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Subgraph of Compatible Action Graph for Finite Cyclic Groups of p -Power Order

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Abstract. Given two groups G and H , then the nonabelian tensor product $G \otimes H$ is the group generated by $g \otimes h$ satisfying the relations $gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$ for all $g, g' \in G$ and $h, h' \in H$. If G and H act on each other and each of which acts on itself by conjugation and satisfying $({}^{g,h} g') = {}^g ({}^{h^{-1}} g')$ and $({}^{h,g} h') = {}^h ({}^{g^{-1}} h')$, then the actions are said to be compatible. The action of G on H , ${}^g h$ is a homomorphism from G to a group of automorphism H . If $({}^g h, {}^h g)$ be a pair of the compatible actions for the nonabelian tensor product of $G \otimes H$, then $\Gamma_{G \otimes H} = (V(\Gamma_{G \otimes H}), (E(\Gamma_{G \otimes H})))$ is a compatible action graph with the set of vertices, $V(\Gamma_{G \otimes H})$ and the set of edges, $E(\Gamma_{G \otimes H})$. In this paper, the necessary and sufficient conditions for the cyclic subgroups of p -power order acting on each other in a compatible way are given. Hence, a subgraph of a compatible action graph is introduced and its properties are given.

1. Introduction

Recently, much interesting research on algebra has been given to study the algebraic structure of a ring or group. This topic becomes a sensational research topic for the study by utilizing the tools and the properties of graph theory to establish a specific type of graphs. In the literature, there are many studies have been investigated the algebraic properties of ring or group using a specific graph, for instance, Erfanian et al. [1] defined the non-commuting graph of rings which is denoted by Γ_R , with two vertices x and y are adjacent whenever $xy \neq yx$. Then, some theoretical properties of this graph have been studied. While, in [2] Erfanian et al. introduced the generalized of the conjugate graph $\Gamma_{(G,n)^c}$, which is the graph whose vertices are all the non-central subsets of G with n elements and two distinct vertices x and y joined by an edge if $x = y^g$ for some $g \in G$. Then, some properties of this graph have been investigated such as chromatic, dominating and independence numbers. Furthermore, Abbas and Erfanian [3] defined the nilpotent conjugacy class graph of the group G which is denoted by $\Gamma(G)$ and



whose vertices are the nontrivial conjugacy classes of G and two distinct vertices x^G and y^G are adjacent whenever there exist two elements $x' \in x^G$ and $y' \in y^G$ such that $\langle x', y' \rangle$ is nilpotent. Meanwhile, Rajkumar and Deviy [4] defined the coprime graph of subgroups of G which is denoted by $P(G)$ and studied some theoretical properties of this graph. Furthermore, Selvakumar and Subajini [5] classified the finite groups whose non-cyclic graphs are outer-planar, while in [6] they have discussed some basic properties of the coprime graph Γ_G and classified all the finite groups whose coprime graphs are toroidal and projective. Moreover, Shahoodh et al. [7] provided the compatible action graph $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$, for the nonabelian tensor product for the finite cyclic groups of the p -power order where p is an odd prime. This graph whose vertices set is the set of $\text{Aut}(G)$ and $\text{Aut}(H)$ with two distinct vertices ρ and ρ' are adjacent whenever they are compatible on each other, where $\rho \in \text{Aut}(G)$ and $\rho' \in \text{Aut}(H)$. In this paper, the subgraph of compatible action graph $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$, for the finite cyclic groups of the p -power order has been studied. The main point in this paper is to find the number of the compatible pairs in the intersection between a group and its subgroup which represent as a compatible action graph and its subgraph.

2. Preliminary results

In this section, some of the previous results on compatible action graph and the compatible actions for the finite cyclic groups of the p -power order, where p is an odd prime are stated. We started with necessary and sufficient conditions for the pair of the actions for such type of groups to act compatibly on each other are given in the following proposition.

