ASM Sc. J., 12, Special Issue 5, 2019 for ICoAIMS2019, 86-90

Development on Mathematical Model of Convective Boundary Layer Flow of Viscoelastic Fluid with Microrotation Effect under Constant Wall Temperature Thermal Condition over a Bluff Body

Laila Amera Aziz[1]*, Abdul Rahman Mohd Kasim[1] and Mohd Zuki Salleh[1]

[1]*Centre for Mathematical Sciences, College of Computing and Applied Sciences, Universiti Malaysia Pahang, 26300, Gambang, Pahang, Malaysia*

This paper presents the development of mathematical model on the flow of viscoelastic fluid associated with microrotation properties under boundary layer approximation. The respective fluid, also known as viscoelastic micropolar fluid is a member of the non-Newtonian fluid family that contains microstructure while displaying the characteristic of being viscous and elastic. Due to the intricate nature of the fluid and enhanced with the fact that the fluid flows over a bluff body, a complex mathematical model is proposed. The governing equations are derived from the three fundamental physical principals upon which all fluid dynamics are based and later expressed as two-dimensional boundary-layer coordinate. The complexity of the model is reduced after undergoing the boundary layer and Boussinesq approximation. Before it is solved numerically, the mathematical equations of the respective model are subjected to another transformation where the associated equations are changed into dimensionless form and can be written in the simplest form of ordinary or partial differential equations.

Keywords: viscoelastic micropolar; horizontal circular cylinder; magnetohydrodynamic

List of symbols

a	radius of cylinder
$\bar{u},\bar{v}$	dimensional velocity component along $\bar{x},\bar{y}$ axes
u,v	non-dimensional velocity component along x,y axes
$\bar{x},\bar{y}$	dimensional boundary layer cording along the surface of cylinder and normal to it
x,y	non-dimensional boundary layer cording along the surface of cylinder and normal to it
k	thermal conductivity
c_p	specific heat
$\mathbf{F}_b$	body force
H	non-dimensional microrotation
$\bar{H}$	dimensional microrotation
j	microinertia density
p	pressure
$\bar{\mathbf{V}}$	dimensional velocity vector
T	dimensional temperature
B	magnetic field strength
g	gravitational force
Re	Reynolds number
M	magnetic parameter
u_e	free stream velocity
T_w	surface temperature
T_∞	ambient temperature

Greek symbols

ρ	fluid density
τ	stress tensor
κ	vortex viscosity
μ	dynamic viscosity
θ	non-dimensional temperature
σ	electrical conductivity
λ	mixed convection parameter
β	thermal expansion coefficient
α	inclination angle

*Corresponding author's e-mail: laila@ump.edu.my

I. INTRODUCTION

Microrotation and viscoelasticity properties of fluid plays significant roles in engineering applications. Micropolar fluid, for example has proven to be better lubricants than other fluid with the same viscosity. According to (Tipei, 1979), lubricants with microrotational properties generates higher pressure and load carrying capacity while enhancing bearing performance. Similar studies on micropolar fluids as lubricants have also been shown in (Prakash & Sinha, 1975; Zu-gan & Zhang-ji, 1987; Das & Guha, 2018). The mathematical concept of micropolar fluid has been discussed in details by (Lukaszewicz, 1999) in his work and besides lubrication theory, other applications in daily life of the fluid has also been highlighted. Among them is how micropolar model can serve to represent biological flows such as the blood flow in our body and lubrication in human joints which is a great contribution in biomedical engineering.

Viscoleastic fluid is viscous fluid with elasticity characteristics such as paints, polymeric fluids, DNA suspension and even some biological fluid (Perez-Reyes et al.,2018). In industry, dampers are crucial for ventilation system and the existence of viscoelasticity in damping agent is able to improve the performance of a typical industrial damper (Ashrafi & Eskafi, 2011). According to (Nie & Kumacheva, 2008), viscoelastic fluid also plays a big role in photolithography process to create pattern on surface such as for semiconductor microelectronics and plastic electronics. Other practical application of viscoelastic fluid involves drag reduction and polymeric suspensions.

