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## Solving boundary problems for biharmonic operator by using Integro-differential operators of fractional order

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**Abstract.** The investigation of the properties of the integro-differential operators will be carried out. Which generalizes the well-known Bavin operators to the fractional value of the parameters. The properties of the defined operators are in the classes of the polyharmonic operators. It is established that the newly defined fractional operators map the polyharmonic functions on the ball to the polyharmonic functions. Also it is proposed that the inverse for the fractional operator and application of the integro-differential fractional operators to solve biharmonic problems with fractional boundary conditions. The sufficient condition for existence and uniqueness of the solution for biharmonic equation with fractional boundary conditions are obtained. The solution of the biharmonic equation is obtained by using the integro-differential fractional operator.

### 1. Introduction

The mathematical models of the various vibrating systems are partial differential equations and finding the solutions of such equations are obtained by developing the theory of eigenfunction expansions of differential operators. The biharmonic equation which is fourth order differential equation is encountered in plane problems of elasticity. It is also used to describe slow flows of viscous incompressible fluids. Many physical processes taking place in real space can be described using the differential operators, particularly biharmonic operator. Biharmonic equations appear in the study of mathematical models in several real-life processes as, among others, radar imaging [1] or incompressible flows [2]. Omitting a huge amount of works devoted to the study of this kind of equations, we refer some of them regarding to their used methods. Difference schemes and variational methods were used in the works [3, 4]. By using numerical and iterative methods, Dirichlet and Neumann boundary problems for biharmonic equations were studied in the papers [5, 6].

Let  $\mathbb{R}^n$  denotes  $n$ -dimensional Euclidean space. Throughout of the paper we assume that  $n$  is fixed and not less than 2. We let  $x = (x_1, x_2, \dots, x_n)$  denote a typical point in  $\mathbb{R}^n$  and let  $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$  denote the Euclidean norm of the  $x$ . Next we define a harmonic function. Let  $D$  be domain in  $\mathbb{R}^n$  with the property that if  $x \in D$ , then  $tx + (1-t)x \in D$  for all  $t, 0 \leq t \leq 1$ . A twice continuously differentiable, complex-valued function  $u(x)$  defined on  $D$  is harmonic on  $D$  if

$$\Delta u(x) = 0, x \in D,$$

where  $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$  -Laplace operator. Next, for given real positive numbers  $\mu_j, j = 1, 2, \dots, m$  we use notation

$\bar{\mu} = (\mu_1, \mu_2, \dots, \mu_m)$ . The properties and applications of the integro-differential operators of the following form



$$\delta_{\mu_1}[u](x) = \sum_{i=1}^n x_i \frac{\partial u(x)}{\partial x_i} + \mu_1 u(x),$$

$$\delta_{\mu_1}^{-1}[u](x) = \int_0^1 t^{\mu_1-1} u(tx) dt$$

and

$$\delta_{\bar{\mu}}^{(m)}[u](x) = \delta_{\mu_1}[\delta_{\mu_2}[\dots\delta_{\mu_m}[u]\dots]](x),$$

$$\delta_{\bar{\mu}}^{(-m)}[u](x) = \delta_{\mu_m}^{-1}[\delta_{\mu_{m-1}}^{-1}[\dots\delta_{\mu_1}^{-1}[u]\dots]](x)$$

are investigated in [7].

The main properties of the operators  $\delta_{\bar{\mu}}^{(m)}$  and  $\delta_{\bar{\mu}}^{(-m)}$  are given in the following

**Lemma 1.** If the function  $u(x)$  is harmonic in the region  $D$ , then  $\delta_{\bar{\mu}}^{(m)}[u](x)$  and  $\delta_{\bar{\mu}}^{(-m)}[u](x)$  are also harmonic functions in the region  $D$ :

$$\Delta u(x) = 0 \Rightarrow \Delta[\delta_{\bar{\mu}}^{(m)}[u](x)] = 0 = \Delta[\delta_{\bar{\mu}}^{(-m)}[u](x)].$$

**Lemma 2.** If the function  $u(x)$  is harmonic in the region  $D$ , then for all  $x \in D$  one has

$$\delta_{\bar{\mu}}^{(m)}[\delta_{\bar{\mu}}^{(-m)}[u]](x) = u(x),$$

$$\delta_{\bar{\mu}}^{(-m)}[\delta_{\bar{\mu}}^{(m)}[u]](x) = u(x).$$

It is clear that from the statement of the Lemma 2 follows that the operators  $\delta_{\bar{\mu}}^{(m)}$  and  $\delta_{\bar{\mu}}^{(-m)}$  are inverse with respect to each other in the classes of harmonic functions in  $D$ .