Proposition 2.1 [8]

Let $G = \langle x \rangle \cong C_{p^\alpha}$ and $H = \langle y \rangle \cong C_{p^\beta}$ be finite cyclic groups. Then there exist mutual actions of G and H on each other such that ${}^y x = x^k$ and ${}^x y = y^l$ for $k, l \in \mathbb{Z}$ if and only if the conditions (i) and (ii) below are satisfied. These actions are compatible if and only if condition (iii) is satisfied as well.

- (i) $\gcd(k, p^\alpha) = \gcd(l, p^\beta) = 1$
- (ii) $k^{p^\beta} \equiv 1 \pmod{p^\alpha}$ and $l^{p^\alpha} \equiv 1 \pmod{p^\beta}$
- (iii) $k^{l-1} \equiv 1 \pmod{p^\alpha}$ and $l^{k-1} \equiv 1 \pmod{p^\beta}$.

In order to prove that the actions are compatible for the abelian groups, it is enough to show the conditions as given in the following proposition.

Proposition 2.2 [8]

Let G and H be groups, which act on each other. If G and H are abelian, then the mutual actions are compatible if and only if ${}^{(g,h)}g' = {}^h g'$ and ${}^{(h,g)}h' = {}^g h'$ for all $g, g' \in G$ and $h, h' \in H$.

The next corollary shows that, if one of the actions is trivial, then the actions are always compatible.

Corollary 2.1 [8]

Let G and H be groups. Furthermore, let G act trivially on H . If G is abelian, then for any action of H on G , the mutual actions are compatible.

The order of the compatible action graph for the finite cyclic groups of the p -power order, where p is an odd prime is presented in the following proposition.

Proposition 2.3 [7]

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of p -power order where p is an odd prime and $\alpha, \beta \geq 3$. Then, the order of the compatible action graph is

- (i) $\left| V(\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p) \right| = (p-1)(p^{\alpha-1} + p^{\beta-1})$ if $G \neq H$.
- (ii) $\left| V(\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p) \right| = (p-1)p^{\alpha-1}$ if $G = H$.

Next, the out-degree for the given vertex in the compatible action graph is given as follows.

Proposition 2.4 [7]

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p -power order where p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v \in V(\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p)$ where $v \in \text{Aut}(G)$ and $|v| = p^k$. Then $\text{deg}^+(v)$ is one of the following:

- (i) $(p-1)p^{\beta-1}$ if $k=0$.
- (ii) $(p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$, with $r = \min\{\alpha, \beta\} - k$ if $k=1, 2, \dots, \alpha-1$.

The following theorem, presented the exact number of the of the compatible pairs of actions that can be identified between the nonabelian tensor product for two finite cyclic groups of the p -power order.

Theorem 2.1 [9]

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be finite cyclic groups of p -power order with p is an odd prime and $\alpha, \beta \geq 3$. Then, there exist $(p-1)p^{\beta-1} + \sum_{k=1}^{\alpha-1} (p-1)p^{k-1} + \sum_{k=1}^{\alpha-1} (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$ compatible pairs of actions when $r = \min\{\alpha, \beta\} - k$. and $d k=1, 2, \dots, \alpha-1$

Next, the number of the compatible pairs of actions for such type of groups, when one of the actions is trivial is given as follows.

Proposition 2.5 [9]

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be finite cyclic groups of the p -power order. Furthermore, let $\rho \in \text{Aut}(G)$ with $|\rho|=1$ and $\alpha, \beta \geq 1$. Then, the number of the compatible pairs of actions is $(p-1)p^{\beta-1}$.

The next theorem, shows the compatibility for two actions that have the p -power order for the two finite cyclic groups of the p -power order.

