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On solvability of some boundary value problems for the non-local polyharmonic equation with boundary operators of the Hadamard Type

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Abstract. In this paper the solvability problems of some boundary value problems for a non-local polyharmonic equation are studied. A non-local polyharmonic equation is represented by using some orthogonal matrix. The properties and examples of such matrices are given. In the current boundary value problem, which being considered in the paper, the fractional order differentiation operators are used as boundary operators. These operators are defined as derivatives of the Hadamard-Caputo type. Note that in particular cases of the parameters of the boundary conditions we obtain well known conditions of the Dirichlet, Neumann, and Robin type problems. For the problems under consideration, theorems on the existence and uniqueness of solutions are proved. The exact solvability conditions for the problem under study are found. In addition, we obtained representation for the solution of the fractional boundary problem for polyharmonic operator.

1. Introduction

The concept of a non-local operator and the related concept of a non-local differential equation started to appear in mathematics quite recently. E.g., in [1], the notion of non-local differential equations incorporated the loaded equations, equations with fractional derivatives of the unknown function, equations with deviating arguments in other words, all equations in which the unknown function and/or its derivatives enter with different values of arguments. A specific type of the non-local differential equations is formed by equations in which the deviation of arguments has an involution character.

It is well known that differential equations containing an involution in the unknown function or its derivative confer model equations with alternating deviation of the argument. In general such equations can be attributed to the class of functional-differential equations. Solvability issues for certain partial differential equations with involution are covered in [2]-[4]. Besides, in [5]-[9], boundary value problems for second- and fourth-order elliptic equations are studied in the case when an involution appears in the boundary conditions.

This paper is devoted to the study of the some boundary value problems with fractional order for a non-local polyharmonic equation.

Let proceed with the statement of the problems.

We consider $n-$dimensional Euclidean space $R^n, n \geq 2$. Let $\Omega = \{x \in R^n : |x| < 1\}$ be a unit ball. The boundary $\partial\Omega$ of the unit ball Ω is unit sphere.

We denote by S a real orthogonal $n \times n$ matrix: $S \cdot S^T = I_n$, where I_n denotes unit $n \times n$ matrix. A matrix is orthogonal exactly when its column vectors have length one, and are pairwise orthogonal; likewise for the row vectors. In short, the columns (or the rows) of an orthogonal matrix are an orthonormal basis of R^n, and any orthonormal basis gives rise to a number of orthogonal matrices. Any orthogonal matrix is invertible, with $S^{-1} = S^T$. Suppose also that there exists a natural number l such that $S^l = I_n$.

Note that, since any orthogonal transform is isometric, any $x \in \Omega$ and any $x \in \partial\Omega$ satisfies the inclusions $S^k x \in \Omega$, and, respectively, $S^k x \in \partial\Omega$ for any positive integer k.

Let us give some simple examples of such mappings S.

Example 1. Let, for any $x \in \Omega$, the mapping S be defined by the relation $Sx = -x$, i.e. $S = -I_n$. Obviously, one has $S \cdot S^T = -I_n\,(-I_n) = I_n$, $S^2 = I_n$, and therefore l equals 2.

Example 2. The mapping S can clearly be a rotation in the space R^n, e.g. S is the product of rotations $S = C^1_{\varphi_1} C^2_{\varphi_2} \cdots C^{n-2}_{\varphi_{n-2}}$ where C^j_φ corresponds to the matrix

$$\begin{pmatrix} I_j & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & I_{n-j-2} \end{pmatrix},$$

I_j is the $j \times j$ unit matrix and $j = \overline{1, n-2}$. Indeed, one has $S^T = C^{n-2}_{-\varphi_{n-2}} \ldots C^2_{-\varphi_2} C^1_{-\varphi_1}$, $C^i_\varphi C^j_\psi = C^j_{\varphi+\psi}$, and therefore

$$SS^T = C^1_{\varphi_1} C^2_{\varphi_2} \ldots C^{n-2}_{\varphi_{n-2}} \cdot C^{n-2}_{-\varphi_{n-2}} \ldots C^2_{-\varphi_2} C^1_{-\varphi_1} = I_n.$$

Moreover, it is necessary to suppose that exists a natural number $l \in N$ such that $S^l = I_n$.

Let $u(x)$ be a smooth function in the ball Ω, let $r = |x|$, let $\theta = x/r$, and let $d = r\frac{d}{dr}$ be the Dirac operator, where $r\frac{d}{dr} = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$.