We note that the operator  $\delta_{\bar{\mu}}^{(-m)}$  can be represented as  $m$  tuple iterated integrals. For certain partial cases of the parameters  $\mu_i, i=1, \dots, m$  the operator  $\delta_{\bar{\mu}}^{(-m)}$  can be represented in more compact form. For example, in the case  $\mu_i = \mu, i=1, \dots, m$  (we refer the readers to [8] for details):

$$\delta_{\bar{\mu}}^{(-m)}[u](x) = \frac{1}{(m-1)!} \int_0^1 t^{\mu-1} \left(\ln \frac{1}{t}\right)^{m-1} u(tx) dt. \quad (1)$$

Futhermore, if  $\mu_1 = \mu, \mu_2 = \mu+1, \dots, \mu_\ell = \mu+m-1, m \geq 1$ , then we have the presentation as follows (see [7] for details)

$$\delta_{\bar{\mu}}^{(-m)}[u](x) = \frac{1}{(m-1)!} \int_0^1 t^{\mu-1} (1-t)^{m-1} u(tx) dt. \quad (2)$$

Onwards, since  $(m-1)! = \Gamma(m), m \geq 1$ , where  $\Gamma(\cdot)$  - Gamma function, then it easy to see that the integrals (1) and (2) are defined for non-integer values of  $m > 0$ . Therefore naturally we arrive to the problem of defining the inverse operator  $\delta_{\bar{\mu}}^{(-m)}$  for such defined operator of fractional order  $m$ . The mentioned problem of defining the inverse for fractional values of  $m$  is suggested in the papers [9,10], where it is shown that if  $m = \alpha$  for all values of  $0 < \alpha < 1$ , then the operators

$$D_{\mu}^{\alpha}[u](x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \left(\ln \frac{1}{t}\right)^{\alpha-1} \frac{u(tx) - u(x)}{t^{1-\mu}} dt + \mu^{\alpha} u(x), \quad (3)$$

and

$$B_{\mu}^{\alpha}[u](x) = \frac{r^{\alpha-\mu}}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^r (r-t)^{-\alpha} t^{\mu} u(t\theta) dt \quad (4)$$

are inverse operators of the (1) and (2), respectively. Moreover, the following identity holds:

$$D_{\mu}^1[u](x) = B_{\mu}^1[u](x) = \delta_{\mu}^1[u](x).$$

We note that the operator (3) is connected with the order  $\alpha$  in Hadamard means, while the operator (4) in Riemann-Liouville means. In the papers [9,10] the authors considered the application of the operators

$$D_{\bar{\mu}}^m[u](x) = D_{\mu_1}^{\alpha_1}[D_{\mu_2}^{\alpha_2}[\dots[D_{\mu_m}^{\alpha_m}[u]\dots]]](x),$$

$$B_{\bar{\mu}}^m[u](x) = B_{\mu_1}^{\alpha_1}[B_{\mu_2}^{\alpha_2}[\dots[B_{\mu_m}^{\alpha_m}[u]\dots]]](x),$$

where  $0 < \alpha_j \leq 1, 0 < \mu_j, j = 1, 2, \dots, m$ , to the problems of solvability of boundary problems. In general case when  $\alpha \in (m-1, m], m = 1, 2, \dots$ , the inverse operator to the operator (2) is constructed in [11] and it is shown that the inverse differential operator can be represented as

$$D_\mu^\alpha [u](x) = \frac{r^{-\alpha}}{\Gamma(m-\alpha)} \int_0^r \left( \ln \frac{r}{t} \right)^{m-\alpha-1} \left( t \frac{d}{dt} \right)^m [t^\mu u(t\theta)] \frac{dt}{t}.$$

Continuing research conducted in the works [9,10], in the current paper we construct the inverse operator of the (3), which is in general differential operator, i.e. for all  $\alpha \in (m-1, m], m = 1, 2, \dots$ . We show the application of the such operators to the problems of solvability of the boundary problems corresponding to Laplace operator.

