle \cong C_{2^n}$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma| = 2$ and $\sigma' \in \mathrm{Aut}(H)$, where $m \geq 2, n \geq 3$.

i. If $\sigma(x) = x^t$ with $t \equiv -1 \pmod{2^m}$ or $t \equiv 2^{m-1} - 1 \pmod{2^m}$, then (σ, σ') is a compatible pair if and only if σ' is the trivial automorphism or $|\sigma'| = 2$.

ii. If $\sigma(x) = x^t$ with $t \equiv 2^{m-1} + 1 \pmod{2^m}$, then (σ, σ') is a compatible pair if and only if $|\sigma'| \leq 2^{s'}$ with $s' \leq m - 1$. In particular, σ is compatible with all $\sigma' \in \mathrm{Aut}(H)$ provided $n \leq m + 1$.

Furthermore, the necessary and sufficient conditions of compatible conditions where one of the actions has an order greater than two are stated in the following theorem.

Theorem 3.4 (Mohamad, 2012)
Let $G = \langle x \rangle \cong C_{2^m}$ and $H = \langle y \rangle \cong C_{2^n}$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma| = 2^s, s \geq 2$ and $\sigma' \in \mathrm{Aut}(H)$, where $m \geq 4, n \geq 1$.

i. If $\sigma(x) = x^t$ with $t \equiv -5^j \pmod{2^m}$, then (σ, σ') is a compatible pair if and only if $\sigma'(y) = y^{t'}$ with $t' \equiv 1 \pmod{2^n}$ or $t' \equiv 2^{n-1} + 1 \pmod{2^m}$.

ii. If $\sigma(x) = x^t$ with $t \equiv 5^j \pmod{2^m}$, then (σ, σ') is a compatible pair if and only if $|\sigma'| \leq 2^{m-s}$ provided $n \leq m - s + 2$.

Proposition 3.9 is a special case of Proposition 3.8 that gives the compatibility conditions when one of the actions is trivial with G and H are cyclic groups.

Proposition 3.9 (Mohamad, 2012)
Let $C_m = \langle x \rangle$ and $C_n = \langle y \rangle$ be finite cyclic groups and act on each other. If one of the actions is trivial, then any pair of actions of $C_m = \langle x \rangle$ and $C_n = \langle y \rangle$ are compatible.

3.4 Graph Theory

The concept of compatible actions of the nonabelian tensor product can be represented by a graph. By letting the elements of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ be the vertices with two vertices of $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ connected by an edge if (σ, σ') is a compatible pair of actions, we obtain a graph which we call the compatible action graph of G and H. In this section, some fundamental concepts that are related to graph theory are given. First, the definition of a graph is stated as follows.

Definition 3.3 : Graph (Rosen, 2012)
A graph $G = (V, E)$ consists of a nonempty set of vertices, V (or nodes) and a set of edges, E. Each edge has either one or two vertices associated with it, which is called its endpoints. An edge is said to connect its endpoints.

The definition of directed graph is given as below.

Definition 3.4 : Directed Graph (Rosen, 2012)
A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

In a graph with directed edges, the degree of the vertex has two types which are the in-degree of a vertex and the out-degree of a vertex. In the following definition, the degree of a vertex for a directed graph is presented.

Definition 3.5 : Degree of a Vertex (Rosen, 2012)
In a graph with directed edges, the in-degree of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes one to both the in-degree and the out-degree of this vertex.)

The definition of bipartite graph is given as follows.

Definition 3.6 : Bipartite Graph (Rosen, 2012)
A graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

The definition of a path for directed graphs is given in Definition 3.7.

Definition 3.7 : Path (Rosen, 2012)
Let n be a non- negative integer and G a directed graph. A path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_1, x_2, \ldots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a circuit or cycle. A path or circuit is called simple if it does not contain the same edge more than once.

The connectivity of a directed graph exists when there is a path which starts and ends at the same vertex. The definition of a connected directed graph is given below.

Definition 3.8 : Connected Graph (Rosen, 2012)
A directed graph is connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

The definition for the order of a graph is given below.

Definition 3.9 : Order of a Graph (Bollobás, 2013)
The order of a graph G is the number of vertices in G. It is denoted by $|G|$. Thus, $|G| = |V(G)|$.

3.5 The GAP Programmes for Compatibility

The GAP software is a free computer software for computational discrete algebra with the main emphasis on computational group theory. The GAP software provides a programming language, a library of functions and algebraic objects such as the ordinary groups. In this section, a GAP algorithm is used to find the total number of compatible pairs of actions. Let G and H be cyclic 2-groups with order 2^m and 2^n, respectively. Then, the outputs of the coding below gives the list of automorphisms with their specific order that satisfy the compatibility conditions and give the total of number of compatible pairs of actions.

```
NumberCompatibleAction:= function(m,n)
local a,b,c,d,e,f,g, xyx,yxy;
   g:=0;
   for a in [0..m-1] do
      for b in [0..n-1] do
         c:=a;
         d:=b;
         if Gcd(m, a)=1 and Gcd(n, b)=1 then
           for e in [2..m-1] do
              if c<>1 then
                 c:=a^e mod m;
              fi;
              if c=1 then
                c:=e;
                break;
              fi;
          od;
      for b in [0..n-1] do
         if d<>1 then
          d:=b^f mod n;
            fi;
          if d=1 then
           d:=f;
            break;
          fi;
         od;
        fi;
     xyx:=a^b mod m;
     yxy:=b^a mod n;
     if xyx=a and yxy=b then
     g:=g+1;
        Print("a=",a," (order action=",c,")",
        ",b=",b," (order action=",d,")");
         Print(" Compatible","\n");
     fi;
   od;
 od;
Print(" No of Compatible = ",g);
end;
```

Some outputs from the routine are given in Appendix A. By Proposition 3.4 and Theorem 3.4, the value of m should start from four so that the actions will have order greater than two and the value of n starts from one. A summary of the outputs is given in Table 3.1 which shows the number of compatible pairs of actions for $C_{2^m} \otimes C_{2^n}$ for some specific values of m and n.

Table 3.1: The Number of Compatible Pairs of Actions for $C_{2^m} \otimes C_{2^n}$

m	n	compatible pair	m	n	compatible pair	m	n	compatible pair
4	1	15	5	1	24	6	1	41
4	2	18	5	2	28	6	2	46
4	3	24	5	3	36	6	3	56
4	4	36	5	4	52	6	4	76
4	5	52	5	5	84	6	5	116
4	6	68	5	6	116	6	6	196
4	7	100	5	7	148	6	7	260
4	8	164	5	8	212	6	8	324
4	9	292	5	9	340	6	9	452

m	n	compatible pair	m	n	compatible pair	m	n	compatible pair
7	1	74	8	1	139	9	1	268
7	2	80	8	2	146	9	2	276
7	3	92	8	3	160	9	3	292
7	4	116	8	4	188	9	4	324
7	5	164	8	5	244	9	5	388
7	6	260	8	6	356	9	6	516
7	7	452	8	7	580	9	7	772
7	8	580	8	8	1028	9	8	1284
7	9	708	8	9	1284	9	9	2308

The example that follows focuses on the inquiry on finding the equality of the number of the compatible pair of actions of 2-power order for $G \otimes H$ is equal with $H \otimes G$ for the case of $G \neq H$.

Example 3.1 Let $G = C_{2^4}$ and $H = C_{2^8}$. Table 3.1 illustrates that there are 164 compatible pairs of actions for $G \otimes H$ wheres there are 188 for $H \otimes G$. Therefore, the number of compatible pairs of actions for $G \otimes H$ and $H \otimes G$ are not necessarily equal when $G \neq H$.

3.6 Conclusion

In this chapter, the works done by other researchers have been discussed. The GAP algorithm has been developed to compute the number of compatible pairs of actions for the given groups, in which the outputs can verify some results of the necessary and sufficient condition for the cyclic 2-groups to act compatibly with each other.

CHAPTER 4

THE EXACT NUMBER OF COMPATIBLE PAIRS OF ACTIONS FOR CYCLIC 2-GROUPS

4.1 Introduction

The main concern of this chapter is to determine the number of compatible pairs of actions for a given pair of cyclic 2-groups. Throughout this chapter, the number of compatible pairs of actions between two cyclic 2-groups has been found by using necessary and sufficient conditions of two cyclic 2-groups to act compatibly on each other. First, the number of compatible pairs of actions with specific order are counted. Then, the total number of compatible pairs of actions are obtained. In addition, some properties of automorphisms are given.

4.2 The Compatible Pairs of Actions with Specific Order for Cyclic 2-Groups

In this section, the compatible pairs of actions with a specific order for the cyclic 2-groups is discussed. There are three separate cases since the order of action is included as one of the equivalent conditions for actions that act compatibly with each other.

We begin by considering the number of compatible pairs of actions when the action has order one.

Proposition 4.1
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 1, n \geq 1$. If G acts trivially on H, then the number of compatible pairs of actions is 2^{n-1}.

Proof

By Proposition 3.8, when G acts trivially on H, for any action of H on G, the mutual actions are compatible. Thus, by Theorem 3.1 the number of compatible pairs of actions is 2^{n-1} since $|\text{Aut}(H)| = 2^{n-1}$. $\square$

Next, the number of compatible pairs of actions where one of the actions has order two is determined.

Proposition 4.2

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 2, n \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2$ and $\sigma' \in \text{Aut}(H)$ such that the pair (σ, σ') acts compatibly with each other.

i. If $\sigma(x) = x^t$ with $t \equiv -1 \pmod{2^m}$ or $t \equiv 2^{m-1} - 1 \pmod{2^m}$, then there are eight compatible pairs (σ, σ').

ii. If $\sigma(x) = x^t$ with $t \equiv 2^{m-1} + 1 \pmod{2^m}$, then there are 2^{r-1} compatible pairs (σ, σ') where $r = \min\{m+1, n\}$.

Proof

By Theorem 3.3, there are two cases to consider.

i. **Case 1:** $\sigma(x) = x^t$ with $t \equiv -1 \pmod{2^m}$ or $t \equiv 2^{m-1} - 1 \pmod{2^m}$. First, if σ' is the trivial automorphism, then the mutual actions are compatible. Thus, two compatible pairs of actions of the form $(\sigma, 1)$ exists. Next, consider $|\sigma'| = 2$, then by Theorem 3.3(i), the pair (σ, σ') is compatible. Since there are 3 possibilities for σ' so that $|\sigma'| = 2$, it follows that the number of compatible pairs in this particular case is six. Next, consider $|\sigma'| > 2$. By Theorem 3.3(i), the pair (σ, σ') is not compatible. Therefore, if $\sigma(x) = x^t$ with $t \equiv -1 \pmod{2^m}$ or $t \equiv 2^{m-1} - 1 \pmod{2^m}$, then there are eight compatible pairs of actions (σ, σ').

ii. **Case 2:** $\sigma(x) = x^t$ with $t \equiv 2^{m-1} + 1 \pmod{2^m}$. By Theorem 3.3(ii), σ is compatible with all σ' provided $s' \leq n$ and $s' \leq m + 1$ or we can assume that $s' \leq \min\{n, m+1\}$. Suppose that $r = \min\{n, m+1\}$. Then, we have that all σ' where $|\sigma'| \leq 2^r$ are compatible with σ when $|\sigma'| = 2^{s'}$. By Theorem 3.1, there are 2^{n-1} number of automorphisms in Aut(H). Therefore, the number of compatible pairs of actions is 2^{r-1} where $r = \min\{m+1, n\}$.