Theorem 2.2 [10]

Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be groups such that $\alpha, \beta \geq 3$. Furthermore, let $\sigma \in \text{Aut}(G)$ with $|\sigma| = p^k$, where $k=1, 2, \dots, \alpha-1$ and $\sigma' \in \text{Aut}(H)$ with $|\sigma'| = p^{k'}$ where $k'=1, 2, \dots, \beta-1$. Then (σ, σ') is a compatible pair of actions if and only if $k+k' \leq \min\{\alpha, \beta\}$.

3. Main results

In this section, the subgraph of compatible action graph for the finite cyclic groups of the p -power order has been introduced. Let C_{p^α} and C_{p^β} be two finite cyclic groups of the p -power order where p is an odd prime such that $\alpha, \beta \geq 3$. Without loss of generality, suppose that $C_{p^{\alpha-i}}$ and $C_{p^{\beta-i}}$ are two subgroups of C_{p^α} and C_{p^β} respectively with $i=1,2,\dots,\min\{\alpha,\beta\}-2$. Then, the subgraph of compatible action graph has been defined for the subgroups $C_{p^{\alpha-i}}$ and $C_{p^{\beta-i}}$ by reducing i values from the power of the groups C_{p^α} and C_{p^β} which is denoted by $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$ where $i=1,2,\dots,\min\{\alpha,\beta\}-2$. Meanwhile, our concern in this paper is to find intersection between the compatible action graph and its subgraph which is the main point in this paper. Thus, the necessary and sufficient conditions for the cyclic subgroups of the p -power order acting on each other in a compatible way when the order of the subgroups are reduced by the same power order from the order of the groups are given in the following proposition.

Proposition 3.1

Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be the finite cyclic groups of the p -power order where p is an odd prime with $\alpha, \beta \geq 3$. Furthermore, let (ρ, ρ') is a compatible pair of actions for $C_{p^\alpha} \otimes C_{p^\beta}$ where $\rho(g) = g^k$ and $\rho'(h) = h^l$ with $k, l \in N$. Then, (ρ, ρ') is a compatible pair of actions for $C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}$ where $\rho(g) = g^{k \bmod p^{\alpha-i}}$ and $\rho'(h) = h^{l \bmod p^{\beta-i}}$ with $i=1,2,\dots,\min\{\alpha,\beta\}-2$.

Proof:

Without loss of generality, assume that $C_{p^{\alpha-i}} \leq C_{p^\alpha}$ and $C_{p^{\beta-i}} \leq C_{p^\beta}$, then $C_{p^{\alpha-i}} = \langle g' \rangle$ and $C_{p^{\beta-i}} = \langle h' \rangle$ for some $g' \in G$ and $h' \in H$. Since (ρ, ρ') is a compatible pair of actions for $C_{p^\alpha} \otimes C_{p^\beta}$, then there exist a mutual actions of G and H on each other such that ${}^h g = g^k$ and ${}^g h = h^l$ for $k, l \in N$. In order to prove $\rho(g) = g^{k \bmod p^{\alpha-i}}$ and $\rho'(h) = h^{l \bmod p^{\beta-i}}$ is a compatible pair of actions for $C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}$, by Proposition 2.1, there are three conditions need to be satisfied as follows.

$$(i) \quad \gcd(k, p^{\alpha-i}) = \gcd(l, p^{\beta-i}) = 1.$$

Define that $\rho: G \rightarrow G$ with $\rho(g) = g^k$ and $k \in N$ is an automorphism if and only if $\gcd(k, p^\alpha) = 1$. Since p^α is an odd number because p is odd, then k must be even. Therefore, $\gcd(k, p^{\alpha-i}) = 1$. Similarly, there exist a mutual actions of G and H such that ${}^g h = h^l$. Since $\gcd(l, p^\beta) = 1$, then $\gcd(l, p^{\beta-i}) = 1$. Hence, $\gcd(k, p^{\alpha-i}) = \gcd(l, p^{\beta-i}) = 1$, and the first condition is satisfied.