Since the viscoelastic and micropolar has been proven to have important applications, the mathematical model of both fluid has been presented and solved independently in various studies since decades ago. The mathematical study on micropolar fluid has been led by (Eringen, 1966) which has then motivated other researchers to explore the fluid flow at various geometrics and effects including ((Nazar et al., 2003) and (Salleh et al., 2012). The development of mathematical model on viscoelastic fluid is also as popular, dated as early as 1960s. Among existing models of the flow of viscoelastic fluid that also considers the MHD effect are (Andersson, 1992; Rashidi et al., 2014; Aziz et al., 2017).

What sets apart the mathematical model in this study and the previous models is that our major interest is in fluid with merging characteristics of elasticity and microrotation effect. The flow of viscoelastic micropolar fluid in circular tube and between rotating coaxial cylinder has been proposed by (Yeremeyev & Zubov, 1999) while (Madhavi et al., 2019) focuses on the heat transfer of similar fluid from a vertical cone. This study on the other hand, will concentrate on the derivation of the flow of viscoelastic micropolar fluid on horizontal circular cylinder with the presence of MHD effect.

II. GOVERNING EQUATIONS

The flow of incompressible fluid can be well described by the famous Navier-Stokes equations which embodied the fundamental physical principles that mass, momentum and energy are conserved (Cebeci & Cousteix, 2005). The viscoelastic micropolar model which consists of four equations, namely the continuity, momentum, angular momentum and energy equation is an extended version of the Navier-Stokes model that can be expressed by the following equations.

Continuity equation:

$$\bar{\nabla}\cdot\bar{\mathbf{V}}=0 \tag{1}$$

Momentum equation:

$$\rho\left(\frac{\partial\bar{\mathbf{V}}}{\partial t}+\bar{\mathbf{V}}\cdot\bar{\nabla}\bar{\mathbf{V}}\right)=-\bar{\nabla}\bar{p}+\bar{\nabla}\cdot\boldsymbol{\tau}+\mathbf{F}_b \tag{2}$$

where $\boldsymbol{\tau}=(\mu+\kappa)(2\boldsymbol{d})-k_0(2\boldsymbol{d})$

Angular momentum equation:

$$\rho j\left(\bar{\mathbf{V}}\cdot\bar{\nabla}\right)\bar{\mathbf{H}}=+\kappa\left(-2\bar{H}+\bar{\nabla}\times\bar{\mathbf{V}}\right)+\gamma\bar{\nabla}^2\bar{\mathbf{H}} \tag{3}$$

Energy equation:

$$\rho c_p\left(\bar{\mathbf{V}}\cdot\bar{\nabla}\right)T=k\bar{\nabla}^2T \tag{4}$$

Comparing this model to the classic viscoelastic model in (Mohd Kasim et al., 2013), an additional angular momentum equation as derived by (Tipei, 1979) and (Eremeyev et al., 2013) is incorporated in our fluid model to describe the rotation and movement of the microstructures in the complex fluid. The microinertia density and the spin gradient viscosity in the equation are defined as $j=\frac{a\nu}{U_\infty}$ and $\gamma=\left(\frac{\mu+\kappa}{2}\right)j$. An extra parameter, κ has also been added in the momentum equation to further explain the characteristics of micropolar fluid.

III. MATHEMATICAL ANALYSIS

A set of governing dimensional equations are derived from equation (1) to (4) and written in two-dimensional Cartesian coordinate system as follows.

Continuity equation:

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{5}$$

x-momentum equation:

$$\begin{aligned} \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\mu+\kappa}{\rho}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \frac{\kappa}{\rho}\frac{\partial \bar{N}}{\partial \bar{y}} \\ &-\frac{k_0}{\rho}\left[\begin{array}{l} \bar{u}\left(\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} + \frac{\partial^3 \bar{u}}{\partial \bar{x}\partial \bar{y}^2}\right) + \bar{v}\left(\frac{\partial^3 \bar{u}}{\partial \bar{x}^2\partial \bar{y}} + \frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right) - \\ \frac{\partial \bar{u}}{\partial \bar{y}}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}}\right) + \frac{\partial \bar{u}}{\partial \bar{x}}\left(3\frac{\partial^2 \bar{v}}{\partial \bar{x}\partial \bar{y}} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) \\ -2\frac{\partial \bar{v}}{\partial \bar{x}}\frac{\partial^2 \bar{u}}{\partial \bar{x}\partial \bar{y}} \end{array}\right] \\ &+g\beta\left(T-T_\infty\right)sin\left(\frac{\bar{x}}{a}\right) - \frac{\sigma}{\rho}\bar{u}B^2\sin^2\alpha \end{aligned} \tag{6}$$