For any $\alpha > 0$, the expression

$$J^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^r \left(ln\frac{r}{\tau}\right)^{\alpha-1} u(\tau\theta)\frac{d\tau}{\tau}$$

is called an integration operator of order a in the sense of Hadamard. By convention, $J^0[u](x) = u(x)$.

Note that the operator J^α cannot be applied even to sufficiently smooth functions $u(x)$ with $u(0) \neq 0$. Therefore, for each $\alpha \in (p-1, p], p = 1, 2, ...,$ we define the fractional differentiation operator as the following modification of the Hadamard operator:

$$D^\alpha[u](x) = J^{p-\alpha}\left[\delta^p u\right](\, x) \equiv \frac{1}{\Gamma(p-\alpha)} \int_0^r \left(ln\frac{r}{\tau}\right)^{p-\alpha-1} \left(\tau\frac{d}{d\tau}\right)^p [u(\tau\theta)]\frac{d\tau}{\tau}.$$

Let $\mu \geq 0$. Set

$$J^\alpha_\mu[u](x) = r^{-\mu} J^\alpha[r^\mu u](x), D^\alpha_\mu[u](x) = r^{-\mu} D^\alpha[r^\mu u](x).$$

Let $m \geq 1, 0 \leq \alpha \leq 1$, $a-$ be some real number. We consider the following boundary value problem in Ω

$$(-\Delta)^m u(x) + a(-\Delta)^m u(Sx) = f(x), \quad x \in \Omega, \tag{1}$$

$$D^{\alpha+k}_\mu u(x) = g_k(x), x \in \partial\Omega, k = 0, 1, ..., m-1. \tag{2}$$

By a solution of the problem equations (1) and (2) we mean a function $u(x) \in C^{2m}(\Omega) \cap C(\bar{\Omega})$ with $D_\mu^{\alpha+k} u(x) \in C(\bar{\Omega}), k = 0, 1, ..., m-1$, satisfying equation (1) and the boundary conditions (2) in the classical sense.

Note that the problem (1) and(2) in the case $a = 0$ were earlier considered in [10] and for the case $\alpha = 0, \mu \geq 0, m = 1$, i.e. for Poisson equation, is considered in [11]. In the case $\alpha = 0, \mu = 0$ we obtain a Dirichlet problem and when $\alpha = 1, \mu = 0$ we have the Neumann type problems.

2. Auxiliary statements

Consider the operator $I_S u(x) = u(Sx) = u(x^*)$. In view of what has been said above, this operator is defined on functions $u(x), x \in \Omega$. Let S_{col}^i and S_{row}^i be the i-th column and i-th row of the matrix S, respectively.

We prove two simple lemmas. Let $u(x)$ be a twice continuously differentiable function in Ω.

Lemma 1. *Operators δ and I_S are commutative $\delta I_S u(x) = I_S \delta u(x)$, and also the equality $\nabla I_S = I_S S^T \nabla$ holds, and operators Δ and I_S are also commutative.*

Proof. We can write the operator δ in the form $\delta u = (x, \nabla) u$. Since

$$\frac{\partial}{\partial x_i} I_S u(x) = \frac{\partial}{\partial x_i} u(Sx) = \frac{\partial}{\partial x_i} u((S_{row}^1, x), \ldots, (S_{row}^n, x))$$
$$= \sum_{j=1}^n s_{ji} I_S u_{x_j}(x) = (S_{col}^i, I_S \nabla u(x)) = I_S (S_{col}^i, \nabla) u(x), \quad (3)$$

then

$$\delta I_S u(x) = \delta u(Sx) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} u(Sx) = \sum_{i=1}^n x_i \left(S_{col}^i, I_S \nabla u(x)\right)$$
$$= \left(\sum_{i=1}^n x_i S_{col}^i, I_S \nabla u(x)\right) = (Sx, I_S \nabla u(x)) = I_S (x, \nabla u(x)) = I_S \delta u(x).$$

Further, due to the formula (3), we find

$$\frac{\partial^2}{\partial x_i^2} I_S u(x) = \frac{\partial}{\partial x_i} I_S (S_{col}^i, \nabla) u(x) = I_S (S_{col}^i, \nabla)^2 u(x)$$

and therefore

$$\Delta I_S u(x) = \sum_{i=1}^n I_S (S_{col}^i, \nabla)^2 u(x) = I_S \left| \left((S_{col}^1, \nabla), \ldots, (S_{col}^n, \nabla)\right) \right|^2 u(x)$$
$$= I_S \left| S^T \nabla \right|^2 u(x) = I_S (S^T \nabla, S^T \nabla) u(x) = I_S (S S^T \nabla, \nabla) u(x) = I_S \Delta u(x).$$

At last,
$$\nabla I_S u(x) = I_S ((S_{col}^1, \nabla), \ldots, (S_{col}^n, \nabla)) u(x) = I_S (S^T \nabla) u(x).$$

Corollary 1. If the function $u(x)$ is polyharmonic in Ω, then the function $u(x^*) = I_S u(x)$ is also polyharmonic in Ω.

