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On the generalized Radimacher-Menchoff Theorem for general spectral decomposition of the elliptic differential operators

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Abstract. The spectral decompositions of elliptic operators are considered. An analogue of the Radimacher-Menchoff Theorem for general spectral expansions corresponding to a self-adjoint extension of elliptic operators is obtained. The theorem obtained allows us to obtain a result on the almost everywhere convergence of the spectral decompositions from the Liouville classes. The estimation of the maximal operator corresponding to the spherical sums of multiple Fourier series from Liouville classes $L_p^\alpha(T^N)$, $1 \le p \le 2$, is derived.

1. Introduction

1.1. Torus

We define Torus as a cube $[-\pi, \pi]^N$:

$$T^N = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : -\pi < x_i \le \pi, i = 1, \cdots, N\},$$

which naturally isomorphic to $\mathbb{R}^N/\mathbb{Z}^N$. By this we mean, for $x, y \in \mathbb{R}^N$ we say that

$$x \equiv y,$$

if $x - y \in 2\pi\mathbb{Z}^N$. Here $\equiv$ is an equivalence relation that partitions $\mathbb{R}^N$ into equivalence classes, where $2\pi\mathbb{Z}^N$ is the additive subgroup of $\mathbb{R}^N$ and $\mathbb{Z}^N$ is integer coordinates.

The T^N in this way can be indicated of as Cartesian product of N copies $S^1 \times \cdots \times S^1$ of the circle. The T^N is an additive group and then zero $0 = (0_1, 0_2, \cdots, 0_N)$ is the identity element of the group. Therefore, the inverse of $x \in T^N$ is denote by $-x = (-x_1, -x_2, \cdots, -x_N)$.

The T^N can be thought of as subset of $\mathbb{C}^N$ such that

$$(e^{ix_1}, e^{ix_2}, \cdots, e^{ix_N}) \in \mathbb{C}^N, \quad (x_1, x_2, \cdots, x_N) \in [-\pi, \pi]^N,$$

this mean the interval $[-\pi, \pi]$ can be visualized as the unit circle in $\mathbb{C}$ once $-\pi$ and π are identified. Now, we say that a function f is $2\pi-$periodic in every coordinate, if

$$f(x_1 + 2\pi n_1, x_2 + 2\pi n_2, \cdots, x_N + 2\pi n_N) = f(x_1, x_2, \cdots, x_N),$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{Z}^N$. Hence, such a function is define on torus T^N. The $N-$dimensional Lebesgue measure (i.e volume in $\mathbb{R}^N$) is restricted to the set $T^N = [-\pi, \pi]^N$ and denoted by dx. By translation invariance of the Lebesgue measure (i.e. $m(E + a) = m(E)$, $E \subset \mathbb{R}^N$, $a \in \mathbb{R}^N$) and the periodicity of functions on T^N, we have

$$\int_{T^N} f(x)\,dx = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1, x_2, \cdots, x_N)\,dx_1 \cdots dx_N = \int\limits_{[-\pi,\pi]^N} f(x)\,dx = \int\limits_{[y,y+2\pi n]^N} f(x)\,dx,$$

for all f on T^N and $y \in \mathbb{R}^N$. Finally, the L_P spaces on T^N are nested such that

$$L_1 \supset \cdots \supset L_2 \supset \cdots \supset L_\infty.$$

Let consider multiple trigonometric series

$$\sum_{n\in\mathbb{Z}^N} c_n e^{i(n,x)} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} c_{n_1 n_2 \cdots n_N} e^{i(n_1x_1+n_2x_2+\cdots+n_Nx_N)}, \tag{1}$$

with arbitrary complex coefficient c_n. We denote by expression (n, x) the inner product. Similar to one dimension case the we define trigonometric polynomial of degree ν as follows:

$$T_\nu(x) = \sum_{|n|\leq\nu} c_n e^{i(n,x)}$$

Indeed, the set of trigonometric polynomial is dense in $L_p(T^N)$.