$\square$

The number of compatible pairs of actions for cyclic 2-groups when one of the actions has an order two is presented in the next theorem.

Theorem 4.1
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 2, n \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2$. Then there are $2^{r-1} + 8$ compatible pairs of actions (σ, σ') where $\sigma' \in \text{Aut}(H)$with $r = \min\{m+1, n\}$.

Proof
By Proposition 4.2, there are $2^{r-1} + 8$ compatible pairs of actions (σ, σ') where $\sigma' \in \text{Aut}(H)$ with $r = \min\{m+1, n\}$. $\square$

Next, the number of compatible pairs of actions for cyclic 2-groups where one of the actions has an order greater than two is determined. By Theorem 3.4, there are two cases for the compatible conditions for any two automorphisms with specific order, namely for $\sigma(g) = g^t$ with $t \equiv -5^j \pmod{2^m}$ and $\sigma(g) = g^t$ with $t \equiv 5^j \pmod{2^m}$ given in Lemmas 4.1 and 4.2 respectively.

Lemma 4.1
Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2^s, s \geq 2$ and $\sigma' \in \text{Aut}(H)$. If $\sigma(g) = g^t$ with $t \equiv -5^j \pmod{2^m}$, then there are 2^s compatible pairs (σ, σ').

Proof
By Proposition 3.4, there exist 2^s automorphisms of G of order 2^s for $s = 2, 3, \ldots, n-2$. Suppose that $\sigma(g) = g^t$ with $t \equiv -5^j \pmod{2^m}$, thus only 2^{s-1} number of automorphisms are considered. Next, by Theorem 3.4(i), (σ, σ') is a compatible pair when $\sigma'(h) = h^{t'}$ with $t' \equiv 1 \pmod{2^n}$ or $t' \equiv 2^{n-1} + 1 \pmod{2^n}$, then the number of compatible pairs (σ, σ') is $2^{s-1} + 2^{s-1} = 2^s$. $\square$

Lemma 4.2

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2^s, s \geq 2$ and $\sigma' \in \text{Aut}(H)$ with $|\sigma'| = 2^{s'}$. If $\sigma(g) = g^t$ with $t \equiv 5^j \pmod{2^m}$, then there are $2^{r'-1}$ compatible pairs (σ, σ') with $r' = \min\{m, n\}$.

Proof

Suppose that $\sigma(g) = g^t$ with $t \equiv 5^j \pmod{2^m}$. Then by Proposition 3.4, only 2^{s-1} automorphisms are considered. By Theorem 3.4(ii), σ is compatible with all σ' provided $s' \leq n - s$ and $s' \leq m - s$ or simply presented as $s' \leq \min\{n - s, m - s\}$. Suppose that $r' = \min\{n, m\}$, hence all σ' where $|\sigma'| \leq 2^{r'-s}$ are compatible with σ when $|\sigma'| = 2^{s'}$. Thus, if all actions of order $2^{s'}$ are considered where $s' \geq 2$, then there are $2^2 + 2^3 + \cdots + 2^{r'-s} = 2^{r'-s+1} - 4$ compatible pairs of actions (σ, σ'). By Theorem 3.4(ii), only $\dfrac{2^{r'-s+1} - 4}{2} = 2^{r'-s} - 2$ compatible pairs of actions (σ, σ') when $\sigma(g) = g^t$ with $t \equiv 5^j \pmod{2^m}$ since 2^{s-1} automorphism are considered under this case.

By Theorem 3.4, σ is also compatible with the trivial action and one action with $|\sigma'| = 2$. Thus, the number of the compatible pair of actions for σ with specific order is $2^{r'-s} - 2 + 2 = 2^{r'-s}$. By Proposition 3.3, there are 2^{s-1} automorphisms of order 2^s. Therefore, the number of compatible pairs of actions for $|\sigma| = 2^s$ is $2^{s-1}(2^{r'-s}) = 2^{r'-1}$ where $r' = \min\{m, n\}$. $\square$

4.3 The Total Number of Compatible Pairs of Actions for Cyclic 2-Groups

This section gives the total number of compatible pairs of actions for two cyclic 2-groups as follows.

Lemma 4.3

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = 2^s$, $s \geq 2$ and $\sigma' \in \text{Aut}(H)$. The number of compatible pairs of actions (σ, σ') is $(2^{m-1} - 4) + (m - 3)(2^{r'-1})$ provided $r' = \min\{m, n\}$.

Proof

Lemmas 4.1 and 4.2 give the number of compatible pairs of actions when one of the actions has an order 2^s for particular s and $s \geq 2$. Now, the total number of (σ, σ') for all $s \geq 2$ is given as follows.

i. By Lemma 4.1, there are 2^s compatible pairs of actions and note that, the highest order of σ in $\mathrm{Aut}(C_{2^m})$ is 2^{m-2}. By Proposition 3.4, there exist 2^s automorphisms of order 2^s. Thus, if all actions of order 2^s are considered where $s = 2, 3, \ldots, m-2$, then the number of the compatible pair of actions is

$$2^2 + 2^3 + 2^4 + \cdots + 2^{m-2} = 2^{m-1} - 4.$$

ii. By Lemma 4.2, there are $2^{r'-1}$ compatible pairs of actions and note that, the highest order of σ in $\mathrm{Aut}(C_{2^m})$ is 2^{m-2}. By Proposition 3.4, there exist 2^s automorphisms of order 2^s. Thus, if all actions order of 2^s are considered where $s = 2, 3, \ldots, m-2$, there are

$$2^{r'-1} + 2^{r'-1} + \cdots + 2^{r'-1} = (m-3)2^{r'-1}$$

compatible pairs of actions (σ, σ') provided r' =$\min\{m, n\}$.

Thus, the number of compatible pairs of actions when one of the actions has an order greater than two for both cases is given as $2^{m-1} - 4 + (m-3)2^{r'-1}$. □

Theorem 4.2

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, there exist

$$(m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1} + 4$$

compatible pairs of actions where $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$. Proposition 4.1, Theorem 4.1 and Lemma 4.3 give the number of compatible pairs (σ, σ') with specific order. Thus, three cases considered which are $|\sigma| = 1, 2$ and 2^s where $s \geq 2$.

i. Suppose that $|\sigma| = 1$. By Proposition 4.1, when one of the actions is trivial, then the number of compatible pairs of actions is 2^{n-1}

ii. Suppose that $|\sigma| = 2$. By Theorem 4.1, there are $2^{r-1} + 8$ compatible pairs of actions where $r = \min\{m+1, n\}$ and $m \geq 2$.

iii. Suppose that $|\sigma| = 2^s$ where $s \geq 2$. By Lemma 4.3, the number of compatible pairs of actions for this case is $2^{m-1} - 4 + (m-3)2^{r'-1}$ where $r' = \min\{m, n\}$ and $m \geq 4$.

Hence, the number of compatible pairs of actions for the cyclic 2-groups is

$$2^{n-1} + 2^{r-1} + 8 + 2^{m-1} - 4 + (m-3)(2^{r'-1}) = (m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1} + 4.$$

□

By using Theorem 4.2, the number of compatible pairs of actions for cyclic 2-groups can be computed. The following example shows the number of compatible pairs of actions for cyclic 2-groups.

Example 4.1

Let $G = C_{2^4}$ and $H = C_{2^6}$ be cyclic groups. Now, consider the action of G and H act on each other such that ${}^h g = g^k$ and ${}^g h = h^l$ for $g \in G$, $h \in H$ and $k, l \in \mathbb{Z}$. Table 4.1 illustrates the compatible pair of actions for $C_{2^4} \otimes C_{2^6}$ given by GAP software. From the table, there are 68 compatible pairs of actions for $C_{2^4} \otimes C_{2^6}$. By Theorem 4.2, the number of compatible pairs of actions is $(4-3)(2^{4-1}) + 2^{5-1} + 2^{6-1} + 2^{4-1} + 4 = 68$. Hence, the result obtained by Theorem 4.2 is the same as that given in Table 4.1 for the case $m = 4, n = 6$.

Table 4.1: Compatible Pairs of Actions for $C_{2^4} \otimes C_{2^6}$

$\lvert g^k \rvert$	k	l	$\lvert h^l \rvert$	$\lvert g^k \rvert$	k	l	$\lvert h^l \rvert$	$\lvert g^k \rvert$	k	l	$\lvert h^l \rvert$
1	1	1	1	1	1	47	4	2	9	41	8
1	1	3	16	1	1	49	4	2	9	47	4
1	1	5	16	1	1	51	16	2	9	49	4
1	1	7	8	1	1	53	16	2	9	55	8
1	1	9	8	1	1	55	8	2	9	57	8
1	1	11	16	1	1	57	8	2	9	63	2
1	1	13	16	1	1	59	16	2	15	1	1
1	1	15	4	1	1	61	16	2	15	31	2
1	1	17	4	1	1	63	2	2	15	33	2
1	1	19	16	2	7	1	1	2	15	63	2
1	1	21	16	2	7	31	2	4	3	1	1
1	1	23	8	2	7	33	2	4	3	33	2
1	1	25	8	2	7	63	2	4	5	1	1
1	1	27	16	2	9	1	1	4	5	17	4
1	1	29	16	2	9	7	8	4	5	33	2
1	1	31	2	2	9	9	8	4	5	49	4
1	1	33	2	2	9	15	4	4	11	1	1
1	1	35	16	2	9	17	4	4	11	33	2
1	1	37	16	2	9	23	8	4	13	1	1
1	1	39	8	2	9	25	8	4	13	17	4
1	1	41	8	2	9	31	2	4	13	33	2
1	1	43	16	2	9	33	2	4	13	49	4
1	1	45	16	2	9	39	8				

4.4 Compatible Pairs of Actions for Cyclic 2-Groups of the Same Order

This section focuses on the total number of compatible pairs of actions between two same cyclic 2-groups with the same order of actions. First, the necessary and sufficient conditions for the cyclic 2-groups of the same order that act compatibly on each other are given.