Let  $\Omega$  be unit ball, and a function  $u(x)$  be smooth function in  $\Omega$ ,  $r = |x|, \theta = x/|x|$ . We consider the Riemann-Liouville operator of integration and differentiation of order  $\alpha \in (m-1, m], m = 1, 2, \dots$ , (see [12] for definition and properties)

$$I^\alpha [u](x) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} u(t\theta) dt,$$

$$D^\alpha [u](x) = \frac{1}{\Gamma(\alpha)} \frac{d^m}{dr^m} \int_0^r (r-t)^{m-\alpha-1} u(t\theta) dt.$$

We introduce the following notations

$$B_\mu^{-\alpha} [u](x) = r^{1-\mu-\alpha} I^\alpha [r^{\mu-1} u(x)],$$

$$B_\mu^\alpha [u](x) = r^{1-\mu} D^\alpha [r^{\mu-1+\alpha} u(x)].$$

Since,  $I^0 u(x) = u(x)$ , then for  $\alpha = m$  we obtain,

$$B_\mu^{-m} [u](x) = r^{1-\mu-m} I^m [r^{\mu-1} u(x)] = \frac{r^{1-\mu-m}}{\Gamma(m)} \int_0^r (r-t)^{m-1} t^{\mu-1} u(t\theta) dt.$$

After the transforming the variables  $t = sr$  the latter integral we represent as follows

$$\frac{r^{1-\mu-m}}{\Gamma(m)} \int_0^r (r-t)^{m-1} t^{\mu-1} u(t\theta) dt = \frac{1}{(m-1)!} \int_0^1 (1-s)^{m-1} s^{\mu-1} u(sx) ds = \delta_\mu^{(-m)} u(x).$$

Similarly, we have

$$B_\mu^m [u](x) = r^{1-\mu} D^m [r^{\mu-1+\alpha} u(x)] = r^{1-\mu} \frac{d^m}{dr^m} I^{m-m} [r^{\mu-1+m} u(x)] = r^{1-\mu} \frac{d^m}{dr^m} [r^{\mu-1+m} u(x)].$$

Let  $m = 1$ . Then one has

$$B_\mu^1 [u](x) = r^{1-\mu} \frac{d}{dr} [r^\mu u(x)] = \left( r \frac{d}{dr} + \mu \right) u(x) \equiv \delta_\mu^1 u(x).$$

We proceed for the case  $m = 2$ ,

$$\begin{aligned} B_\mu^2 [u](x) &= r^{1-\mu} D^2 [r^{\mu-1+2} u(x)] = r^{1-\mu} \frac{d^2}{dr^2} [r^{\mu+1} u(x)] = r^{1-\mu} \frac{d}{dr} \left[ (\mu+1) r^\mu u(x) + r^{\mu+1} \frac{d}{dr} \right] \\ &= r^{1-\mu} \frac{d}{dr} \left[ r^\mu \left( r \frac{d}{dr} + (\mu+1) \right) u(x) \right] = r^{1-\mu} \left[ \mu r^{\mu-1} \delta_{\mu+1} [u](x) + r^\mu \frac{d}{dr} \delta_{\mu+1} [u](x) \right] \\ &= \mu \delta_{\mu+1} [u](x) r^{1-\mu} + r \frac{d}{dr} \delta_{\mu+1} [u](x) = \left( \mu + r \frac{d}{dr} \right) \delta_{\mu+1} [u](x) = \delta_\mu [\delta_{\mu+1} [u]](x). \end{aligned}$$

In general case using the methods of mathematical induction one can derive

$$B_\mu^m [u](x) = \left( r \frac{d}{dr} + \mu \right) \left( r \frac{d}{dr} + \mu + 1 \right) \dots \left( r \frac{d}{dr} + \mu + m - 1 \right) u(x) \equiv \delta_\mu \delta_{\mu+1} \dots \delta_{\mu+m-1} u(x) \equiv \delta_\mu^{(m)} u(x)$$

Consequently, in the case when parameters are not integer like the case of integer values of parameters  $\alpha$  the operator  $B_\mu^\alpha$  coincides with the operator (2) for  $\mu_1 = \mu, \mu_2 = \mu+1, \dots, \mu_\ell = \mu+m-1, m \geq 1$ . We constructed the fractional analogous of the operators  $\delta_\mu^{(m)}$  and  $\delta_\mu^{(-m)}$  for the case  $\bar{\mu} = (\mu, \mu+1, \dots, \mu+m-1)$ ,  $m \geq 1$ .

## 2. Properties of the operators $B_\mu^\alpha$ and $B_\mu^{-\alpha}$ .

In this section we investigate the properties of the operators  $B_\mu^\alpha$  and  $B_\mu^{-\alpha}$ . In the calculation here in we assume that  $m-1 < \alpha \leq m, m=1,2,\dots, \mu > 0$ .