In order to focus on the Fourier series, one may determine the coefficient c_n in the following manner, assume that the series (1) converge to a function $f(x)$ in the sense that allows one to integrate the series term by term, as it is the case of uniform convergence and convergence in $L_p(T^N)$, then we have

$$f(x) = \sum_{n\in\mathbb{Z}^N} c_n e^{i(n,x)},$$

now by multiplying the both side with $e^{-i(n,x)}$ and integrating over T^N we obtain

$$\hat{f}(n) = (2\pi)^{-N} \int_{T^N} f(x) e^{-i(n,x)} dx, \tag{2}$$

where $\hat{f}(n) = c_n$ is called the Fourier coefficients of $f(x)$. The nth Fourier coefficients are defined if a complex-valued function $f \in L_1(T^N)$, moreover $\hat{f}(n) \to 0$ as $|n| \to 0$.

Hence, one can associate each function $f \in L_p(T^N),\ 1 \leq p \leq \infty$ with a multiple trigonometric series (Fourier series) such that

$$\sum_{n\in\mathbb{Z}^N} \hat{f}(n) e^{i(n,x)}, \tag{3}$$

so-called the Fourier series of the function $f(x)$. The main and natural questions arise here: does the Fourier series converge to f? What is the effect of dimensions on behave of the Fourier series? Also under which classes of functions the convergence may be true? The study of these questions release a important field so-called Fourier analysis a sub-domain of the harmonic analysis.

1.2. Spectral Theory of the Elliptic Differential Operators
Let consider an arbitrary differential operator with constant coefficients:

$$A(D) = \sum_{|\alpha|\le m} a_\alpha\, D^\alpha, \tag{4}$$

where $D^\alpha = D_1^{\alpha_1}\dots D_N^{\alpha_N}$, $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, and as above, α is a multi-index.

The polynomial $A(\xi)$ is associated with differential operator $A(D)$ by replaced D with $\xi \in \mathbb{R}^N$ and it is called a symbol of operator $A(D)$, the homogeneous polynomial

$$A_h(\xi) = \sum_{|\alpha|=m} a_\alpha\, \xi^\alpha$$

is called its principle symbol. The operator $A(D)$ is said to be elliptic of order m if its principle symbol satisfies $|A_h(\xi)| > 0$ for all $\xi \in \mathbb{R}^N,\ \xi \ne 0$.

The operator $A(D)$ is considered in the Hilbert space $L_2(T^N)$ as an unbounded operator with domain $C^\infty(T^N)$ the class of infinitely differentiable functions on T^N. In case the coefficients are real, the $A(D)$ will satisfy the symmetric condition:

$$(Au, v) = (u, Av),\ \ \forall\, u, v \in C^\infty(T^N). \tag{5}$$

In addition, since the operator $A(D)$ is elliptic, then by Gårding's inequality the operator $A(D)$ is semi-bounded

$$(Au, u) \ge \lambda_A(u, u), \quad \forall\, u \in C^\infty(T^N), \tag{6}$$

where nonnegative constant λ_A is called lower bound of A. Hence, Friedrichs's theorem asserts that for every symmetric semi-bounded operator there are at least one self-adjoint extension with the same lower bound, then there is a self-adjoint extension $\bar{A}$ in $L_2(T^N)$ of operator $A(D)$ which, indeed, its closure, and they are coincided on the domain of definition i. e. $\bar{A}u = Au,\ u \in C^\infty(T^N)$. By von Neumann's spectral theorem, the operator $\bar{A}$ has a spectral decomposition of unity $\{E_\lambda\}$, and then it can be represented in the following form

$$\bar{A} = \int_{\lambda_A}^{\infty} \lambda\, dE_\lambda,$$

the projections E_λ increase monotonically, and continuous on the left, moreover

$$\lim_{\lambda\to\infty} \|E_\lambda u - u\|_{L_2(T^N)} = 0, \quad u \in L_2(T^N).$$