Proposition 4.3

Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, where $m \geq 4$. Furthermore, let $\sigma \in \text{Aut}(G)$ and $\sigma' \in$Aut(H) with $|\sigma| = |\sigma'| = 2^k$ where $k \geq 0$. Then, (σ, σ') is a compatible pairs of actions if $2k \leq m$.

Proof

Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, where $m \geq 1$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$. Then, there are three cases to be considered as follows:

i. Assume that $k = 0$. Then $|\sigma| = |\sigma'| = 2^0 = 1$. By Proposition 3.8, when one of the actions is trivial, then it will be compatible with all actions. Therefore, (σ, σ') is a compatible pairs of actions if $k = 0$.

ii. Assume that $k = 1$. Then $|\sigma| = |\sigma'| = 2^1 = 2$. Furthermore, let $G = \langle g \rangle \cong C_m$ and $H = \langle h \rangle \cong C_n$ where m and n are even integers with the actions of h on g and g on h given by

$$^{h}g = g^k \text{ and } {}^{g}h = h^l$$

for $g \in G$, $h \in H$ and $k, l \in \mathbb{Z}$. The values of k and l must be odd since $\gcd(2, k) = \gcd(2, l) = 1$ for automorphisms. Thus, $k = 2a + 1$ and $l = 2b + 1$ for positive integers a and b and hence $k \equiv 1 \pmod 2$ and $l \equiv 1 \pmod 2$. Since the actions both have order two, by Proposition 3.6 it follows that ${}^{({}^{g}h)}g' = {}^{h}g'$ and ${}^{({}^{h}g)}h' = {}^{g}h'$. Therefore, the actions always act compatibly if they have order two.

iii. Assume that $k \geq 2$. Then $|\sigma| = |\sigma'| = 2^k$ where $k \geq 0$. Let ${}^{h}g = g^t$, $t = \delta 2^{m-k} + 1$ and ${}^{g}h = h^s$, $s = \varepsilon 2^{m-k} + 1$. Then, $\gcd(\delta, 2) = \gcd(\varepsilon, 2) = 1$. We have $t = 2^k \cdot \delta 2^{m-2k} + 1$. Thus

$${}^{{}^{h}g}h = {}^{g^t}h = {}^{g}(\underbrace{g^{2^k}(\ldots(g^{2^k}}_{x-times}h))) = {}^{g}h$$

with $x = \delta 2^{m-2k}$, $2k \leq m$.

Then, let ${}^{{}^{h}g}h = {}^{g}h$. Then ${}^{g^t}h = {}^{g}h$ or equivalently ${}^{g^{t-1}}h = h$. Hence $t - 1 \equiv 0 \bmod 2^k$ or $m - k \geq k$.

Next, suppose $s = \varepsilon 2^{m-k} + 1 = 2^k \cdot \varepsilon 2^{m-2k} + 1$ with $m - 2k \geq 0$, since both same groups. Thus,

$$^{^g h}g = {}^{h^s}g = {}^h(\underbrace{^{h^{g^k}}(\ldots(^{h^{2^k}}}_{y-times} g))) = {}^h g$$

with $y = \varepsilon 2^{m-2k}$.

Now suppose $^{^g h}g = {}^h g$. Then $^{h^s}g = {}^h g$ or equivalently $^{h^{s-1}}g = g$. Hence $s - 1 \equiv 0 \bmod 2^k$ or $m - k \geq k$ or $2k \leq m$. Therefore, (σ, σ') is compatible pairs of action if $2k \leq m$ where $k \geq 2$.

As conclusion, (σ, σ') is a compatible pair of actions if $2k \leq m$ where $k \geq 0$. $\square$

By using Proposition 4.3, the number of compatible pairs of nontrivial actions between two cyclic 2-groups with the same order of actions of the same order are determined. The result is given in Proposition 4.4.

Proposition 4.4

Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 4$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$.

i. If $k = 0$, then the number of compatible pair of actions is one.

ii. If $k = 1$, then the number of compatible pairs of actions is nine.

iii. If $k \geq 2$, then the number of compatible pairs of actions is 2^{2k-2}.

Proof

Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 4$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$.

i. Let $k = 0$. There is one automorphism of order one for each σ and σ'. Thus, there is one compatible pair of actions only.

ii. Let $k = 1$. There are three actions that have order two and by Proposition 4.3, all actions are always compatible with both actions that have order two. Thus, there are nine compatible pairs of actions.

iii. Let $k \geq 2$. By Proposition 3.4, there exist 2^k automorphisms of order 2^k and only half of them are considered by Theorem 3.4. Then, the particular number of automorphisms under this case are 2^{k-1}. Furthermore, by Proposition 4.3, (σ, σ') is a compatible pair of actions if $k + k \leq m$. Thus, $2^{k-1} \cdot 2^{k-1} = 2^{2k-2}$ is the number of compatible pairs of actions that have order 2^k when $k \geq 2$.

$\square$

Particularly, the number of compatible pairs of actions when both actions have order greater than two for 2-cyclic groups of the same order with the same order of actions is given as follows:

Lemma 4.4

Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 4$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k \geq 2$. The total number of compatible pairs of actions (σ, σ') is $\sum_{k=2}^{m-3} 2^{2k-2}$.

Proof

Let $G = H = \langle g \rangle \cong C_{2^m}$ be cyclic groups, $m \geq 4$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k \geq 2$. By Proposition 4.4, there are 2^{2k-2} compatible pairs of actions when $k \geq 2$. Consider the highest order of σ in $\mathrm{Aut}(C_{2^m})$ is 2^{m-3}. Hence, the number of compatible pairs of actions is

$$2^{2(2)-2} + 2^{2(3)-2} + 2^{2(4)-2} + \cdots + 2^{2(m-3)-2} = \sum_{k=2}^{m-3} 2^{2k-2}$$

Therefore, the number of compatible pairs of actions is $\sum_{k=2}^{m-3} 2^{2k-2}$. $\square$

As a result, the total number of compatible pairs of actions for two same cyclic 2-groups with the same order of actions is stated in the following theorem.

Theorem 4.3

Let $G = H = \langle g \rangle \cong C_{2^m}$ be groups, $m \geq 4$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k = 0, 1, \ldots, 2^{m-3}$. Then, there exist

$$\sum_{k=2}^{m-3} 2^{2k-2} + 10$$

compatible pairs of actions (σ, σ').

Proof

Let $G = H = \langle g \rangle \cong C_{2^m}$ be groups, $m \geq 4$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma| = |\sigma'| = 2^k$ for $k = 0, 1, \ldots, 2^{m-3}$. By Proposition 4.4, there are three cases represented by $k = 0$, $k = 1$ and $k \geq 2$. Thus, three cases are considered:

i. By Proposition 4.4(i), only one compatible pairs of actions exists when $k = 0$.

ii. By Proposition 4.4(ii), there are nine compatible pairs of actions when $k = 1$.

iii. When $k \geq 2$, the total number of compatible pairs of actions is $\sum_{k=2}^{m-3} 2^{2k-2}$ by Lemma 4.4.

Therefore, the number of compatible pairs of actions for two cyclic 2-groups with the same order of actions is $1 + 9 + \sum_{k=2}^{m-3} 2^{2k-2} = \sum_{k=2}^{m-3} 2^{2k-2} + 10$. □

Next, a corollary that gives the presentations of all automorphisms of the cyclic 2-groups with specific order is given as below.

Corollary 4.1

Let $G = \langle g \rangle \cong C_{2^n}$, $n \geq 4$. Every $\sigma \in \mathrm{Aut}(G)$ can be represented as $\sigma = \tau^i \rho^j$ with $i = 0, 1$ and $j = 0, 1, \ldots, 2^{n-2} - 1$ and $\tau^i \rho^j (g) = g^t$ where $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$.

i. If $i = 0$ and $j = 0$, then $t \equiv 1 \pmod{2^n}$ and $|\sigma| = 1$.

ii. If $i = 1$ and $j = 0$, then $t \equiv -1 \pmod{2^n}$ and $|\sigma| = 2$.

iii. If $i = 0$ and $j = 2^{n-3}$, then $t \equiv 2^{n-1} - 1 \pmod{2^n}$ and $|\sigma| = 2$.

iv. If $i = 1$ and $j = 2^{n-3}$, then $t \equiv 2^{n-1} + 1 \pmod{2^n}$ and $|\sigma| = 2$.

v. If $i = 0$ and $j \neq 2^{n-3}$ and $j \neq 0$, then $|\sigma| > 2$.

vi. If $i = 1$ and $j \neq 2^{n-3}$ and $j \neq 0$, then $|\sigma| > 2$.

Proof

By Theorem 3.2, $\sigma \in \mathrm{Aut}(G)$ can be represented as $\sigma = \tau^i\rho^j$ with $i = 0, 1$ and $j = 0, 1, \ldots, 2^{n-2} - 1$ and $\tau^i\rho^j(g) = g^t$ where $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$. Let $i = 0$ and $j = 0$. Then, $t \equiv (-1)^0 \cdot 5^0 \pmod{2^n} \equiv 1 \pmod{2^n}$ which gives $|\sigma| = 1$. Assume σ be an automorphisms of order two. By Corollary 3.5, there exist three automorphisms of order two which are $t \equiv -1 \pmod{2^n}$ when $i = 1$ and $j = 0$, $t \equiv 2^{n-1} - 1 \pmod{2^n}$ when $i = 0$ and $j = 2^{n-3}$ and $t \equiv 2^{n-1} + 1 \pmod{2^n}$ when $i = 1$ and $j = 2^{n-3}$. Then, by letting $i = 0$ or $i = 1$ with $j \neq 2^{n-3}$ and $j \neq 0$, the automorphisms have order greater than two. $\square$

4.5 Conclusion

The number of compatible pairs of actions for two cyclic 2-groups have been computed. From the results, the number of the compatible pairs of actions between two cyclic 2-groups,C_{2^m} and C_{2^n} is $4 + (m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1}$ where $m \geq 4, n \geq 3$ provided $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$. Next, the total number of compatible pairs of actions for same cyclic 2-groups with the same order of actions is $\sum_{k=2}^{m-3} 2^{2k-2} + 10$ where k is the order of actions exist in $\mathrm{Aut}(C_{2^m})$. The results in this chapter will be used in the next chapter in introducing a new graph, namely the compatible action graph.

CHAPTER 5

THE COMPATIBLE ACTION GRAPH

5.1 Introduction

In this chapter, a new graph namely the compatible action graph is introduced by extending the results on the compatible pairs of actions for cyclic 2-groups. Some properties of compatible action graphs and special types of compatible action graphs are investigated.