$$(ii) \quad k^{p^{\beta-i}} \equiv 1 \pmod{p^{\alpha-i}} \text{ and } l^{p^{\alpha-i}} \equiv 1 \pmod{p^{\beta-i}}.$$

Let H acts on G , then there exist a mutual action of H on G such that ${}^h g = g^k$, then $g = {}^{1H} g = {}^{h^{p^{\beta-i}}} g = g^{k^{p^{\beta-i}}}$. Thus $k^{p^{\beta-i}} \equiv 1 \pmod{p^{\alpha-i}}$. Similarly, if G acts on H , there exist a mutual action of G on H such that ${}^g h = h^l$, then $l^{p^{\alpha-i}} \equiv 1 \pmod{p^{\beta-i}}$. Hence, the second condition is satisfied.

$$(iii) \quad k^{l-1} \equiv 1 \pmod{p^{\alpha-i}} \text{ and } l^{k-1} \equiv 1 \pmod{p^{\beta-i}}.$$

By Proposition 2,2 G and H act compatibly on each other if and only if ${}^{(g h)} g = {}^h g$ and ${}^{(h g)} h = {}^g h$. From the first condition, ${}^g h = h^l = g^{k^l}$ and ${}^h g = g^k$. Thus $k^l \equiv k \pmod{p^{\alpha-i}}$ or equivalently

$k^{l-1} \equiv 1 \pmod{p^{\alpha-i}}$ since $\gcd(k, p^{\alpha-i}) = 1$. Similarly for the second condition is $l^k \equiv l \pmod{p^{\beta-i}}$ or equivalently $l^{k-1} \equiv 1 \pmod{p^{\beta-i}}$ since $\gcd(l, p^{\beta-i}) = 1$. Hence, the third condition is hold. Therefore, $\rho(g) = g^{k \pmod{p^{\alpha-i}}}$ and $\rho'(h) = h^{l \pmod{p^{\beta-i}}}$ is a compatible pair of actions for $C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}$. \square

Next, the order of the subgraph of compatible action graph is investigated.

Proposition 3.2

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p -power order where p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$ be two compatible action graphs with $i = 1, 2, \dots, \min\{\alpha, \beta\} - 2$. Then, the order of the subgraph of compatible action graph is

$$\left| \Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p \right| = \frac{1}{p^i} \left| \Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \right|.$$

Proof:

From Proposition 2.3, the order of the compatible action graph have been considered into two cases which are $G = H$ and $G \neq H$. Thus, two cases are considered as follows.

Case I: Suppose that $G \neq H$. By Proposition 2.3(i), $\left| \Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \right| = (p-1)(p^{\alpha-1} + p^{\beta-1})$. Thus,

$$\left| \Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p \right| = (p-1)(p^{\alpha-i-1} + p^{\beta-i-1}) = \frac{(p-1)}{p^i} (p^{\alpha-1} + p^{\beta-1}) = \frac{1}{p^i} \left| \Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \right|.$$

Case II: Suppose that $G = H$. From Proposition 2.3(ii), $\left| \Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \right| = (p-1)p^{\alpha-1}$. Thus,

$$\left| \Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p \right| = (p-1)p^{\alpha-i-1} = \frac{(p-1)}{p^i} p^{\alpha-1} = \frac{1}{p^i} \left| \Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \right|. \quad \square$$

The next proposition shows the number of the edges of the subgraph of compatible action graph. From Theorem 2.1, there are two cases are considered as follows.

Proposition 3.3

Let C_{p^α} and C_{p^β} be the finite cyclic groups of the p -power order where p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$ be two compatible action graphs where $i = 1, 2, \dots, \min\{\alpha, \beta\} - 2$.

(i) If $\deg^+(v) = (p-1)p^{\beta-1}$, then $\left| E(\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p) \right| = \frac{(p-1)}{p^i} p^{\beta-1}$.