The operator $\bar{A}$ has a complete orthonormal system of eigenfunctions $\{(2\pi)^{-N/2}e^{inx}\}$ in $L_2(T^N)$ corresponding to the eigenvalues $A(n),\ n \in \mathbb{Z}^{\mathbb{N}}$. Thus, the spectral decomposition of $f \in L_2(T^N)$ coincides with partial sums of the multiple Fourier series of function f related to $A(n)$:

$$E_\lambda f(x) = \sum_{A(n)<\lambda} \hat{f}(n)\, e^{i(n,x)}.$$

Some important properties are collected in the following Lemma (for the proof of the properties we refer the readers to [3]).

Lemma 1.1. *Let $f \in L_2(T^N)$. Then one has*

(i) $(E_\lambda f, f) = \sum_{A(n)<\lambda} |\hat{f}(n)|^2$;

(ii) $E_\lambda^2 f = E_\lambda f$;
(iii) $(E_\lambda f, f) \leq (E_\eta f, f), \;\; \lambda \leq \eta$;
(iv) $E_\lambda = 0, \;\; \forall \lambda : \lambda \leq \lambda_A$;
(v) $\lim\limits_{\lambda\to\infty} \|E_\lambda f - f\|_{L_2(T^N)} = 0.$

An interesting fact that the lower order coefficients $a_\alpha, |\alpha| < m$ of $A(D)$ do not influence the convergence of the spectral decomposition $E_\lambda f$ provided the function f is sufficiently smooth. Then one can reduce the study of convergence for partial sum to the study of simpler case, that is, applying the summation over expanding its principle symbol $A_h(n)$.

2. Main result

The spectral decomposition E_λ can be written as an integral operator:

$$E_\lambda f(x) = \int_{T^N} \Theta_\lambda(x,y) f(y)\, dy,$$

where the kernel

$$\Theta_\lambda(x,y) = (2\pi)^{-N} \sum_{A(n)<\lambda} e^{i(n,x-y)} \tag{7}$$

is called the spectral function of operator $\bar{A}$. Indeed, the operator E_λ is a convolution operation $\Theta_\lambda * f$, such that

$$E_\lambda f(x) = \int_{T^N} \Theta_\lambda(x-y) f(y)\, dy. \tag{8}$$

In this paper we deal with problem on the almost everywhere convergence of the multiple Fourier series in the classes of Liouville functions $f \in L_p^\alpha(T^N)$. This type of convergence is a consequence of estimation of the maximal operator:

$$E_* f(x) = \sup_{\lambda>0} |E_\lambda f(x)|\,.$$

The following is the main result of this paper.

Theorem 2.1. *Let $f \in L_2(T^N)$ and E_λ be spectral decomposition of unity corresponding to the self adjoint extension of the elliptic differential operator $A(D)$ (4) on the torus. If*

$$\sum_{n\in\mathbb{Z}^N} |\hat{f}(n)|^2 \log^2(1+A(n)) < \infty,$$

then the elliptic partial sums of f

$$E_\lambda f(x) = \sum_{A(n)<\lambda} \hat{f}(n) e^{i(n,x)} \tag{9}$$

converges almost everywhere to the function $f(x)$ on the torus T^N. Moreover, for the maximal operator $E_* f(x)$ one has

$$\|E_* f\|_{L_2(T^N)} \leq C \left(\sum_{n\in\mathbb{Z}^N} |\hat{f}(n)|^2 \log^2(1+A(n)) \right)^{\frac{1}{2}}. \tag{10}$$

Remark 2.2. *One can visualize the summation of (9) by taking the sum over all integer vectors $n \in \mathbb{Z}^N$ such that they are a solutions of $A(n) = \lambda$, where λ runs through the nonnegative part of real line. In case that there is no integer solutions for certain λ, then they are considered neglected.*

3. Maximal operator

A maximal operator

$$E_*f(x) = \sup_{\lambda>0} |E_\lambda f(x)|.$$

plays important role in the theory of almost everywhere convergence of the Fourier series. If the maximal operator is a bounded from $L_2(T^N)$ to $L_2(T^N)$, then the Fourier series of any function from $L_2(T^N)$ will be almost everywhere convergent.