5.2 Motivation of Compatible Action Graph

The idea which made us investigate the compatible action graph for subgroups of the cyclic 2-groups is that, usually we think that the compatible pairs of actions for the subgroup H from the group G should exist in the group G, but it is not necessary. The following example is given to show that there are compatible pairs of actions which exist in the subgroup $C_{2^4} \otimes C_{2^4}$ but not in the group $C_{2^5} \otimes C_{2^5}$

Example 5.1

Let $G = \langle g \rangle \cong C_{2^5}$ and $H = \langle h \rangle \cong C_{2^5}$ be cyclic 2-groups. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$ be two actions such that $\sigma(g) = g^5$ and $\sigma'(h) = h^5$ or equivalently ${}^{g}h = h^5$ and ${}^{h}g = g^5$.

Then, for the first compatibility condition,

$$\begin{aligned} {}^{{}^{h}g}h &= {}^{g^5}h \\ &= h^{5^5} \\ &= h^{21} \quad \text{since} \quad 5^5 \equiv 21 \pmod{2^5} \\ &\neq {}^{g}h. \end{aligned}$$

Thus, (σ, σ') is not compatible in $C_{2^5} \otimes C_{2^5}$

Now, let $G = \langle g \rangle \cong C_{2^4}$ and $H = \langle h \rangle \cong C_{2^4}$ be cyclic 2-groups. Furthermore, let $\sigma(g) = g^5$ and $\sigma'(h) = h^5$ or equivalently ${}^{g}h = h^5$ and ${}^{h}g = g^5$. Then, for the first compatibility condition,

$$\begin{aligned} {}^{{}^{h}g}h &= {}^{g^5}h \\ &= h^{5^5} \\ &= h^{5} \quad \text{since} \quad 5^5 \equiv 5 \pmod{2^4} \\ &= {}^{g}h. \end{aligned}$$

Similarly, the second compatibility conditions is also satisfied. Thus, (σ, σ') is compatible in $C_{2^4} \otimes C_{2^4}$

Generally, Example 5.2 illustrated the idea and the intersection between the group and the subgroup.

Example 5.2

Let $G \cong C_{2^5}$ be a cyclic 2-groups. Furthermore, let $H \cong C_{2^4}$ be a subgroup of G. For the nonabelian tensor product of the subgroup, $C_{2^4} \otimes C_{2^4}$, there are three compatible pairs of actions (σ, σ'), which are (g^5, h^5), (g^7, h^7) and (g^{15}, h^{15}) are compatible in $C_{2^4} \otimes C_{2^4}$ but not in $C_{2^5} \otimes C_{2^5}$.

However, there are some compatible pairs of actions (σ, σ') such as $(g^5, h^9), (g^9, h^5), (g^9, h^9), (g^9, h^{13})and(g^{13}, h^9)$ are compatible in $C_{2^4} \otimes C_{2^4}$ and also

$C_{2^5} \otimes C_{2^5}$. Thus, all of the compatible pairs of actions represented as intersection between both nonabelian tensor product.

Therefore, to find the number of compatible pairs of actions that exist in group G as well as its subgroup H, the compatible action graph has been introduced.

5.3 Properties of Compatible Action Graph

In this section, a graph namely compatible action graph is introduced and its properties has been studied. The definition of compatible action graph is given as follows.

Definition 5.1 : Compatible Action Graph

Let G and H be finite cyclic 2-groups and (σ, σ') be a pair of compatible actions for the nonabelian tensor product $G \otimes H$, where $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$. Then,

$$\Gamma_{G\otimes H} = (\ V(\Gamma_{G\otimes H}),\ E(\Gamma_{G\otimes H}))$$

is a compatible action graph with the set of the vertices $V(\Gamma_{G\otimes H})$, which is the union of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$, and the set of edges, $E(\Gamma_{G\otimes H})$ that connect these vertices which is the set of all compatible pairs of actions (σ, σ'). That is

$$V(\Gamma_{G\otimes H}) = \begin{cases} \mathrm{Aut}(G) \cup \mathrm{Aut}(H) & \text{if } G \neq H \\ \mathrm{Aut}(G) & \text{if } G = H. \end{cases}$$

Furthermore, the vertices σ and σ' are adjacent if they are compatible.

Next, the order of a compatible action graph for the cyclic 2-groups are studied. By Definition 3.9, the order of Γ is defined as the cardinality of the vertex set of Γ. Since the vertices of the compatible action graph are elements of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$, then the order of the compatible action graph is the number of automorphisms of G and H. Therefore, $|\Gamma_{G\otimes H}| = |V(\Gamma_{G\otimes H})|$.

The order of the compatible action graph is considered for two cases namely $m \neq n$ and $m = n$. Thus, the following proposition gives the order of the compatible action graph for the cyclic 2-groups.

Proposition 5.1

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, the order of the compatible action graph of G and H is

i. $|\Gamma_{G \otimes H}| = 2^{m-1} + 2^{n-1}$ if $m \neq n$.

ii. $|\Gamma_{G \otimes H}| = 2^{m-1}$ if $m = n$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. By Definition 3.9, $|\Gamma_{G \otimes H}| = |V(\Gamma_{G \otimes H})|$. Furthermore, by Definition 5.1, $V(\Gamma_{G \otimes H})$ is the set of Aut(G) and Aut(H). There are two cases which are $m \neq n$ and $m = n$ need to be considered.

i. Let $m \neq n$ and note that $|V(\Gamma_{G \otimes H})| = |\text{Aut}(G)| + |\text{Aut}(H)|$ where $|\text{Aut}(G)| = 2^{m-1}$ and $|\text{Aut}(H)| = 2^{n-1}$. Then, $|\Gamma_{G \otimes H}| = 2^{m-1} + 2^{n-1}$.

ii. Let $m = n$. Without loss of generality, let $|G| = |H| = 2^m$. Since $|V(\Gamma_{G \otimes H})| = |\text{Aut}(G)|$ and $|\text{Aut}(G)| = 2^{m-1}$. Then, $|\Gamma_{G \otimes H}| = 2^{m-1}$.

□

The compatible action graph of cyclic 2-groups is a directed graph since an action of G on H is a mapping $\Phi : G \rightarrow \text{Aut}(H)$. The compatible action graph may contain a loop and the loop is only present for the case $G = H$. Thus, the loop contributes one to both of the in-degree and the out-degree of the vertex. The compatible action graph may not contain multiple directed edges because the presentation of automorphism are not repeated given by the necessary and sufficient conditions of the compatible on each other for cyclic 2-groups.

The following proposition gives the cardinality of the edge set for the compatible action graph for the cyclic 2-groups.

Proposition 5.2

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then,

$$|E(\Gamma_{G\otimes H})| = (m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1} + 4$$

where $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. By Definition 5.1, the $E(\Gamma_{G\otimes H}))$ is a nonempty set of all pairs of (σ, σ') . Thus, by Theorem 4.2, $|E(\Gamma_{G\otimes H})| = (m-3)(2^{r'-1}) + 2^{r-1} + 2^{m-1} + 2^{n-1} + 4$ provided $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$. □

In the terminology of graphs, directed edges reflect the fact that the edges in a directed graph have directions. The beginning vertex of direction is called the initial vertex and the ending vertex is the terminal vertex. Thus, (σ, σ') can be defined as the edge of the compatible action graph with the directed edge and by Definition 3.6, the vertex σ is considered as initial vertex and σ' as terminal vertex. In addition, σ is to be adjacent to σ' and σ' is to be adjacent from σ.

The following proposition presented the out-degree of vertex v for compatible action graph.

Proposition 5.3

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $v \in V(\Gamma_{G\otimes H})$ where $v \in \mathrm{Aut}(G)$ and $v(g) = g^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^m}$ where $i = 0, 1$ and $j = 0, 1, \ldots, 2^{m-2} - 1$ and $|v| = 2^s, s = 0, 1, \ldots, m-2$. Then, $\deg^+(v)$ is exactly one of the following

i. 2^{n-1} if $i = 0$ and $j = 0$.

ii. 4 if $i = 0$ or $i = 1$ and $j = 2^{m-3}$

iii. 2^{r-1} if $i = 1$ and $j = 0$ provided $r = \min\{m+1, n\}$.

iv. 2^s if $i = 1$, $j \neq 0$ and $j \neq 2^{m-3}$.

v. $2^{r'-1}$ if $i = 0$, $j \neq 0$ and $j \neq 2^{m-3}$ provided $r' = \min\{m, n\}$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $v \in V(\Gamma_{G \otimes H})$ where $v \in \mathrm{Aut}(G)$ and $v(g) = g^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^m}$ where $i = 0, 1$ and $j = 0, 1, \ldots, 2^{m-2} - 1$ and $|v| = 2^s, s = 0, 1, \ldots, m-2$. There are five cases according to the $\deg^+(v)$.

i. Let $i = 0$ and $j = 0$, then the action is trivial. By Proposition 4.1, the action is compatible when the action of H on G is trivial. Thus, $\deg^+(v) = 2^{n-1}$.

ii. By Proposition 4.2(i), there are eight compatible pairs of actions. Let $i = 0$ and $j = 2^{m-3}$, then there are four compatible pairs of actions. The number of compatible pairs of actions is the same when $i = 1$ and $j = 2^{m-3}$. Particularly, the compatible pairs of actions are four for $i = 0$ and four for $i = 1$. Therefore, $\deg^+(v) = 4$ where $i = 0$ or $i = 1$ and $j = 2^{m-3}$.

iii. By Proposition 4.2(ii), there are 2^{r-1} compatible pairs of actions for $t \equiv 2^{m-1} + 1 \pmod{2^m}$ provided $r = \min\{m+1, n\}$. Thus, $\deg^+(v) = 2^{r-1}$ where $i = 1$ and $j = 0$.

iv. By Lemma 4.1, the number of compatible pairs of actions is 2^s. Therefore, $\deg^+(v) = 2^s$ when $i = 1$, $j \neq 0$ and $j \neq 2^{n-3}$.

v. By Lemma 4.2, there are $2^{r'-1}$ number of the compatible pairs of actions where $i = 0$, $j \neq 0$ and $j \neq 2^{m-3}$ provided $r' = \min\{m, n\}$. Thus, the $\deg^+(v) = 2^{r'-1}$.

$\square$

By Definition 3.4, the graph with directed edges the in-degree of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. In the next proposition, the in-degree of vertex v is presented.