**Lemma 3.** Let  $H_k(x)$  be homogeneous polynomial of degree  $k \in N_0 = \{0,1,\dots\}$ . Then one has

$$B_\mu^{-\alpha} [H_k](x) = \gamma_{k,\mu} H_k(x), \quad (5)$$

$$B_\mu^\alpha [H_k](x) = 1/\gamma_{k,\mu} H_k(x), \quad (6)$$

where  $\gamma_{k,\mu} = \frac{\Gamma(k+\mu)}{\Gamma(k+\mu+\alpha)}$ .

**Proof.** Let prove the equation (5). From definition

$$\begin{aligned} B_\mu^{-\alpha} [H_k](x) &= \frac{r^{1-\mu-\alpha} H_k(\theta)}{\Gamma(\alpha)} \int_0^r (r-t)^{m-1-\alpha} t^{k+\mu-1} dt = \frac{r^k H_k(\theta)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{k+\mu-1} ds \\ &= \frac{r^k H_k(\theta)}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(k+\mu)}{\Gamma(k+\mu+\alpha)} = \frac{\Gamma(k+\mu)}{\Gamma(k+\mu+\alpha)} H_k(x) = \gamma_{k,\mu} H_k(x). \end{aligned}$$

Let prove second equation (6). Using the definition of the operator  $B_\mu^\alpha$  we derive

$$\begin{aligned} B_\mu^\alpha [H_k](x) &= r^{1-\mu} D^\alpha [r^{\mu-1+\alpha} H_k(x)] = \frac{r^{1-\mu}}{\Gamma(m-\alpha)} \frac{d^m}{dr^m} \int_0^r (r-t)^{m-\alpha-1} t^{\mu-1+\alpha} H_k(t\theta) dt \\ &= \frac{r^{1-\mu} H_k(\theta)}{\Gamma(m-\alpha)} \frac{d^m}{dr^m} \int_0^r (r-t)^{m-\alpha-1} t^{\mu-1+\alpha+k} dt = \frac{r^{1-\mu} H_k(\theta)}{\Gamma(m-\alpha)} \int_0^1 (1-s)^{m-\alpha-1} s^{\mu+\alpha+k-1} ds \frac{d^m}{dr^m} (r^{\mu+k+m-1}) \\ &= \frac{r^{1-\mu} H_k(\theta)}{\Gamma(m-\alpha)} \frac{\Gamma(m-\alpha)\Gamma(\mu+\alpha+k)}{\Gamma(\mu+m+k)} (\mu+k+m-1)(\mu+k+m-2)\dots(\mu+k) r^{\mu+k-1} \\ &= \frac{\Gamma(\mu+\alpha+k)}{\Gamma(\mu+k)} r^k H_k(\theta) = \frac{\Gamma(\mu+\alpha+k)}{\Gamma(\mu+k)} H_k(x). \end{aligned}$$

**Lemma 4.** Let  $H_k(x)$  be harmonic polynomial of degree  $k \in N_0$ . Then one has

$$B_\mu^{-\alpha} [ |x|^{2j} H_k ](x) = \gamma_{k+2j,\mu} |x|^{2j} H_k(x), \quad (7)$$

$$B_\mu^\alpha [H_k](x) = 1/\gamma_{k+2j,\mu} |x|^{2j} H_k(x). \quad (8)$$

The proof of the Lemma 4 can be conducted by similar way as in the proof of the Lemma 3.

**Consequence 1.** Let  $H_k(x)$  be harmonic polynomial of degree  $k \in N_0$ . Then we have

$$B_\mu^{-\alpha} [B_\mu^\alpha [ |x|^{2j} H_k ]](x) = B_\mu^\alpha [B_\mu^{-\alpha} [ |x|^{2j} H_k ]](x) = |x|^{2j} H_k(x). \quad (9)$$

**Lemma 5.** If  $u(x)$  is polyharmonic function in the domain  $\Omega$ , then functions  $B_\mu^{-\alpha} [u](x)$  and  $B_\mu^\alpha [u](x)$  are also polyharmonic functions in the domain  $\Omega$ .

**Proof.** Let  $u(x)$  be polyharmonic function in the domain  $\Omega$ . Using the theorem of Almanzi ([13], p.208) we conclude that there exist a harmonic functions  $u_j(x)$ ,  $j=0,1,\dots,m-1$  in the domain  $\Omega$  such that  $u(x)$  is represented in the following form

$$u(x) = u_0(x) + |x|^2 u_1(x) + \dots + |x|^{2(m-1)} u_{m-1}(x) \quad (10)$$

The harmonic functions  $u_j(x)$  have a representation

$$u_j(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{h_k} u_{k,j}^{(i)} H_k^{(i)}(x), \quad (11)$$

where  $\{H_k^{(i)}(x), i=1, \dots, h_k\}$  - is a complete system of the harmonic polynomials of degree  $k \in N_0$ , we denote by  $u_{k,j}^{(i)}$  coefficients of the expansion (10). It is well known that (see [14], p.489), that the series (10) absolutely

and uniformly convergent for all  $x: |x| \leq \rho < 1$ . From the representation (10) and (11) it follows that  $u(x)$  can be represented in the form

$$u(x) = \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} \sum_{i=1}^{h_k} u_{k,j}^{(i)} |x|^{2j} H_k^{(i)}(x) \tag{12}$$