y-momentum equation:

$$\begin{aligned} \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\mu+\kappa}{\rho}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right) + \frac{\kappa}{\rho}\frac{\partial \bar{N}}{\partial \bar{x}} \\ &-\frac{k_0}{\rho}\left[\begin{array}{l} \bar{u}\left(\frac{\partial^3 \bar{v}}{\partial \bar{x}^3} + \frac{\partial^3 \bar{v}}{\partial \bar{x}\partial \bar{y}^2}\right) + \bar{v}\left(\frac{\partial^3 \bar{v}}{\partial \bar{x}^2\partial \bar{y}} + \frac{\partial^3 \bar{v}}{\partial \bar{y}^3}\right) \\ -\frac{\partial \bar{v}}{\partial \bar{x}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{x}\partial \bar{y}}\right) - \frac{\partial \bar{v}}{\partial \bar{y}}\left(3\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} - \frac{\partial^2 v}{\partial x^2}\right) \\ -2\frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{x}\partial \bar{y}} \end{array}\right] \\ &-g\beta\left(T-T_\infty\right)cos\left(\frac{\bar{x}}{a}\right) - \frac{\sigma}{\rho}\bar{v}B^2\sin^2\alpha \end{aligned} \tag{7}$$

Angular momentum equation:

$$\rho j\left(\bar{u}\frac{\partial \bar{H}}{\partial \bar{x}} + v\frac{\partial \bar{H}}{\partial \bar{y}}\right) = -\kappa\left(2\bar{H} + \frac{\partial \bar{u}}{\partial \bar{y}} - \frac{\partial \bar{v}}{\partial \bar{x}}\right) + \gamma\left(\frac{\partial^2 \bar{H}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{H}}{\partial \bar{y}^2}\right) \tag{8}$$

Energy equation:

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \alpha\left(\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2}\right) \tag{9}$$

Then equation (5) to (9) are transformed to dimensionless form by using these variables (Kasim, 2014; Mohammad 2015) to simplify our computation by eliminating the unit of each variable and parameter involved.

$$\begin{aligned} &x = \frac{\bar{x}}{a}, y = Re^{1/2}\frac{\bar{y}}{a}, u = \frac{\bar{u}}{U_\infty}, v = Re^{1/2}\frac{\bar{v}}{U_\infty}, \\ &H = Re^{-1/2}\frac{a}{U_\infty}\bar{H}, p = \frac{\bar{p}}{\rho U^2{}_\infty}, \theta = \frac{T-T_\infty}{T_w-T_\infty} \end{aligned} \tag{10}$$

and

$$\begin{aligned} &M = \frac{\sigma B^2 a}{\rho U_\infty}, Re = \frac{U_\infty a}{\nu}, Gr = \frac{g\beta\left(T_w-T_\infty\right)a^3}{\nu^2}, \\ &\lambda = \frac{Gr}{Re^2}, K_1 = \frac{\kappa}{\mu}, K = \frac{k_0\,Re}{\rho a^2} \end{aligned}$$

A. Dimensionless Equation

As a result, these dimensionless equations are produced.

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11}$$

x-momentum equation:

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\left(1+K_1\right)}{Re}\left(\frac{\partial^2 u}{\partial x^2} + Re\frac{\partial^2 u}{\partial y^2}\right) \\ &+K_1\frac{\partial N}{\partial y} + \lambda\theta\sin x - Mu\sin^2\alpha \\ &-K\left[\begin{array}{l} \frac{1}{Re}\left(u\frac{\partial^3 u}{\partial x^3} + v\frac{\partial^3 u}{\partial x^2\partial y} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\right) \\ +\left(u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right) \\ -\frac{2}{Re}\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y} \end{array}\right] \end{aligned} \tag{12}$$

y-momentum equation:

$$\begin{aligned} \frac{1}{Re}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) &= -\frac{\partial p}{\partial y} + \frac{\left(1+K_1\right)}{Re}\left(\frac{1}{Re}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ &-\frac{K_1}{Re}\frac{\partial H}{\partial x} + \frac{\lambda}{Re^{1/2}}\theta\cos x - \frac{Mv}{Re}\sin^2\alpha \\ &-K\left[\begin{array}{l} \frac{1}{Re^2}\left(u\frac{\partial^3 v}{\partial x^3} + v\frac{\partial^3 v}{\partial x^2\partial y} - \frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2}\right) + \\ \frac{1}{Re}\left(\begin{array}{l} u\frac{\partial^3 v}{\partial x\partial y^2} + v\frac{\partial^3 v}{\partial y^3} - \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2} \\ -2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x\partial y} \end{array}\right) \end{array}\right] \end{aligned} \tag{13}$$

Angular momentum equation:

$$u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}=-K_1\left(2H+\frac{\partial u}{\partial y}-\frac{1}{Re}\frac{\partial v}{\partial x}\right)+\left(\frac{1+K_1}{2}\right)\left(\frac{1}{Re}\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}\right) \tag{14}$$

Energy equation:

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\left(\frac{1}{Re}\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right) \tag{15}$$

B. Boundary Layer Approximation

Due to the complex nature of the fluid, the governing equations are fairly as complicated. With the purpose to reduce the complexity, some of the partial derivative terms are eliminated by using the boundary layer approximation. Since $Re \to \infty$ in boundary layer flows, hence $\frac{1}{Re} \to 0$. As a result, all terms with reciprocal of the Reynold numbers can be discarded from our model.

Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{16}$$

Momentum equation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+(1+K_1)\frac{\partial^2 u}{\partial y^2}+K_1\frac{\partial N}{\partial y}-K\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right]+\lambda\theta\sin x-Mu\sin^2\alpha \tag{17}$$

Angular momentum equation:

$$u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}=-K_1\left(2H+\frac{\partial u}{\partial y}\right)+\left(\frac{1+K_1}{2}\right)\frac{\partial^2 H}{\partial y^2} \tag{18}$$

Energy equation:

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2} \tag{19}$$

For the momentum equation, only one equation remains in the model since for y-momentum equation, $-\frac{\partial p}{\partial y}=0$. This implies that the pressure of the flow is only dependant on x.

Considering the case of mixed convection, referring to (Mohammad ,2015), the momentum equation outside the boundary layer region is given by

$$-\frac{\partial p}{\partial x}=u_e\frac{du_e}{dx}+Mu_e \tag{20}$$

where the velocity for the region is represented by u_e.

Substituting equation (20) into (17), the momentum equation can be rewritten as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e\frac{du_e}{dx}+(1+K_1)\frac{\partial^2 u}{\partial y^2}+K_1\frac{\partial N}{\partial y}-K\left[u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right]+\lambda\theta\sin x-M(u-u_e)\sin^2\alpha \tag{21}$$

IV. CONCLUSION

Equation (16), (18) - (21) are now ready to be applied with stream function which will further reduced the complexity of the model by reducing the number of variables in our computation. Then, these equations can be reduced to ordinary differential equation as shown by (Aziz et al., 2017) and solved using finite difference method.

This study is particularly important to demonstrate every single step involved in order to come out with the viscoelastic micropolar model. Without a valid model, any solution or results produced would be impractical. Moreover, this model could not only represent viscoelastic fluid with microrotation effect, but can also model the flow of viscoelastic and micropolar fluid when both fluids are considered separately.

V. ACKNOWLEDGEMENT

The authors would like to express our gratitude for the financial support from Universiti Malaysia Pahang for RDU 1703258 & RDU 170328.

VI. REFEREENCES

Andersson, HI 1992, 'MHD flow of a viscoelastic fluid past a stretching surface', *Acta Mechanica*, vol.95, no.1, pp. 227-230.

Ashrafi, N and Eskafi, MR 2011, 'Application of Viscoelastic Fluids in Industrial Dampers', no. 54921, pp. 189-195.

Aziz, L A, Mohd Kasim, A R, Salleh, M Z, Yusof, N S and Shafie, S 2017, 'Magnetohydrodynamics effect on convective boundary layer flow and heat transfer of viscoelastic micropolar fluid past a sphere', *Journal of Physics: Conference Series,* vol. 890.