Proposition 5.4

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $v \in V(\Gamma_{G \otimes H})$ where $v \in \text{Aut}(H)$ and $v(h) = h^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$ where $i = 0, 1$ and $j = 0, 1, \ldots, 2^{n-2} - 1$ and $|v| = 2^s, s = 0, 1, \ldots, n-2$. Then, $\deg^-(v)$ is exactly one of the following

i. 2^{m-1} if $i = 0$ and $j = 0$.

ii. 4 if $i = 0$ or $i = 1$ and $j = 2^{n-3}$.

iii. 2^{r-1} if $i = 1$ and $j = 0$ provided $r = \min\{m+1, n\}$.

iv. $2^{s'}$ if $i = 1$, $j \neq 0$ and $j \neq 2^{n-3}$.

v. $2^{r'-1}$ if $i = 0$, $j \neq 0$ and $j \neq 2^{n-3}$ provided $r' = \min\{m, n\}$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $v \in V(\Gamma_{G \otimes H})$ where $v \in \text{Aut}(H)$ and $v(h) = h^t$ with $t \equiv (-1)^i \cdot 5^j \pmod{2^n}$ where $i = 0, 1$ and $j = 0, 1, \ldots, 2^{m-2} - 1$ and $|v| = 2^s, s = 0, 1, \ldots, m-2$. There are five cases according to $\deg^-(v)$.

i. Let $i = 0$ and $j = 0$, the action is trivial. By Proposition 4.1, the action is compatible when the action of G on H is trivial. Thus, $\deg^-(v) = 2^{m-1}$.

ii. By Proposition 4.2(i), there are eight compatible pairs of actions. Let $i = 0$ and $j = 2^{n-3}$, then there are four compatible pairs of actions. The number of compatible pairs of actions is the same when $i = 1$ and $j = 2^{n-3}$. Particularly, the compatible pairs of actions are four for $i = 0$ and four for $i = 1$. Therefore, $\deg^-(v) = 4$ where $i = 0$ or $i = 1$ and $j = 2^{n-3}$.

iii. By Proposition 4.2(ii), there are 2^{r-1} compatible pairs of actions provided $r = \min\{m+1, n\}$. Thus, $\deg^-(v) = 2^{r-1}$ where $i = 1$ and $j = 0$.

iv. By Lemma 4.1, the number of the compatible pairs of actions is $2^{s'}$. Thus, $\deg^-(v) = 2^{s'}$ where $i = 1$, $j \neq 0$ and $j \neq 2^{n-3}$.

v. By Lemma 4.2, given that there are $2^{r'-1}$ number of the compatible pairs of actions where $i = 0$, $j \neq 0$ and $j \neq 2^{n-3}$ provided $r' = \min\{m, n\}$. Thus, $\deg^-(v) = 2^{r'-1}$.

$\square$

Particulary, the following corollary shows that the out-degree of vertex v and the in-degree of vertex v are equal for the compatible action graph when $G = H$.

Corollary 5.1

Let $G \cong C_{2^m}$ be a cyclic group where $m \geq 4$. Then $\deg^-(v) = \deg^+(v)$ for $\Gamma_{G\otimes G}$.

Proof

Let $G \cong C_{2^m}$ be a cyclic group where $m \geq 4$. By Propositions 5.3 and 5.4, the $\deg^-(v) = \deg^+(v)$ for any $v \in V(\Gamma_{G\otimes G})$. $\square$

In the next section, properties of compatible action graph such as connectivity are given. In addition, some results on the special types of graph such as the bipartite graph and complete graph are also provided.

5.4 Types of Compatible Action Graphs

The previous section gives some properties of compatible action graphs according to vertices and edges only. In this section, the connectivity, bipartite and complete of compatible action graph are provided.

First, the connectivity of compatible actions graphs is investigated. A compatible action graph is connected when there is a path between any pair of vertices. The connectivity of the compatible action graph for the cyclic 2-groups is given in the following theorem.

Theorem 5.1

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then,

$\Gamma_{G\otimes H}$ is a connected graph.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 1$. Furthermore, let $v_1 \in V(\Gamma_{G\otimes H})$ with $v_1 \in \text{Aut}(G)$ and $\deg^-(v_1) = 2^{n-1}$. Then, v_1 is compatible with every $v_2 \in \text{Aut}(H)$ since $|\text{Aut}(H)| = 2^{n-1}$.

Next, let $v_2 \in V(\Gamma_{G\otimes H})$ with $v_2 \in \text{Aut}(H)$ and $\deg^+(v_2) = 2^{m-1}$. Then, v_2 is compatible with every $v_1 \in \text{Aut}(G)$ since $|\text{Aut}(G)| = 2^{m-1}$. Thus, $\Gamma_{G\otimes H}$ is a connected graph. $\square$

The above result shows that the compatible action graph is a connected graph. Now, assume that $G \neq H$. Then, the compatible action graph has the property where the vertex can be partitioned into two disjoints sets namely V_1 and V_2 or equivalently the graph is bipartite. This result is provided in the next theorem.

Theorem 5.2

Let $G \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, $\Gamma_{G\otimes H}$ is a bipartite graph if and only if $m \neq n$.

Proof

Let $G \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. First, we need to show that if $\Gamma_{G\otimes H}$ is bipartite, then $m \neq n$. By using contradiction method, assume that $\Gamma_{G\otimes H}$ is bipartite and $m = n$ is true. Assume that $m = n$, then $\text{Aut}(C_{2^m}) = \text{Aut}(C_{2^n})$. Thus, there exist a loop which cannot be partitioned into two disjoint sets, which contradicts on the assumption. Therefore, $m \neq n$.

Next, let $m \neq n$. By definition of compatible pairs of actions, if $m \neq n$, then any $v \in \text{Aut}(C_{2^m})$ only compatible with some $v' \in \text{Aut}(C_{2^n})$. Thus, clearly it can be partitioned into two disjoint sets $\text{Aut}(C_{2^m})$ and $\text{Aut}(C_{2^n})$ respectively. Therefore, $\Gamma_{G\otimes H}$ is a bipartite graph. $\square$

Next, the special type of graph, which is called as a complete graph is investigated. The complete graph contains exactly one edge between each pair of vertices. The result shows the compatible action graphs is not the complete graph. The result is given as follows

Theorem 5.3

Let $G \cong C_{2^m}$ and $H \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Then, $\Gamma_{G \otimes H}$ is not a complete graph.

Proof

By Proposition 5.2, the number of vertices for compatible action graphs is $2^{m-1} + 2^{n-1}$. If a compatible action graph is a complete graph, the number of edges is $[2^{m-1} + 2^{n-1}]^2$. By Proposition 5.1, there exist $|E(\Gamma_{G \otimes H})| = 2^{n-1} + 2^{r-1} + 4 + 2^{m-1} + (m-3)\, 2^{r'-1}$ edges where $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$. Thus,

$$|E(\Gamma_{G \otimes H})| = 2^{n-1} + 2^{r-1} + 4 + 2^{m-1} + (m-3)\, 2^{r'-1} \leq [2^{m-1} + 2^{n-1}]^2.$$

Therefore, the compatible action graph is not a complete graph. □

5.5 Conclusion

In this chapter, the compatible action graph is introduced. Consequently, some properties of the compatible actions graph are stated such that the cardinality of the edge set, order of a compatible action graph, the number of directed edges in the in-degree and out-degree of a vertex and the connectivity, bipartite and the complete graph of compatible action graph are given. In the next chapter, a subgraph of the compatible action graphs of the cyclic 2-groups are investigated.

CHAPTER 6

SOME SUBGRAPHS OF THE COMPATIBLE ACTION GRAPH

6.1 Introduction

Let C_{2^m} and C_{2^n} be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $C_{2^{m-i}}$ and $C_{2^{n-i}}$ be two subgroups of C_{2^m} and C_{2^n} where $i = 1, 2, \ldots, \min\{m, n\} - 2$. Then, the subgraph of compatible action graph has been defined for the subgroups $C_{2^{m-i}}$ and $C_{2^{n-i}}$ by reducing two to the power of i of the order of the groups C_{2^m} and C_{2^n}. In addition, the necessary and sufficient conditions for the cyclic 2-subgroups act on each other in a compatible way are also given. Next, the intersection between the compatible action graph and its subgraph which are $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ are found. Some of the properties of the corresponding subgraphs of the compatible action graph are provided.

6.2 Compatibility Conditions for Subgroups of Cyclic 2-Groups

In this section, the necessary and sufficient conditions for the subgraphs of cyclic 2-groups act on each other in a compatible way for $C_{2^{m-i}} \otimes C_{2^{n-i}}$ for $i = 1, 2, \ldots, \min\{m, n\} - 2$ is given. Throughout this section, the reduction of 2-power order or i is the same for both groups.

The following proposition gives necessary and sufficient conditions for $C_{2^{m-i}}$ and $C_{2^{n-i}}$ to act compatibly on each other.

Proposition 6.1

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let (σ, σ') be a compatible pair of actions for $C_{2^m} \otimes C_{2^n}$ where $\sigma(g) = g^k$ and $\sigma'(h) = h^l$ with k and l are odd integers. Then, (σ, σ') is a compatible pair of actions for $C_{2^{m-i}} \otimes C_{2^{n-i}}$ where $\sigma(g) = g^{k \pmod{2^{m-i}}}$ and $\sigma'(h) = h^{l \pmod{2^{n-i}}}$ with $i = 1, 2, \ldots, \min\{m, n\} - 2$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let (σ, σ') be a compatible pair of actions for $C_{2^m} \otimes C_{2^n}$ where $\sigma(g) = g^k$ and $\sigma'(h) = h^l$ with k and l are odd integers. Without loss of generality, let $C_{2^{m-i}} \leq C_{2^m}$, $C_{2^{n-i}} \leq C_{2^n}$ and $C_{2^{m-i}} = \langle g' \rangle$, $C_{2^{n-i}} = \langle h' \rangle$ where $g' \in G$ and $h' \in H$. Since (σ, σ') is a compatible pair of actions for $C_{2^m} \otimes C_{2^n}$, then there exist mutual actions of G and H on each other such that ${}^h g = g^k$ and ${}^g h = h^l$. By Proposition 3.7, in order to show $\sigma(g) = g^{k \pmod{2^{m-i}}}$ and $\sigma'(h) = h^{l \pmod{2^{n-i}}}$ is compatible pair of actions for $C_{2^{m-i}} \otimes C_{2^{n-i}}$, there are three conditions need to be satisfied as well.

By Proposition 3.7(i), the first condition that needs to be shown is $\gcd(k, 2^{m-i}) = \gcd(l, 2^{n-i}) = 1$. Define $\sigma : G \to G$ by $\sigma(g) = g^k$, then σ is an automorphism if and only if $\gcd(k, 2^m) = 1$. Since k is an odd number and 2^{m-i} is an even number because it is a 2-power, therefore $\gcd(k, 2^{m-i}) = 1$. Similarly, there exists a mutual action of G on H such that ${}^g h = h^l$. Since $\gcd(l, 2^n) = 1$, then $\gcd(l, 2^{n-i}) = 1$. Hence, $\gcd(k, 2^{m-i}) = \gcd(l, 2^{n-i}) = 1$ and the first condition is satisfied.