Lemma 3.1. *Let $f \in L_2(T^N)$. Then the operator*

$$E_*f(x) = \sup_k |E_{2^k} f(x)|$$

is bounded in the following means

$$\|E_*f\|_{L_2(T^N)} \leq C \sum_{n\in\mathbb{Z}^N} |\hat{f}(n)|^2 \log_2 A(n).$$

Proof. Let $f \in L_2(T^N)$. Then for the operator E_{2^k} we have

$$\|E_{2^k} f\|_{L_2(T^N)} \leq \|E_{2^k} f - f\|_{L_2(T^N)} + \|f\|_{L_2(T^N)}.$$

By taking the square from both sides and applying the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we derive

$$\|E_{2^k} f\|_{L_2(T^N)} \leq 2\|f\|^2_{L_2(T^N)} + 2\|E_{2^k} f - f\|^2_{L_2(T^N)}.$$

Taking the supreme

$$\|E_*f\|_{L_2(T^N)} \leq 2\|f\|^2_{L_2(T^N)} + 2\sum_{k=0}^{\infty} \|E_{2^k} f - f\|^2_{L_2(T^N)},$$

the second term is estimated by using the properties of the projector operators as follow

$$\begin{aligned}\sum_{k=2}^{\infty} \|E_{2^k} f - f\|^2_{L_2(T^N)} &= \sum_{k=2}^{\infty} (E_{2^k} f - f, E_{2^k} f - f) \\ &= \sum_{k=2}^{\infty} \left[\|E_{2^k} f\|^2_{L_2(T^N)} - 2(E_{2^k} f, f) + \|f\|^2_{L_2(T^N)}\right].\end{aligned}$$

Taking into account that the operators E_λ are self-adjoint one has

$$\sum_{k=0}^{\infty} \|E_{2^k} f - f\|^2_{L_2(T^N)} = \sum_{k=0}^{\infty} \left[\|f\|^2_{L_2(T^N)} - \|E_{2^k} f\|^2_{L_2(T^N)}\right].$$

Which is equivalent to the following

$$\sum_{k=0}^{\infty} \|E_{2^k} f - f\|^2_{L_2(T^N)} = \sum_{k=0}^{\infty} \sum_{A(n)\geq 2^k} |\hat{f}(n)|^2.$$

The internal sum on the right side of the latter equation we can make first summation over elliptic levels $\{n \in \mathbb{Z}^N : A(n) = \nu\}$, then make summation by $\{\nu : \nu \geq 2^k\} = \{\nu : \log_2 \nu \geq k\}$:

$$\sum_{k=0}^{\infty}\sum_{A(n)\geq 2^k}|\hat{f}(n)|^2=\sum_{A(n)\geq 1}\sum_{k=0}^{[\log_2 A(n)]}|\hat{f}(n)|^2=\sum_{\nu=1}^{\infty}\sum_{k=0}^{[\log_2 \nu]}\sum_{A(n)=\nu}|\hat{f}(n)|^2.$$

We know that $\sum\limits_{k=0}^{[\log_2 \nu]} 1 = 1+[\log_2 \nu]$. Then one easily conclude

$$\sum_{k=0}^{\infty}\|E_{2^k}f-f\|^2_{L_2(T^N)}\leq\sum_{\nu=1}^{\infty}(1+\log_2\nu)\sum_{A(n)=\nu}|\hat{f}(n)|^2\leq C\sum_{n\in\mathbb{Z}^N}|\hat{f}(n)|^2\log_2 A(n).$$

Finally taking into account

$$\|f\|^2_{L_2(T^N)}=\sum_{n\in\mathbb{Z}^N}|\hat{f}(n)|^2\leq\sum_{n\in\mathbb{Z}^N}|\hat{f}(n)|^2\log_2 A(n),$$

we have

$$\|E_*f\|_{L_2(T^N)}\leq C\sum_{n\in\mathbb{Z}^N}|\hat{f}(n)|^2\log_2 A(n),$$

which proves the statement of Lemma 3.1.