Aziz, LA Kasim, ARM Salleh, MZ Shafie, S and Yusoff, WNSW 2017, 'Boundary layer flow of mixed convection viscoelastic micropolar fluid over a horizontal circular cylinder with aligned magnetohydrodynamic effect', *Malaysian Journal of Fundamental and Applied Sciences*, vol.13, no.4, pp 567-571.

Cebeci,T and Cousteix, J 2005, 'Conservation Equations for Mass and Momentum for Incompressible Flows', *Modeling and Computation of Boundary-Layer Flows*, Berlin Heidelberg Springer, pp. 17-25.

Das, S and Guha, S K 2018, 'Non-linear stability analysis of micropolar fluid lubricated journal bearings with turbulent effect' *Industrial Lubrication and Tribology*, vol. 71 no.1, pp. 31-39.

Eremeyev, VA Lebedev, LP and Altenbach, H 2013, *Foundations of Micropolar Mechanics*, 1 edn, Springer-Verlag Berlin Heidelberg.

Eringen, AC 1966, Theory Of Micropolar Fluids, Journal of Mathematics and Mechanics, vol.16, no.1, pp. 1-18.

Kasim, ARM 2014, Convective Boundary Layer Flow of Viscoelastic Fluid, PhD thesis, Faculty of Science Universiti Teknologi Malaysia, Malaysia.

Lukaszewicz, G 1999, *Micropolar Fluids Theory and Applications*, 1 edn, Birkhäuser Basel.

Madhavi, K Ramachandra Prasad, V Subba Rao, A Anwar Bég, O and Kadir, A 2019, 'Numerical Study of Viscoelastic Micropolar Heat Transfer from a Vertical Cone for Thermal Polymer Coating', *Nonlinear Engineering*, vol.8, no.1, pp. 449-460.

Mohammad, NF 2015, Magnetohydrodynamic Flow Past Asphere In A Viscous And Micropolar Fluids For Unsteady Free And Mixed Convective Boundary Layer, PhD thesis, Faculty of Science Universiti Teknologi Malaysia, Malaysia.

Mohd Kasim, AR Mohammad, NF Shafie, S and Pop, I 2013, 'Constant heat flux solution for mixed convection boundary layer viscoelastic fluid', *Heat and Mass Transfer*, vol.49 no.2, pp. 163-171.

Nazar, R, Amin, N and Pop, I 2003, 'Mixed convection boundary layer flow about an isothermal sphere in a micropolar fluid', *International Journal of Thermal Sciences*, vol.42, no.3, pp. 283-293.

Nie, Z and Kumacheva, E 2008, 'Patterning surfaces with functional polymers', *Nature Materials* vol. 7, pp. 277-290.

Perez-Reyes,I Vargas-Aquilar,RO Perez-Vega,SB and Ortiz-Perez, AS 2018, 'Applications of Viscoelastic Fluids Involving Hydrodynamic Stability and Heat Transfer', *Polymer Rheology*, IntechOpen.

Prakash, J and Sinha, P 1975 'Lubrication theory for micropolar fluids and its application to a journal bearing', *International Journal of Engineering Science*, vol. 13 no.3, pp. 217-232.

Rashidi, M Ali, M Freidoonimehr, N Rostami ,B and Hossain, MA 2014 'Mixed convective heat transfer for MHD viscoelastic fluid flow over a porous wedge with thermal radiation' *Advances in Mechanical Engineering*, vol.6.

Salleh, MZ, Nazar, R and Pop, I 2012, 'Numerical solutions of free convection boundary layer flow on a solid sphere with Newtonian heating in a micropolar fluid', *Meccanica* vol.47, no.5, pp. 1261-1269.

Tipei, N 1979, 'Lubrication With Micropolar Liquids and Its Application to Short Bearings', *Journal of Lubrication Technology*, vol. 101 no.3, pp. 356-363.

Yeremeyev, VA and Zubov, LM 1999, 'The theory of elastic and viscoelastic micropolar liquids', *Journal of Applied Mathematics and Mechanics* vol.63, no.5, pp 755-767.

Zu-gan, Q and Zhang-ji, L 1987, 'Lubrication theory for micropolar fluids and its application to a journal bearing with finite length', *Applied Mathematics and Mechanics,* vol. 8 no.7, pp. 655-665.