By Proposition 3.7(ii), the second condition that needs to be satisfied are $k^{2^{n-i}} \equiv 1 \pmod{2^{m-i}}$ and $l^{2^{m-i}} \equiv 1 \pmod{2^{n-i}}$. There exist mutual action of H on G such that ${}^h g = g^k$, then $g = {}^{1_H} g = {}^{h^{2^{n-i}}} g = g^{k^{2^{n-i}}}$. Thus, $k^{2^{n-i}} \equiv 1 \pmod{2^{m-i}}$. Similarly, there exist mutual action of G on H such that ${}^g h = h^l$, then $l^{2^{m-i}} \equiv 1 \pmod{2^{n-i}}$. Hence, the second condition is satisfied.

By Proposition 3.7(iii), the third condition that needs to be considered is $k^{l-1} \equiv 1 \pmod{2^{m-i}}$ and $l^{k-1} \equiv 1 \pmod{2^{n-i}}$. By Proposition 3.6, G and H act compatibly on each other if and only if ${}^{({}^{g}h)}g = {}^{h}g$ and ${}^{({}^{h}g)}h = {}^{g}h$. From the first relation, ${}^{({}^{g}h)}g = {}^{h^{l}}g = g^{k^{l}}$ and ${}^{h}g = g^{k}$. Thus, $k^{l} \equiv k \pmod{2^{m-i}}$ or equivalently $k^{l-1} \equiv 1 \pmod{2^{m-i}}$. Similarly, for the second relations, $l^{k} \equiv l \pmod{2^{n-i}}$ or equivalently $l^{k-1} \equiv 1 \pmod{2^{n-i}}$. Hence, the third condition is satisfied.

Conclusively, $(g^{k \bmod 2^{m-i}}, h^{l \bmod 2^{n-i}})$ is a compatible pair of actions for $C_{2^{m-i}} \otimes C_{2^{n-i}}$. □

The properties for a subgraph of the compatible action graph are given in the next section.

6.3 Properties of a Subgraph of the Compatible Action Graph

In this section, the number of the compatible pairs of actions for the intersection between two subgroups can be presented as the intersection of the compatible action graph and its subgraph has been determined. The properties of a subgraph of compatible actions graph on the order of $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ are first investigated. By Proposition 5.1, the order of $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ is considered for two cases namely $m \neq n$ and $m = n$. Thus, the following proposition gives the order of $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ for both cases.

Proposition 6.2

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $i = 1, 2, \ldots, \min\{m, n\} - 2$. Then, $|\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}| = \frac{1}{2^i} |\Gamma_{C_{2^m} \otimes C_{2^n}}|$.

Proof

Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups where $m \geq 4, n \geq 3$. Furthermore, let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $i = 1, 2, \ldots, \min\{m, n\} - 2$.

By Proposition 5.1, the two cases are considered namely $m = n$ and $m \neq n$. Thus,

i. Suppose that $m \neq n$. By Proposition 5.1(i), $|\Gamma_{C_{2^m} \otimes C_{2^n}}| = 2^{m-1} + 2^{n-1}$. Thus,

$$\begin{aligned}|\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}| &= 2^{m-i-1} + 2^{n-i-1} \\ &= \frac{1}{2^i}(2^{m-1} + 2^{n-1}) \\ &= \frac{1}{2^i}|\Gamma_{C_{2^m} \otimes C_{2^n}}|.\end{aligned}$$

ii. Suppose that $m = n$. By Proposition 5.1(ii), $|\Gamma_{C_{2^m} \otimes C_{2^n}}| = 2^{m-1}$. Thus,

$$\begin{aligned}|\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}| &= 2^{m-i-1} \\ &= \frac{1}{2^i}(2^{m-1}) \\ &= \frac{1}{2^i}|\Gamma_{C_{2^m} \otimes C_{2^n}}|.\end{aligned}$$

In conclusion, $|\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}| = \dfrac{1}{2^i}|\Gamma_{C_{2^m} \otimes C_{2^n}}|$ for both cases. □

The cardinality of the edge set needs to be found in order to investigate some properties of $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$. By Proposition 5.3, there are five cases depend on $\deg^+(v)$ which need to be considered.

Case I: $\deg^+(v)$ is 2^{n-1} if $i = 0$ and $j = 0$.

Lemma 6.1

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ such that $\deg^+(v) = 2^{n-1}$ and $|v| = 1$. Then, $|E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| = \dfrac{1}{2^i}(2^{n-1})$.

Proof

Let v be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ such that $\deg^+(v) = 2^{n-1}$. By Proposition 4.1, there exist 2^{n-i-1} number of compatible pairs of actions for trivial actions. Thus,

$$\begin{aligned}|E(\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}) \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| &= 2^{n-i-1} \\ &= \frac{1}{2^i}(2^{n-1}).\end{aligned}$$

Hence, $|E(\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}) \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}})| = \frac{1}{2^i}(2^{n-1})$ when $\deg^+(v) = 2^{n-1}$. □

Case II: $\deg^+(v)$ is 4 if $i = 0$ or $i = 1$ and $j = 2^{m-3}$.

Lemma 6.2

Let $\Gamma_{C_{2^m}\otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m}\otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ such that $\deg^+(v) = 4$ and $|v| = 2$. Then, $|E(\Gamma_{C_{2^m}\otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}})| = 2$.

Proof

Let $\Gamma_{C_{2^m}\otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m}\otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ such that $\deg^+(v) = 4$ and $|v| = 2$. Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups. Then, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma| = 2$ and $\sigma' \in \mathrm{Aut}(H)$. By Proposition 3.5, there are three automorphisms of order two. However, only two automorphisms are covered in this case, $\sigma(g) = g^t$ where $t \equiv 2^{m-1} - 1 \pmod{2^m}$ or $t \equiv -1 \pmod{2^m}$.

Assume that $\sigma(g) = g^{2^{m-1}-1 \pmod{2^m}}$ and $\sigma(g) = g^{-1 \pmod{2^m}}$. By Theorem 3.3, the actions are compatible with $|\sigma'| = 2$ where $\sigma'(h) = h^{t'}$ with $t' \equiv 2^{n-1} + 1 \pmod{2^n}$, $t' \equiv 2^{n-1} - 1 \pmod{2^n}$ or $t' \equiv -1 \pmod{2^n}$ and $|\sigma'| = 1$. Thus, (σ, σ') is compatible pairs of actions for $C_{2^m} \otimes C_{2^n}$.

By Proposition 6.1, there are some of compatible pairs of actions that compatible in $C_{2^m} \otimes C_{2^n}$ but not compatible in $C_{2^{m-i}} \otimes C_{2^{n-i}}$ because of $\sigma(g) = g^{-1 \pmod{2^m}) > 2^{m-i}}$, $\sigma' = h^{-1 \pmod{2^n}) > 2^{n-i}}$ and $\sigma' = h^{2^{n-1}+1 \pmod{2^n}) > 2^{n-i}}$. A summary of intersection between two compatible action graphs for $\deg^+(v) = 4$ are illustrated in Table 6.1.

Table 6.1: The intersection between two compatible action graphs for $\deg^{+}(v) = 4$

$C_{2^m} \otimes C_{2^n}$	$C_{2^{m-i}} \otimes C_{2^{n-i}}$	Intersection
$(2^{m-1} - 1 \pmod{2^m}, 2^{m-1} - 1 \pmod{2^m})$	$(2^{m-1} - 1 \pmod{2^{m-i}}, 2^{m-1} - 1 \pmod{2^{m-i}})$	No
	$(2^{m-1} - 1 \pmod{2^{m-i}}, 2^{m-1} + 1 \pmod{2^{m-i}})$	No
	$(2^{m-1} - 1 \pmod{2^{m-i}}, -1 \pmod{2^{m-i}})$	No
	$(2^{m-1} - 1 \pmod{2^{m-i}}, 1 \pmod{2^{m-i}})$	No
	$(-1 \pmod{2^{m-i}}, 2^{m-1} - 1 \pmod{2^{m-i}})$	No
	$(-1 \pmod{2^{m-i}}, 2^{m-1} + 1 \pmod{2^{m-i}})$	No
	$(-1 \pmod{2^{m-i}}, -1 \pmod{2^{m-i}})$	Yes
	$(-1 \pmod{2^{m-i}}, 1 \pmod{2^{m-i}})$	Yes

Therefore, there exists two edges in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ which are $(g^{-1 \pmod{2^{m-i}}}, h^{-1 \pmod{2^{n-i}}})$ and $(g^{-1 \pmod{2^{m-i}}}, h^{1 \pmod{2^{n-i}}})$. □

Case III: $\deg^{+}(v)$ is 2^{r-1} if $i = 1$ and $j = 0$ provided $r = \min\{m + 1, n\}$.

Lemma 6.3

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ such that $\deg^{+}(v) = 2^{r-1}$ provided $r = \min\{m + 1, n\}$ and $|v| = 2$. Then, $|E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| = 0$.

Proof

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let v be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ such that $\deg^{+}(v) = 2^{r-1}$ and $|v| = 2$. Let $G = \langle g \rangle \cong C_{2^m}$ and $H = \langle h \rangle \cong C_{2^n}$ be cyclic groups. Then, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma| = 2$ and $\sigma' \in \mathrm{Aut}(H)$.

By Proposition 3.5, there are three automorphisms of order two. However, only one automorphism are covered in this case, $\sigma(g) = g^t$ where $t \equiv 2^{m-1} + 1 \pmod{2^m}$.

By Theorem 3.3, the actions are compatible with $|\sigma'| = 2$ where $\sigma' = h^{t'}$ with $t' \equiv 2^{n-1} + 1 \pmod{2^n}$, $t' \equiv 2^{n-1} - 1 \pmod{2^n}$ or $t' \equiv -1 \pmod{2^n}$ and $|\sigma'| = 1$. Thus, (σ, σ') is compatible pairs of actions for $C_{2^m} \otimes C_{2^n}$.

By Proposition 6.1, there are some of compatible pairs of actions that compatible in $C_{2^m} \otimes C_{2^n}$ but not compatible in $C_{2^{m-i}} \otimes C_{2^{n-i}}$ because of $\sigma(g) = g^{2^{m-1}+1 \pmod{2^m} > 2^{m-i}}$. Therefore, there is no compatible pairs of actions for this case. □

Case IV: $\deg^+(v)$ is 2^s if $i = 1$, $j \neq 0$ and $j \neq 2^{m-3}$.