The following lemma is helping in our proceed in prove the main result in this chapter.

Lemma 3.2. *For any complex numbers $z_k,\ k=0,1,2,\cdots,2^\gamma-1$ and for any natural number $l:\ 1\leq l<2^\gamma$ we have*

$$|z_l-z_0|^2\leq\gamma\sum_{j=0}^{\gamma-1}\sum_{i=1}^{2^{\gamma-j}-1}|z_{i2^j}-z_{(i-1)2^j}|^2$$

Proof. Any natural number $l:\ 1\leq l<2^\gamma$ can be represented as a sum of numbers in dyadic system:

$$l=\epsilon_0+\epsilon_1 2^1+\epsilon_2 2^2+\cdots+\epsilon_{\gamma-1}2^{\gamma-1},$$

where $\epsilon_r=0$ or $\epsilon_r=1,\ 0\leq r\leq\gamma-1$. For the $q:q=1,2,\cdots,\gamma$ we define

$$l_q=\sum_{r=\gamma-q}^{\gamma-1}\epsilon_r 2^r.$$

Then it is easy to see that

$$l_q-l_{q-1}=\sum_{r=\gamma-q}^{\gamma-1}\epsilon_r 2^r-\sum_{r=\gamma-q+1}^{\gamma-1}\epsilon_r 2^r=\epsilon_{\gamma-q}2^{\gamma-q} \tag{11}$$

The difference z_l-z_0 can be written as follows

$$z_l-z_0=\sum_{q=1}^{\gamma}(z_{l_q}-z_{l_{q-1}}),$$

where $l_\gamma=l$ and $l_0=0$. Since identity (11) and the fact that l_q is multiple of $2^{\gamma-q}$, then when $\epsilon_{\gamma-q}=1$, we set $j=\gamma-q$ to have $l_{q-1}=(i-1)2^j$. Thus $z_{l_q}-z_{l_{q-1}}$ is equal to $z_{i2^j}-z_{(i-1)2^j}$. Applying Cauchy-Swartz inequality we obtain

$$|z_l-z_0|^2\leq\gamma\sum_{q=1}^{\gamma}|z_{l_q}-z_{l_{q-1}}|^2\leq\gamma\sum_{j=0}^{\gamma-1}\sum_{i=1}^{2^{\gamma-j}-1}|z_{i2^j}-z_{(i-1)2^j}|^2.$$

The statement of Lemma is established.

Lemma 3.3. *Let $f \in L_2(T^N)$. Then for the spectral decompositions E_λ one has*

$$\left\| \sup_{0\le k<2^\gamma} |E_k f(x)| \right\|^2_{L_2(T^N)} \le \gamma^2 \sum_{A(n)<2^\gamma-1} |\hat{f}(n)|^2.$$

Proof. Let $f \in L_2(T^N)$. In order to apply Lemma (3.2) we denote $z_k = E_k f(x),\ k = 1, 2, 3, \cdots, 2^\gamma - 1$. We take $z_0 = 0$, then we obtain

$$\begin{aligned}
\left\| \sup_{0\le k<2^\gamma} |E_k f(x)| \right\|^2_{L_2(T^N)} &\le \gamma \sum_{j=0}^{\gamma-1} \sum_{i=1}^{2^{\gamma-j}-1} \left\| E_{i2^j} f(x) - E_{(i-1)2^j} f(x) \right\|^2_{L_2(T^N)} \\
&\le \gamma \sum_{j=0}^{\gamma-1} \sum_{i=1}^{2^{\gamma-j}-1} \sum_{(i-1)2^j \le A(n) \le i2^j} |\hat{f}(n)|^2 \\
&\le \gamma \sum_{j=0}^{\gamma-1} \sum_{A(n)<2^\gamma-1} |\hat{f}(n)|^2.
\end{aligned}$$