(ii) If $\deg^+(v) = (p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$, then

$$\left| E(\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p) \right| = \sum_{k=1}^{\alpha-1} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-1} \sum_{i=1}^r (p-1)p^{i-1} \right] \text{ where } r = \min\{\alpha-i, \beta-i\} - k.$$

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p -power order where p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$ be two compatible action graphs where $i = 1, 2, \dots, \min\{\alpha, \beta\} - 2$ and $v \in V(\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p)$, then two cases are considered as follows.

Case I: By Proposition 2.4(i), $\deg^+(v) = (p-1)p^{\beta-1}$ and v represent the trivial automorphism. By Corollary 2.1, v is compatible with any vertex and by Proposition 2.5, there exist $(p-1)p^{\beta-1}$ compatible pairs of actions. Thus,

$$\left| E(\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-i}}}^p) \right| = (p-1)p^{\beta-i-1} = \frac{(p-1)}{p^i} p^{\beta-1}.$$

Case II: By Theorem 2.2, the actions are compatible when $k + k' \leq \min\{\alpha, \beta\}$. By Proposition 2.4(ii),

$\deg^+(v) = (p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$, where $r = \min\{\alpha, \beta\} - k$. From the assumption we

have $v \in \Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$. Thus, $\left| E(\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p) \right| = \sum_{k=1}^{\alpha-1} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-1} \sum_{i=1}^r (p-1)p^{i-1} \right]$, with

$$r = \min\{\alpha - i, \beta - i\} - k. \quad \square$$

Next, the number of the compatible pairs of actions in the intersection between two subgroups can be presented as the intersection of the compatible action graph and its subgraph which has been determined. Thus, for this research we let $i = 1$ as a reduce for the power of the subgroups of the finite cyclic groups of p -power order. Therefore, when one of the actions is trivial, then the number of the edges in the intersection between the compatible action graph and its subgraph is presented in the following lemma.

Lemma 3.1

Let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ be two compatible action graphs where p is an odd prime and $\alpha, \beta \geq 3$.

Furthermore, let v be a vertex in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ such that $\deg^+(v) = (p-1)p^{\beta-1}$. Then, there are

$$\frac{(p-1)p^{\beta-1}}{p} \text{ number of edges in } \Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p.$$

Proof:

Let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ be two compatible action graphs where p is an odd prime and $\alpha, \beta \geq 3$.

Furthermore, let v be a vertex in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$. By Proposition 2.4(i), $\deg^+(v) = (p-1)p^{\beta-1}$.

Since $C_{p^{\alpha-1}}$ and $C_{p^{\beta-1}}$ are subgroups from C_{p^α} and C_{p^β} , then, the vertex v must be in $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$. By

Proposition 3.3(i), when one of the vertex has out-degree $(p-1)p^{\beta-1}$, then there are $\frac{(p-1)}{p^i} p^{\beta-1}$ number

of edges in $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}^p$. Since $i=1$, then there are $\frac{(p-1)}{p} p^{\beta-1}$ number of edges in

$$\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p. \quad \square$$

The next lemma gives the number of compatible pairs of actions in the intersection between the compatible action graph and its subgraph when one of the actions has p -power order. Thus, this case can be represented as Lemma 3.1, but now we consider the vertex $v \in \text{Aut}(G)$ and $|v| = p^k$, where $k = 1, 2, \dots, \alpha - 1$.

Lemma 3.2

Let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ be two compatible action graphs where p is an odd prime and $\alpha, \beta \geq 3$.

Furthermore, let v be a vertex in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ such that

$\deg^+(v) = (p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$ with $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, \dots, \alpha - 1$. Then,

there are $\sum_{k=1}^{\alpha-2} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-2} \sum_{i=1}^r (p-1)p^{i-1} \right]$ number of edges in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$, where

$r = \min\{\alpha, \beta\} - k - 1$ and $k = 1, 2, \dots, \alpha - 2$.