By applying to (10) the operator  $B_{\mu}^{-\alpha}$  taking the account (7) and (8) we obtain

$$B_{\mu}^{-\alpha}[u](x) = \sum_{j=0}^{m-1} |x|^{2j} \sum_{k=0}^{\infty} \sum_{i=1}^{h_k} \gamma_{k+2j,\mu} u_{k,j}^{(i)} H_k^{(i)}(x) = \sum_{j=0}^{m-1} |x|^{2j} w_j(x),$$

where

$$w_j(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{h_k} \gamma_{k+2j,\mu} u_{k,j}^{(i)} H_k^{(i)}(x). \tag{13}$$

Using the asymptotical estimation  $\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \sim p^{\alpha}$ ,  $p \rightarrow \infty$  we obtain that  $\lim_{k \rightarrow \infty} \sqrt[k]{\gamma_{k+2j,\mu}} = 1$ . Then we conclude that the radius of convergent of the series (13) and (11) are same, therefore functions  $w_j(x)$  are harmonic in the region  $\Omega$  for every  $j=0,1,\dots,m-1$ . Applying Almanzi theorem we derive that the function  $B_{\mu}^{-\alpha}[u](x)$  is polyharmonic in  $\Omega$ . Similarly, we can show that the function  $B_{\mu}^{\alpha}[u](x)$  is also polyharmonic in  $\Omega$ . The proof of Lemma 5 is completed.

Using the equations (12) and (9) we obtain:

**Lemma 6.** Let  $u(x)$  be polyharmonic function in the domain  $\Omega$ , then for every  $x \in \Omega$  one has

$$B_{\mu}^{-\alpha}[B_{\mu}^{\alpha}[u]](x) = B_{\mu}^{\alpha}[B_{\mu}^{-\alpha}[u]](x) = u(x). \tag{14}$$

It follows that the operators  $B_{\mu}^{-\alpha}$  and  $B_{\mu}^{\alpha}$  are inverse to each other in the class of polyharmonic operators in the ball. The following is the more general statement on the mentioned relation between the operators  $B_{\mu}^{-\alpha}$  and  $B_{\mu}^{\alpha}$ .

**Lemma 7.** Let  $u(x)$  be a smooth function in the domain  $\Omega$ . Then for all  $x \in \Omega$  the equalities (14) hold.

**Proof.** From the definition of the operator  $B_{\mu}^{-\alpha}$  we obtain

$$B_{\mu}^{-\alpha}[B_{\mu}^{\alpha}[u]](x) = \frac{r^{1-\mu-\alpha}}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} t^{\mu-1} B_{\mu}^{\alpha}[u](t\theta) dt.$$

It is not difficult to show that

$$\begin{aligned} \frac{r^{1-\mu-\alpha}}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} t^{\mu-1} B_{\mu}^{\alpha}[u](t\theta) dt &= r^{1-\mu-\alpha} \left\{ \frac{1}{\Gamma(\alpha)} \frac{d}{dr} \int_0^r \frac{(r-t)^{\alpha}}{\alpha} t^{\mu-1} B_{\mu}^{\alpha}[u](t\theta) dt \right\} \\ &= \dots = r^{1-\mu-\alpha} \left\{ \frac{1}{\Gamma(\alpha)} \frac{d^m}{dr^m} \int_0^r \frac{(r-t)^{\alpha+m-1}}{\alpha(\alpha+1)\dots(\alpha+m-1)} t^{\mu-1} B_{\mu}^{\alpha}[u](t\theta) dt \right\}. \end{aligned}$$