And from the following fact $\sum\limits_{j=0}^{\gamma-1} 1 = \gamma$, we conclude

$$\left\| \sup_{0\le k<2^\gamma} |E_k f(x)| \right\|^2_{L_2(T^N)} \le \gamma^2 \sum_{A(n)<2^\gamma-1} |\hat{f}(n)|^2.$$

4. Proof of the main result

Proof. An arbitrary integer number $k > 1$ can be represented as $k = 2^\gamma + l$, where $\gamma \in \mathbb{N}$ and l is an integer such that $0 \le l < 2^\gamma$. Therefore one can derive

$$\begin{aligned}
E_* f(x) = \sup_{k\ge1} |E_k f(x)| \le \sup_{\gamma\ge1} \sup_{0\le l<2^\gamma} |E_{2^\gamma+l} f(x) - E_{2^\gamma} f(x)| + \\
+ \sup_{\gamma\ge0} |E_{2^\gamma} f(x)| = I_1 + I_2.
\end{aligned}$$

First we estimate I_1, taking into account the inequality

$$I_1^2 \le \sum_{\gamma=1}^{\infty} \sup_{0\le l<2^\gamma} |E_{2^\gamma+l} f(x) - E_{2^\gamma} f(x)|^2,$$

and using Lemma 3.3 the norm of I_1 in $L_2(T^N)$ is estimated as follows

$$\begin{aligned}
\|I_1\|^2_{L_2} &\le \sum_{\gamma=1}^{\infty} \left\| \sup_{0\le l<2^\gamma} |E_{2^\gamma+l} f - E_{2^\gamma} f| \right\|^2_{L_2(T^N)} \\
&\le \sum_{\gamma=1}^{\infty} \gamma^2 \sum_{2^\gamma \le A(n) \le 2^{\gamma+1}-1} \left| \hat{f}(n) \right|^2 \\
&\le \sum_{k=3}^{\infty} \log_2^2 k \sum_{k-1\le A(n)<k} \left| \hat{f}(n) \right|^2.
\end{aligned}$$

So we have

$$\|I_1\|_{L_2}^2 \leq \sum_{n\in\mathbb{Z}^N} \log_2^2(1+A(n)) \left|\hat{f}(n)\right|^2.$$

Let now proceed with the estimation of I_2, we note that

$$E_{2^\gamma}f(x) = E_1 f(x) + \sum_{k=1}^{\gamma} \left[E_{2^k}f(x) - E_{2^{k-1}}f(x)\right].$$

From this we conclude that

$$\begin{aligned} I_2 = \sup_{\gamma\geq 0} |E_{2^\gamma}f(x)| &\leq |E_1 f(x)| + \sum_{\gamma=1}^{\infty} |E_{2^\gamma}f(x) - E_{2^{\gamma-1}}f(x)| \\ &= |E_1 f(x)| + |E_2 f(x) - E_1 f(x)| + \sum_{\gamma=2}^{\infty} \frac{1}{\gamma-1}(\gamma-1)|E_{2^\gamma}f(x) - E_{2^{\gamma-1}}f(x)| \end{aligned}$$

By applying the Cauchy-Swartz inequality one has

$$\begin{aligned} I_2^2 \leq &\left(2 + \sum_{\gamma=2}^{\infty} \frac{1}{(\gamma-1)^2}\right) \times \\ &\times \left(|E_1 f(x)|^2 + |E_2 f(x) - E_1 f(x)|^2 + \sum_{\gamma=2}^{\infty}(\gamma-1)^2 |E_{2^\gamma}f(x) - E_{2^{\gamma-1}}f(x)|^2\right). \end{aligned}$$