Lemma 6.4

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ where $\deg^+(v) = 2^s$ and $|v| = 2^s$ where $s = 2, 3, \ldots, m - 2$. Then, $|E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| = \frac{1}{2^{i+1}}(2^{m-1} - 2^{i+2} + (m - i - 3)2^{r'-1})$ provided $r' = \min\{m, n\}$.

Proof

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ where $\deg^+(v) = 2^s$ and $|v| = 2^s$ where $s = 2, 4, \ldots, m - 2$.

By Lemma 4.3, the cardinality of the edge set for compatible action graph is $(2^{m-1} - 4) + (m - 3)(2^{r'-1})$ provided $r' = \min\{m, n\}$. Thus,

$$\begin{aligned}
|E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| &= \frac{1}{2}(2^{m-i-1} - 4) + (m - i - 3)(2^{r'-i-1}) \\
&= \frac{1}{2^{i+1}}(2^{m-1} - 4(2^i) + (m - i - 3)2^{r'-1}) \\
&= \frac{1}{2^{i+1}}(2^{m-1} - 2^{i+2} + (m - i - 3)2^{r'-1}
\end{aligned}$$

where $s = 3, 4, \ldots, m-2$ provided $r' = \min\{m, n\}$. Nota that, $\min\{m-i, n-i\} = \min\{m, n\} - i = r' - i$. Hence the lemma have been proven. $\square$

Case V: $\deg^+(v)$ is $2^{r'-1}$ if $i = 0$, $j \neq 0$ and $j \neq 2^{m-3}$ provided $r' = \min\{m, n\}$.

Lemma 6.5

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ where $\deg^+(v) = 2^{r'-1}$ provided $r' = \min\{m, n\}$ and $|v| = 2^s$ where $s = i+1$. Then, $|E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| = \dfrac{1}{2^{i+1}}(2^{r'-1} + 2^{i+1})$.

Proof

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let v be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ where $\deg^+(v) = 2^{r'-1}$ provided $r' = \min\{m, n\}$ and $|v| = 2^s$ where $s = i+1$.

By Lemmas 4.1 and 4.2, the number of compatible pairs of actions is $2^{r'-1} + 2^s$ where $s = i+1$. Thus,

$$\begin{aligned} |E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| &= \frac{1}{2}(2^{r'-i-1} + (2^{s-i}) \\ &= \frac{1}{2^{i+1}}(2^{r'-1} + 2^s) \\ &= \frac{1}{2^{i+1}}(2^{r'-1} + 2^{i+1}). \end{aligned}$$

Hence, the lemma has been proven. $\square$

In general, the following theorem gives the cardinality of the edge set for the intersection between the compatible action graph and its subgraph.

Theorem 6.1

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ and $r' = \min\{m, n\}$. Then,

$$|E(\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}})| = \frac{1}{2^{i+1}}(2^{i+1} + 2^{m-1} + 2^n + (m-i-2)2^{r'-1}).$$

Proof

Let $\Gamma_{C_{2^m}\otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 2$. Furthermore, let v be a vertex in $\Gamma_{C_{2^m}\otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$. Lemmas 6.1, 6.2 6.3, 6.4, and 6.5 give the cardinality of the edge set with specific $\deg^+(v)$. Thus, the five cases are considered.

i. Suppose that $\deg^+(v) = 2^{n-1}$. By Lemma 6.1, the cardinality of the edge set is $\frac{1}{2^i}(2^{n-1})$.

ii. Suppose that $\deg^+(v) = 4$. By Lemma 6.2, the cardinality of the edge set is two.

iii. Suppose that $\deg^+(v) = 2^{r-1}$ provided $r = \min\{m+1, n\}$. By Lemma 6.3, there is no cardinality of the edge set.

iv. Suppose that $\deg^+(v) = 2^s$. By Lemma 6.4, the cardinality of the edge set is $\frac{1}{2^{i+1}}(2^{r'-1} + 2^{i+1})$.

v. Suppose that $\deg^+(v) = 2^{r'-1}$ provided $r' = \min\{m, n\}$. By Lemma 6.5, there exists $\frac{1}{2^{i+1}}(2^{m-1} - 2^{i+2} + (m-i-3)2^{r'-1})$ cardinality of the edge set.

Therefore,

$$
\begin{aligned}
&|E(\Gamma_{C_{2^m}\otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}})|\\
&= \frac{1}{2^i}(2^{n-1} + 2 + \frac{1}{2^{i+1}}(2^{m-1} + 2^{i+2} + (m-i-3)2^{r'-1}) + \frac{1}{2^{i+1}}(2^{r'-1} + 2^{i+1})\\
&= \frac{1}{2^{i+1}}(2^n + 2(2^{i+1}) + 2^{m-1} - 2^{i+2} + (m-i-3)2^{r'-1} + 2^{r'-1} + 2^{i+1})\\
&= \frac{1}{2^{i+1}}(2^{i+1} + 2^{m-1} + 2^n + (m-i-2)2^{r'-1})
\end{aligned}
$$

provided $r' = \min\{m, n\}$. Hence, the theorem has been proven. $\square$

The connectivity of $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ is given in the following theorem.

Theorem 6.2

Let $\Gamma_{C_{2^m}\otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 3$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m}\otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$. Then, $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}} \cap \Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ is a connected graph.

Proof

Let $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ be two compatible action graphs with $m \geq 4, n \geq 3$ and $i = 1, 2, \ldots, \min\{m, n\} - 3$. Furthermore, let $v \in \mathrm{Aut}(G)$ be a vertex in $\Gamma_{C_{2^m} \otimes C_{2^n}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ and given that $|v| = 1$. Then, v is identity automorphism and it is compatible with every other vertex in $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$. Therefore, $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}} \cap \Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ is a connected graph. $\square$

6.4 Conclusion

In this chapter, the intersection between the compatible action graph and its subgraph which are $\Gamma_{C_{2^m} \otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}} \otimes C_{2^{n-i}}}$ are discussed. The necessary and sufficient conditions for the cyclic 2-subgroups act on each other in a compatible way and some of the properties of the corresponding subgraphs of the compatible action graph are provided.

CHAPTER 7

SUMMARY AND CONCLUSION

7.1 Summary of the Research

This thesis begins with an introduction which contains research background, problem statement, research objectives, research scope and research significance.

Chapter 2 focuses on the literature review. All related research done by other researchers regarding nonabelian tensor products and graph theory were given.

Some definitions and preparatory results on the automorphism groups, compatibility conditions, graph theory and GAP algorithms were given in Chapter 3. By using the GAP, the number of compatible pairs of actions have been computed.

The number of compatible pairs of actions for the cyclic 2-groups have been computed in Chapter 4. Three cases were considered, when one of the actions has order one, two and greater than two. From the results, there exist $(m-3)(2^{r'-1}) + 2^{r-1} + 2^{n-1} + 2^{m-1} + 4$ compatible pairs of actions between C_{2^m} and C_{2^n} with $m \geq 4, n \geq 3$ where $r = \min\{m+1, n\}$ and $r' = \min\{m, n\}$. Then, the total number of the compatible pairs of actions for 2-cyclic groups of same order with same order of actions is $10 + \sum_{k=2}^{m-3} 2^{2k-2}$ where $m \geq 4$ and $k = 0, 1, \ldots, m-2$. In this chapter, the first objective which is to determine the number of compatible pairs of actions between the cyclic 2-groups is achieved.

Next, the compatible actions graph is introduced and denoted by $\Gamma_{G\otimes H}$. Properties of the compatible actions graph have been proven such that the cardinality of the edge set, order of a compatible action graph, the number of directed edges, the degree of vertex v and also the connectivity of a compatible action graph. Furthermore, the some types of graph are presented such as bipartite graph and complete graph. Thus, the second objective is achieved in Chapter 5.

In Chapter 6, the intersection between the compatible action graph and its subgraph which are $\Gamma_{C_{2^m}\otimes C_{2^n}}$ and $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-i}}}$ are discussed. The necessary and sufficient conditions for the cyclic 2-subgroups act on each other in a compatible way are given. Then, some of the properties of the corresponding subgraphs of the compatible action graph such as the order of the graph, the cardinality of the edge set and the the connectivity of corresponding subgraphs are provided.

7.2 Recommendation for Future Research

This research only focuses on determining the compatible pairs of actions as stated in the objective of research without finding the nonabelian tensor product. Hence, the extension from this research can be done by determining the exact number of the nonabelian tensor product since a different compatible pair of actions will give a different nonabelian tensor product even for the same group.

This research focused on the cyclic groups of 2-power order only. It may also be done for the cyclic groups of p-power order, where p is an odd prime, namely to determine the compatible pair of actions for the cyclic groups of the p-power order since the homomorphism image of automorphism groups of the cyclic groups of p-power are different.

In this research, our concern on the properties of the subgraph of the compatible action graph, $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-j}}}$ where $i = j$. Thus, further research can be done to find the properties of subgraph of compatible action graph where $i \neq j$ which give the new properties of the subgraph of the compatible action graph, $\Gamma_{C_{2^{m-i}}\otimes C_{2^{n-j}}}$.