Then by using the definition of the operator  $B_{\mu}^{\alpha}$  we conclude that

$$\begin{aligned} B_{\mu}^{-\alpha}[B_{\mu}^{\alpha}[u]](x) &= \frac{r^{1-\mu-\alpha}}{\Gamma(\alpha)} \frac{d^m}{dr^m} \int_0^r \frac{(r-t)^{\alpha+m-1}}{\alpha(\alpha+1)\dots(\alpha+m-1)} t^{\mu-1} \left( \frac{t^{\mu-1}}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} s^{\mu-1+\alpha} u(s\theta) ds \right) dt \\ &= \frac{r^{1-\mu-\alpha}}{\Gamma(\alpha)} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dr^m} \int_0^r \frac{(r-t)^{\alpha+m-1}}{\alpha(\alpha+1)\dots(\alpha+m-1)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} s^{\mu-1+\alpha} u(s\theta) ds dt. \end{aligned}$$

If we denote

$$w(t\theta) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} s^{\mu-1+\alpha} u(s\theta) ds,$$

then after integrating by parts  $m$  times and using the identities  $I^{\alpha} \cdot I^{m-\alpha} = I^m, D^m I^m = E$  one has

$$\begin{aligned}
 B_{\mu}^{-\alpha} [B_{\mu}^{\alpha} [u]](x) &= \frac{r^{1-\mu-\alpha}}{\Gamma(\alpha)} \frac{d^m}{dr^m} \int_0^r (r-t)^{\alpha-1} w(t\theta) dt = r^{1-\mu-\alpha} \frac{d^m}{dr^m} I^{\alpha} [I^{m-\alpha} [t^{\mu-1+\alpha} u]](x) \\
 &= r^{1-\mu-\alpha} \frac{d^m}{dr^m} I^m [t^{\mu-1+\alpha} u](x) = r^{1-\mu-\alpha} r^{\mu-1+\alpha} u(x) = u(x),
 \end{aligned}$$

which proves first equality in (14). The following proves the second equality in (14):

$$B_{\mu}^{\alpha} [B_{\mu}^{-\alpha} [u]](x) = r^{1-\mu} \frac{d^m}{dr^m} I^{m-\alpha} [r^{\mu-1+\alpha} \cdot r^{1-\mu-\alpha} I^{\alpha} [r^{\mu-1} u(x)]] = r^{1-\mu} \frac{d^m}{dr^m} I^{m-\alpha} [I^{\alpha} [r^{\mu-1} u(x)]] = u(x).$$

The proof of the Lemma 7 is completed.

**Lemma 8.** Let  $0 < \alpha \leq 1, \mu > 0$  and  $\Delta u(x) = f(x), x \in \Omega$ , where  $f(x)$  – smooth function in  $\bar{\Omega}$ . Then for all  $x \in \Omega$  one has

$$\begin{aligned}
 \Delta B_{\mu}^{-\alpha} [u](x) &= B_{\mu+2}^{-\alpha} [f](x) = u(x), \quad x \in \Omega, \\
 \Delta B_{\mu}^{\alpha} [u](x) &= B_{\mu+2}^{\alpha} [f](x) = u(x), \quad x \in \Omega.
 \end{aligned}$$

**Proof.** We note that the function  $B_{\mu}^{-\alpha} [u](x)$  can be represented as follows

$$B_{\mu}^{-\alpha} [u](x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\mu-1} u(tx) dt.$$

In fact from definition of the operator  $B_{\mu}^{-\alpha}$  we see that

$$B_{\mu}^{-\alpha} [u](x) = r^{1-\mu-\alpha} I^{\alpha} [r^{\mu-1} u(x)] = \frac{r^{1-\mu-\alpha}}{\Gamma(\alpha)} \int_0^r (r-s)^{\alpha-1} s^{\mu-1} u(s\theta) ds,$$

where after using the substitution  $s = rt$  we have the following presentation for the function  $B_{\mu}^{-\alpha} [u](x)$

$$B_{\mu}^{-\alpha} [u](x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\mu-1} u(tx) dt.$$

Thus we have

$$\Delta B_{\mu}^{-\alpha} [u](x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\mu+2-\alpha} f(tx) dt \equiv B_{\mu+2}^{-\alpha} [f](x).$$

By similar method we can obtain representation for the function  $B_{\mu}^{\alpha} [u](x)$

$$\begin{aligned}
 B_{\mu}^{\alpha} [u](x) &= \frac{r^{1-\mu}}{\Gamma(1-\alpha)} \frac{d}{dr} \int_0^r (r-\tau)^{-\alpha} \tau^{\mu-1+\alpha} u(\tau\theta) d\tau = \frac{r^{1-\mu}}{\Gamma(1-\alpha)} \frac{d}{dr} \left[ r^{\mu} \int_0^1 (1-s)^{-\alpha} s^{\mu-1+\alpha} u(sx) ds \right] \\
 &= \frac{r^{1-\mu}}{\Gamma(1-\alpha)} \left[ \mu r^{\mu-1} \int_0^1 (1-s)^{-\alpha} s^{\mu-1+\alpha} u(sx) ds + r^{\mu} \frac{d}{dr} \left( \int_0^1 (1-s)^{-\alpha} s^{\mu-1+\alpha} u(sx) ds \right) \right] \\
 &= \left( r \frac{d}{dr} + \mu \right) \int_0^1 \frac{(1-s)^{-\alpha}}{\Gamma(1-\alpha)} s^{\mu-1+\alpha} u(sx) ds.
 \end{aligned}$$