In the norm of $L_2(T^N)$ space we have

$$\|I_2\|_{L_2(T^N)}^2 \leq 4\left(\|E_1 f\|_{L_2(T^N)}^2 + \|E_2 f - E_1 f\|_{L_2(T^N)}^2 + \sum_{\gamma=2}^{\infty}(\gamma-1)^2 \|E_{2^\gamma}f - E_{2^{\gamma-1}}f\|_{L_2(T^N)}^2\right).$$

Using Lemma 3.3 one has

$$\begin{aligned} \|I_2\|_{L_2(T^N)}^2 &\leq 4\left(\sum_{A(n)<1} |\hat{f}(n)|^2 + \sum_{1\leq A(n)<2} |\hat{f}(n)|^2 + \sum_{\gamma=2}^{\infty}(\gamma-1)^2 \sum_{2^{\gamma-1}\leq A(n)<2^\gamma} |\hat{f}(n)|^2\right) \\ &\leq 4 \sum_{n\in\mathbb{Z}^N} \log_2^2(1+A(n))|\hat{f}(n)|^2. \end{aligned}$$

Collecting the estimations for I_1 and I_2

$$\|E_* f\|_{L_2(T^N)} \leq \|I_1\|_{L_2(T^N)} + \|I_2\|_{L_2(T^N)} \leq C \left(\sum_{n\in\mathbb{Z}^N} |\hat{f}(n)|^2 \log^2(1+A(n))\right)^{\frac{1}{2}}.$$

This prove Inequality (10), and as consequence the Theorem 2.1 is proved.

We say that a function $f \in L_p(T^N),\ p \geq 1$ belongs to the Liouville class $L_p^\alpha(T^N)$, if

$$\left\| \sum_n (1+|n|^2)^{\frac{\alpha}{2}} f_n e^{i(n,x)} \right\|_{L_p(T^N)} < \infty,$$

where the expression on the left side is called a norm of $f \in L_p^\alpha(T^N)$ and denoted by $\|f\|_{L_p^\alpha(T^N)}$.

We will now investigate the conditions for almost everywhere convergence of the Fourier series of functions from the classes $L_2^\alpha(T^N)$. Unlike to the case $L_1^\alpha(T^N)$, the sufficient condition for almost everywhere convergence in the case $L_2^\alpha(T^N)$ is $\alpha > 0$. Hence, we have the following:

Lemma 4.1. *Let $\alpha > 0$. Then the maximal operator E_* is bounded:*

$$\|E_* f\|_{L_2(T^N)} \leq C_{N,\alpha} \|f\|_{L_2^\alpha(T^N)}, \quad \forall f \in L_2^\alpha(T^N). \tag{12}$$

Proof. To prove this fact we use the generalized theorem of Menchoff-Rademacher.

$$\begin{aligned}
\|E_* f(x)\|^2_{L_2(T^N)} &\leq C \sum_{n\in\mathbb{Z}^N} |\hat{f}(n)|^2 \log^2(1+A(n)) \\
&= C \sum_{n\in\mathbb{Z}^N} \left|\hat{f}(n)\right|^2 (1+A(n))^\alpha \frac{\log^2\left(1+|n|^2\right)}{(1+A(n))^\alpha} \\
&\leq C \sup_{n\in\mathbb{Z}^N} \left(\frac{\log^2\left(1+|n|^2\right)}{(1+A(n))^\alpha} \right) \sum_{n\in\mathbb{Z}^N} \left|\hat{f}(n)\right|^2 (1+A(n))^\alpha .
\end{aligned}$$

Taking into account the condition that $\alpha > 0$ one easily can obtain

$$\sup_{n\in Z^N} \left(\frac{\log^2(1+|n|^2)}{(1+A(n))^\alpha} \right) \leq C_\alpha^2.$$

Finally we derive

$$\|E_* f(x)\|_{L_2(T^N)} \leq C_{\alpha,N} \|f\|_{L_2^\alpha(T^N)}.$$

The proof of Lemma 4.1 is completed. □

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