Proof:

Since $C_{p^{\alpha-1}}$ and $C_{p^{\beta-1}}$ are subgroups from C_{p^α} and C_{p^β} , then, the vertex v must be in $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$. By

Proposition 3.4(ii), $\deg^+(v) = (p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$, with $r = \min\{\alpha, \beta\} - k$ and

$k = 1, 2, \dots, \alpha - 1$. By Proposition 3.3(ii), when $C_{p^\alpha} \nu \in V(\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p)$ with $\nu \in \text{Aut}(G)$ and $|\nu| = p^k$,

where $k = 1, 2, \dots, \alpha - 1$, then the number of the edges in the subgraph $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ is

$\sum_{k=1}^{\alpha-1} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-1} \sum_{i=1}^r (p-1)p^{i-1} \right]$ where $k = 1, 2, \dots, \alpha - 1$ and $r = \min\{\alpha - i, \beta - i\} - k$. Since k is

hold for each values of $1, 2, \dots, \alpha - 1$, then k is also hold for $1, 2, \dots, \alpha - 2$. Since the order of the actions is reduced, then the bound represent the power of the order of β and α where $r = \min\{\alpha, \beta\} - k - i$,

the subgroups. Therefore, there are $\sum_{k=1}^{\alpha-2} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-2} \sum_{i=1}^r (p-1)p^{i-1} \right]$ number of edges in

$\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ where $r = \min\{\alpha, \beta\} - k - 1$ and $k = 1, 2, \dots, \alpha - 2$. □

In general, the number of the edges in the intersection between the compatible action graph and its subgraph for the finite cyclic groups of the p -power order where p is an odd prime has been found. Thus, the following theorem shows the number of the compatible pairs of actions that exist in the groups G as well as its subgroup.

Theorem 3.1

Let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ be two compatible action graphs where p is an odd prime and $\alpha, \beta \geq 3$.

Then, there are $\frac{(p-1)p^{\beta-1}}{p} + \sum_{k=1}^{\alpha-2} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-2} \sum_{i=1}^r (p-1)p^{i-1} \right]$ number of edges in

$\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ where $r = \min\{\alpha, \beta\} - k - 1$ and $k = 1, 2, \dots, \alpha - 2$.

Proof:

Let $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p$ and $\Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ be two compatible action graphs where p is an odd prime and $\alpha, \beta \geq 3$.

The number of the edges in the intersection between the compatible action graph and its subgraph can be determined by separating into two cases as follows.

Case I: Suppose that the vertex v in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ such that $\deg^+(v) = (p-1)p^{\beta-1}$. By Lemma

3.1, there are $\frac{(p-1)}{p} p^{\beta-1}$ number of edges in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$.

Case II: Suppose that the vertex v in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$ such that $\deg^+(v) = (p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^r (p-1)p^{i-1}$. By Lemma 3.2, there are $\sum_{k=1}^{\alpha-2} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-2} \sum_{i=1}^r (p-1)p^{i-1} \right]$ number of edges in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$. Thus, in total, there are $\frac{(p-1)p^{\beta-1}}{p} + \sum_{k=1}^{\alpha-2} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-2} \sum_{i=1}^r (p-1)p^{i-1} \right]$ number of edges in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$, where $r = \min\{\alpha, \beta\} - k - 1$ and $k = 1, 2, \dots, \alpha - 2$. \square .

4. Conclusion

The subgraph of the compatible action graph for the finite cyclic groups of the p -power order, where p is an odd prime has been studied. The obtained results concluded that, there are $\frac{(p-1)p^{\beta-1}}{p} + \sum_{k=1}^{\alpha-2} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-2} \sum_{i=1}^r (p-1)p^{i-1} \right]$ compatible pairs of actions in $\Gamma_{C_{p^\alpha} \otimes C_{p^\beta}}^p \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}^p$, where $r = \min\{\alpha, \beta\} - k - 1$ and $k = 1, 2, \dots, \alpha - 2$.

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