Furthermore, since  $\Delta \left( r \frac{d}{dr} v(x) \right) = \left( r \frac{d}{dr} + 2 \right) v(x)$  and  $\Delta u(x) = f(x)$ , then

$$\Delta B_{\mu}^{\alpha} [u](x) = \left( r \frac{d}{dr} + \mu + 2 \right) \int_0^1 \frac{(1-s)^{-\alpha}}{\Gamma(1-\alpha)} s^{\mu+1+\alpha} f(sx) ds.$$

After the substitution  $s = r^{-1} \xi$  we have

$$\int_0^1 (1-s)^{-\alpha} s^{\mu+1+\alpha} f(sx) ds = \int_0^r (r-\xi)^{-\alpha} \xi^{\mu+1+\alpha} f(\xi\theta) d\xi \cdot r^{-(\mu+2)}.$$

Consequently

$$\Delta B_{\mu}^{\alpha} [u](x) = \frac{1}{\Gamma(1-\alpha)} \left( r \frac{d}{dr} + \mu + 2 \right) \left[ r^{-(\mu+2)} \cdot \int_0^r (r-\xi)^{-\alpha} \xi^{\mu+1+\alpha} f(\xi\theta) d\xi \right]$$

$$\begin{aligned}
&= \left[ -(\mu+2)r^{-(\mu+2)} + (\mu+2)r^{-(\mu+2)} \right] \int_0^r \frac{(r-\xi)^{-\alpha}}{\Gamma(1-\alpha)} \xi^{\mu+1+\alpha} f(\xi\theta) d\xi \\
&+ r^{-(\mu+1)} \frac{d}{dr} \int_0^r \frac{(r-\xi)^{-\alpha}}{\Gamma(1-\alpha)} \xi^{\mu+1+\alpha} f(\xi\theta) d\xi = r^{-(\mu+1)} D^\alpha \left[ r^{\mu+1+\alpha} f(x) \right] \\
&= r^{1-(\mu+2)} D^\alpha \left[ r^{(\mu+2)-1+\alpha} f(x) \right] \equiv B_{\mu+2}^\alpha [f](x).
\end{aligned}$$

The proof of the Lemma 8 is completed.

**Consequence 2.3.** Let  $0 < \alpha \leq 1$ ,  $\mu > 0$  and  $\Delta^m u(x) = f(x)$ ,  $x \in \Omega$ ,  $m \geq 2$ , where  $f(x)$  – smooth function in  $\bar{\Omega}$ . Then for all  $x \in \Omega$  we have

$$\Delta^m B_\mu^{-\alpha} [u](x) = B_{\mu+2m}^{-\alpha} [f](x) = u(x), \quad x \in \Omega, \quad (15)$$

$$\Delta^m B_\mu^\alpha [u](x) = B_{\mu+2m}^\alpha [f](x) = u(x), \quad x \in \Omega. \quad (16)$$

### 3. Boundary problems for biharmonic equations

In this section, we investigate the boundary problems for the biharmonic equations with fractional operators on the boundary.

**Theorem.** Let  $0 < \alpha \leq 1$ ,  $\mu > 0$ ,  $f(x)$ ,  $g_1(x)$  and  $g_2(x)$  are sufficiently smooth functions. Then the solution of the biharmonic equation

$$\Delta^2 u(x) = f(x), \quad x \in \Omega, \quad (17)$$

in the class  $C^4(\Omega) \cap C(\bar{\Omega})$ , such that  $B_\mu^\alpha [u](x)$ ,  $r \frac{\partial}{\partial r} B_\mu^\alpha [u](x) \in C(\bar{\Omega})$ , satisfying the boundary conditions

$$B_\mu^\alpha [u](x) = g_1(x), \quad x \in \partial\Omega, \quad (18)$$

$$\frac{\partial}{\partial r} B_\mu^\alpha [u](x) = g_2(x), \quad x \in \partial\Omega, \quad (19)$$

exists, unique and is represented in the following form

$$u(x) = B_\mu^{-\alpha} [v](x), \quad (20)$$

where  $v(x)$  is the solution of the Dirichlet problem:

$$\begin{cases} \Delta^2 v(x) = B_{\mu+4}^\alpha [f](x), & x \in \Omega \\ v(x)|_{\partial\Omega} = g_1(x); \quad \frac{\partial v(x)}{\partial \nu} \Big|_{\partial\Omega} = g_2(x) \end{cases} \quad (21)$$