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APPENDIX A

THE OUTPUT OF GAP SOFTWARE

The outputs for the GAP coding given on pages 24 and 25 are stated below. This output represents the list of automorphisms with their specific order that satisfied the compatible condition and the total of number of compatible pairs of actions.

```
gap> CompatibleAction(16,16);
k=3 (order action=4),l=9 (order action=2) Compatible
k=5 (order action=4),l=5 (order action=4) Compatible
k=5 (order action=4),l=9 (order action=2) Compatible
k=5 (order action=4),l=13 (order action=4) Compatible
k=7 (order action=2),l=7 (order action=2) Compatible
k=7 (order action=2),l=9 (order action=2) Compatible
k=7 (order action=2),l=15 (order action=2) Compatible
k=9 (order action=2),l=3 (order action=4) Compatible
k=9 (order action=2),l=5 (order action=4) Compatible
k=9 (order action=2),l=7 (order action=2) Compatible
k=9 (order action=2),l=9 (order action=2) Compatible
k=9 (order action=2),l=11 (order action=4) Compatible
k=9 (order action=2),l=13 (order action=4) Compatible
k=9 (order action=2),l=15 (order action=2) Compatible
k=11 (order action=4),l=9 (order action=2) Compatible
k=13 (order action=4),l=5 (order action=4) Compatible
k=13 (order action=4),l=9 (order action=2) Compatible
k=13 (order action=4),l=13 (order action=4) Compatible
```

```
k=15 (order action=2),l=7 (order action=2) Compatible
k=15 (order action=2),l=9 (order action=2) Compatible
k=15 (order action=2),l=15 (order action=2) Compatible
 No of Compatible = 21
gap> CompatibleAction(32,32);
k=3 (order action=8),l=17 (order action=2) Compatible
k=5 (order action=8),l=9 (order action=4) Compatible
k=5 (order action=8),l=17 (order action=2) Compatible
k=5 (order action=8),l=25 (order action=4) Compatible
k=7 (order action=4),l=17 (order action=2) Compatible
k=9 (order action=4),l=5 (order action=8) Compatible
k=9 (order action=4),l=9 (order action=4) Compatible
k=9 (order action=4),l=13 (order action=8) Compatible
k=9 (order action=4),l=17 (order action=2) Compatible
k=9 (order action=4),l=21 (order action=8) Compatible
k=9 (order action=4),l=25 (order action=4) Compatible
k=9 (order action=4),l=29 (order action=8) Compatible
k=11 (order action=8),l=17 (order action=2) Compatible
k=13 (order action=8),l=9 (order action=4) Compatible
k=13 (order action=8),l=17 (order action=2) Compatible
k=13 (order action=8),l=25 (order action=4) Compatible
k=15 (order action=2),l=15 (order action=2) Compatible
k=15 (order action=2),l=17 (order action=2) Compatible
k=15 (order action=2),l=31 (order action=2) Compatible
k=17 (order action=2),l=3 (order action=8) Compatible
k=17 (order action=2),l=5 (order action=8) Compatible
k=17 (order action=2),l=7 (order action=4) Compatible
k=17 (order action=2),l=9 (order action=4) Compatible
k=17 (order action=2),l=11 (order action=8) Compatible
k=17 (order action=2),l=13 (order action=8) Compatible
k=17 (order action=2),l=15 (order action=2) Compatible
k=17 (order action=2),l=17 (order action=2) Compatible
```

```
k=17 (order action=2),l=19 (order action=8) Compatible
k=17 (order action=2),l=21 (order action=8) Compatible
k=17 (order action=2),l=23 (order action=4) Compatible
k=17 (order action=2),l=25 (order action=4) Compatible
k=17 (order action=2),l=27 (order action=8) Compatible
k=17 (order action=2),l=29 (order action=8) Compatible
k=17 (order action=2),l=31 (order action=2) Compatible
k=19 (order action=8),l=17 (order action=2) Compatible
k=21 (order action=8),l=9 (order action=4) Compatible
k=21 (order action=8),l=17 (order action=2) Compatible
k=21 (order action=8),l=25 (order action=4) Compatible
k=23 (order action=4),l=17 (order action=2) Compatible
k=25 (order action=4),l=5 (order action=8) Compatible
k=25 (order action=4),l=9 (order action=4) Compatible
k=25 (order action=4),l=13 (order action=8) Compatible
k=25 (order action=4),l=17 (order action=2) Compatible
k=25 (order action=4),l=21 (order action=8) Compatible
k=25 (order action=4),l=25 (order action=4) Compatible
k=25 (order action=4),l=29 (order action=8) Compatible
k=27 (order action=8),l=17 (order action=2) Compatible
k=29 (order action=8),l=9 (order action=4) Compatible
k=29 (order action=8),l=17 (order action=2) Compatible
k=29 (order action=8),l=25 (order action=4) Compatible
k=31 (order action=2),l=15 (order action=2) Compatible
k=31 (order action=2),l=17 (order action=2) Compatible
k=31 (order action=2),l=31 (order action=2) Compatible
 No of Compatible = 53
gap> CompatibleAction(64,64);
k=3 (order action=16),l=33 (order action=2) Compatible
k=5 (order action=16),l=17 (order action=4) Compatible
k=5 (order action=16),l=33 (order action=2) Compatible
k=5 (order action=16),l=49 (order action=4) Compatible
```

```
k=7 (order action=8),l=33 (order action=2) Compatible
k=9 (order action=8),l=9 (order action=8) Compatible
k=9 (order action=8),l=17 (order action=4) Compatible
k=9 (order action=8),l=25 (order action=8) Compatible
k=9 (order action=8),l=33 (order action=2) Compatible
k=9 (order action=8),l=41 (order action=8) Compatible
k=9 (order action=8),l=49 (order action=4) Compatible
k=9 (order action=8),l=57 (order action=8) Compatible
k=11 (order action=16),l=33 (order action=2) Compatible
k=13 (order action=16),l=17 (order action=4) Compatible
k=13 (order action=16),l=33 (order action=2) Compatible
k=13 (order action=16),l=49 (order action=4) Compatible
k=15 (order action=4),l=33 (order action=2) Compatible
k=17 (order action=4),l=5 (order action=16) Compatible
k=17 (order action=4),l=9 (order action=8) Compatible
k=17 (order action=4),l=13 (order action=16) Compatible
k=17 (order action=4),l=17 (order action=4) Compatible
k=17 (order action=4),l=21 (order action=16) Compatible
k=17 (order action=4),l=25 (order action=8) Compatible
k=17 (order action=4),l=29 (order action=16) Compatible
k=17 (order action=4),l=33 (order action=2) Compatible
k=17 (order action=4),l=37 (order action=16) Compatible
k=17 (order action=4),l=41 (order action=8) Compatible
k=17 (order action=4),l=45 (order action=16) Compatible
k=17 (order action=4),l=49 (order action=4) Compatible
k=17 (order action=4),l=53 (order action=16) Compatible
k=17 (order action=4),l=57 (order action=8) Compatible
k=17 (order action=4),l=61 (order action=16) Compatible
k=19 (order action=16),l=33 (order action=2) Compatible
k=21 (order action=16),l=17 (order action=4) Compatible
k=21 (order action=16),l=33 (order action=2) Compatible
k=21 (order action=16),l=49 (order action=4) Compatible
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k=23 (order action=8),l=33 (order action=2) Compatible
k=25 (order action=8),l=9 (order action=8) Compatible
k=25 (order action=8),l=17 (order action=4) Compatible
k=25 (order action=8),l=25 (order action=8) Compatible
k=25 (order action=8),l=33 (order action=2) Compatible
k=25 (order action=8),l=41 (order action=8) Compatible
k=25 (order action=8),l=49 (order action=4) Compatible
k=25 (order action=8),l=57 (order action=8) Compatible
k=27 (order action=16),l=33 (order action=2) Compatible
k=29 (order action=16),l=17 (order action=4) Compatible
k=29 (order action=16),l=33 (order action=2) Compatible
k=29 (order action=16),l=49 (order action=4) Compatible
k=31 (order action=2),l=31 (order action=2) Compatible
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APPENDIX B

PUBLICATIONS/ PRESENTATION IN CONFERENCES

Some results of this thesis conducted have been published/ presented in seminar/ conferences or submitted as listed in the following.

P1. **Sahimel Azwal Sulaiman**, Mohd Sham Mohamad, Yuhani Yusof, Nor Haniza Sarmin, Nor Muhainiah Mohd Ali, Tan Lit Ken and Tahir Ahmad. (2015). Compatible Pair of Non-Trivial Actions for Some Cyclic Groups of 2-Power Order. AIP Conference Proceedings. 2nd ISM International Statistical Conference 2014 (ISM-II): Empowering the Applications of Statistical and Mathematical. 1643, 700-705. MS Garden Hotel, Kuantan, Pahang, Malaysia: 12-14 August 2014. ISBN: 978-0-7354-1281-1. ISI indexed. DOI: 10.1063/1.4907515. (Published: 03/02/15).

P2. **Sahimel Azwal Sulaiman**, Mohd Sham Mohamad, Yuhani Yusof, Tan Lit Ken and Tahir Ahmad. (2015). An Exact Number of Compatible Pair of Nontrivial Actions for Cyclic Groups of 2-Power Order. AIP Conference Proceedings. Simposium Kebangsaan Sains Matematik (SKSM23): Memacu Transformasi Inovasi Negara Melalui Sains Matematik. Pulai Springs Resort, Johor Bharu, Johor, Malaysia: 24-26 November 2015.

P3. **Sahimel Azwal Sulaiman**, Mohd Sham Mohamad, Yuhani Yusof and Mohammed Khalid Shahoodh. (2016). Number of Compatible Pair for Nontrivial Actions of Finite Cyclic 2-groups. Procedding of 4th International Science Postgraduate Conference 2016 (ISPC2016): Research and Innovation in Science and Technology for Sustainable Society. 161-165. Center for Sustainable

Nanomaterials ISIR Universiti Teknologi Malaysia, Johor Bharu, Johor, Malaysia: 22-24 February 2016. ISBN: 978-967-0194-54-7. Proceeding Conference.

P4. **Sahimel Azwal Sulaiman**, Mohd Sham Mohamad, Yuhani Yusof and Mohammed Khalid Shahoodh. (2016). Compatible Pair of Nontrivial Action for Finite Cyclic 2-groups. Proceeding of 3rd National Conference for Postgraduate Research (NCON-PGR2016):Knowledge Discovery For Wealth Creations. Universiti Malaysia Pahang, Pekan, Pahang, Malaysia:24-25 September 2016. Proceeding Conference.

P5. Mohammed Khalid Shahoodh, Mohd Sham Mohamad, Yuhani Yusof and **Sahimel Azwal Sulaiman**. (2016). Compatible Pair of Nontrivial Actions for Cyclic Groups of 3-Power Order. Proceeding of 3rd National Conference for Postgraduate Research (NCON-PGR2016):Knowledge Discovery For Wealth Creations. Universiti Malaysia Pahang, Pekan, Pahang, Malaysia:24-25 September 2016. Proceeding Conference.

P6. **Sahimel Azwal Sulaiman**, Mohd Sham Mohamad, Yuhani Yusof and Mohammed Khalid Shahoodh. (2017). The Number of Compatible Pair of Actions For Cyclic Groups of 2-Power Order. International Journal of Simulation, System, Sciences and Technology. Vol 18, No. 4, December 2017. ISSN: 1473-804x Online, 1473-8031 Print. SCOPUS Index.

P7. Mohammed Khalid Shahoodh, Mohd Sham Mohamad, Yuhani Yusof and **Sahimel Azwal Sulaiman**. (2017). Number of Compatible Pair of Actions for Finite Cyclic Groups of 3-Power Order. International Journal of Simulation, System, Sciences and Technology. Vol 18, No. 4, December 2017. ISSN: 1473-804x Online, 1473-8031 Print. SCOPUS Index.

P8. Mohd Sham Mohamad, **Sahimel Azwal Sulaiman**, Yuhani Yusof and Mohammed Khalid Shahoodh. (2017). Compatible Pair of Nontrivial Action for Finite Cyclic 2-Groups. Journal of Physics: Conference Series. 1st International Conference on Applied & Industrial Mathematics and Statistics 2017 (ICoAIMS 2017) Universiti Malaysia Pahang. Vol. 890, 011001. SCOPUS Index. DOI: https://doi.org/10.1088/1742-6596/890/1/011001