Note that in the case  $\alpha = 1$  and  $\mu > 0$  this result is similar to the third boundary problem for biharmonic equation. Therefore in the general case we have fractional analogue of the Robin problem for non homogeneous biharmonic equation. Note that some boundary problems for the biharmonic equation with boundary operators of fractional order were investigated in the works [15,16].

**Proof of the Theorem.** First, we assume that the solution of the biharmonic equation (17) satisfying the boundary conditions (18), (19) exists in the class  $C^4(\Omega) \cap C(\bar{\Omega})$  exists. Let denote it by  $u(x)$ . We note that

$B_\mu^\alpha [u](x) \in C(\bar{\Omega})$  and  $r \frac{\partial}{\partial r} B_\mu^\alpha [u](x) \in C(\bar{\Omega})$ . We use notation  $v(x) = B_\mu^\alpha [u](x)$ , then it is easy to see that

$$\Delta^2 v(x) = \Delta^2 \left[ B_\mu^\alpha [u](x) \right] = B_{\mu+4}^\alpha [f](x).$$

In addition, from the boundary conditions (18) and (19) it is followed that

$$v(x)|_{\partial\Omega} = B_\mu^\alpha [u](x)|_{\partial\Omega} = g_1(x)$$

and

$$\frac{\partial v(x)}{\partial r} \Big|_{\partial\Omega} = \frac{\partial B_\mu^\alpha [u](x)}{\partial r} \Big|_{\partial\Omega} = g_2(x).$$



So if the function  $u(x)$  is the solution of the biharmonic equation (17) with boundary conditions (18), (19), then the function  $v(x) = B_\mu^\alpha[u](x)$  is the solution of the Dirichlet problem (21). It is well known that if the functions  $f(x), g_1(x)$  and  $g_2(x)$  are sufficiently smooth, then the solution of the Dirichlet problem exists and unique. By applying the inverse operator  $B_\mu^{-\alpha}$  to the equation  $v(x) = B_\mu^\alpha[u](x)$  we see that  $B_\mu^{-\alpha}[v](x) = B_\mu^{-\alpha}[B_\mu^\alpha[u]](x) = u(x)$ , which proves the representation for the solution of the biharmonic equation (17) satisfying the boundary conditions (18) and (19).

Let now assume that the function  $v(x)$  is the solution of the Dirichlet problem (21). Consider a function  $u(x) = B_\mu^{-\alpha}[v](x)$ . We show that the such defined function  $u(x)$  satisfies all conditions of the theorem. Indeed, using the properties in (15) and (14) for all  $x \in \Omega$  we obtain

$$\Delta^2 u(x) = \Delta^2 B_\mu^{-\alpha}[v](x) = B_{\mu+4}^{-\alpha}[B_{\mu+4}^\alpha[f]](x) = f(x),$$

i.e. the function  $u(x) = B_\mu^{-\alpha}[v](x)$  satisfies the equation (17). After application of the operator  $B_\mu^\alpha$  to  $u(x) = B_\mu^{-\alpha}[v](x)$ . By virtue of the second equation (14) we obtain,  $B_\mu^\alpha[u](x) = B_\mu^\alpha[B_\mu^{-\alpha}[v]](x) = v(x)$ .

Then,  $B_\mu^\alpha[u](x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = g_1(x)$  and  $\frac{\partial}{\partial r} B_\mu^\alpha[u](x)|_{\partial\Omega} = \frac{\partial v(x)}{\partial r}|_{\partial\Omega} = g_2(x)$ , which proves that the boundary

conditions also are satisfied. The uniqueness of the solution can be derived from the uniqueness of the solution of the Dirichlet problem. The proof of the Theorem is completed.

#### 4. Conclusion

The obtained properties of the integro-differential operators are very important in application to the theory fractional differential equations. One of the interesting findings of the paper is that the newly defined fractional operators, which are generalization of the Bavrin operators to the fractional value of the parameters, map a class of polyharmonic functions on the ball to the class of polyharmonic functions. As an application of the obtained properties of the fractional integro-differential operators, it is shown that the solution of the biharmonic equation with fractional boundary conditions exists and unique.

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