Indeed, due to Lemma 1, $\Delta^m u(x) = 0 \Rightarrow \Delta^m I_S u(x) = I_S \Delta^m u(x) = 0$.

Lemma 2. The operator $1 + a I_S$, when $(-a)^l \neq 1$ is invertible and the operator

$$G_a = \frac{1}{1 - (-a)^l} \sum_{k=0}^{l-1} (-a)^k I_S^k$$

is inverse to $1 + aI_S$, i.e.

$$G_a\,(1 + aI_S) = (1 + aI_S)\,G_a \equiv E. \tag{4}$$

Proof. It is easy to see that

$$\left(\sum_{k=0}^{l-1}(-a)^k I_S^k\right)(1 + aI_S)u(x) = \left(\sum_{k=0}^{l-1}(-a)^k I_S^k - \sum_{k=1}^{l}(-a)^k I_S^k\right)u(x)$$
$$= \left(E - (-a)^l I_S^l\right)u(x) = (1 - (-a)^l)u(x).$$

Thus, if $(-a)^l \neq 1$ then we can divide both sides of the equality by $1 - (-a)^l$ and hence the operator G_a is inverse to $1 + aI_S$. The proof of Lemma 2 is completed.

3. Auxiliary problems

In this section we study the following problem:

$$\Delta^m v(x) = f(x), x \in \Omega, \tag{5}$$

$$D_\mu^{\alpha+k} v(x) = \psi_k(x), x \in \partial\Omega, k = 0, 1, ..., m-1. \tag{6}$$

The following assertions were proved in [10] for problem (5) and (6).

Theorem 1. Let $\mu > 0, 0 < \lambda < 1, 0 < \alpha \leq 1, f(x) \in C^{\lambda+1}(\bar{\Omega})$, and $\psi_k(x) \in C^{\lambda+2m-k}(\partial\Omega), k = 0, 1, ..., m-1$. Then there exists a unique solution of problem (5),(6), which belongs to the class $C^{\lambda+2m}(\bar{\Omega})$ and can be represented in the form

$$v(x) = J_\mu^\alpha[w](x),$$

where $w(x)$ is the solution of problem

$$\Delta^m w(x) = F(x), x \in \Omega;\ \ \delta_\mu^k[u](x) = \psi_k(x), x \in \partial\Omega, k = 0, 1, ..., m-1, \tag{7}$$

with the function $F(x) = D_{\mu+2m}^\alpha[f](x)$.

Next let consider a following matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 & ... & 1 \\ 0 & 2 & 4 & ... & 2(m-1) \\ 0 & 2^2 & 4^2 & ... & [2(m-1)]^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 2^{m-1} & 4^{m-1} & ... & [2(m-1)]^{m-1} \end{pmatrix}.$$

We denote by $\Delta_j, j = 0, ..., m-1$ the determinants of the matrix obtained from the matrix A by the deletion of the first column and the $(j+1)$ st row. In particular, $\Delta_0 = |A| = detA$. One can readily show that $|A| \neq 0$. Let $0 < \alpha \leq 1$.

For the problem (5) and (6) in case of $\mu = 0, 0 < \alpha \leq 1$ we have

Theorem 2. Let $\mu = 0, 0 < \alpha \leq 1$. If $f(x) \in C^{\lambda+1}(\bar{\Omega})$, $\psi_k(x) \in C^{\lambda+2m-k}(\partial\Omega), k = 0, 1, ...m-1, 0 < \lambda < 1$, then problem (5) and (6) is solvable if and only if the following condition hold

$$\int\limits_{\partial\Omega}\left(1 - |x|^2\right)^{m-1} J_{2m}^{1-\alpha}[f](x)dx = \sum_{k=0}^{m-1} c_{k,m}\int\limits_{\partial\Omega}\psi_k(x)dS_x \tag{8}$$

where

$$c_{k,m} = (-1)^{k+1}\Delta_k \frac{4^{m-1}((m-1)!)^2}{|A|}. \tag{9}$$

If there exists a solution of the problem, then it is unique up to a constant term, belongs to the class $C^{\lambda+2m}(\bar{\Omega})$, and can be represented in the form

$$v(x) = C + J_0^{\alpha}[w](x)$$

where $w(x)$ is the solution of problem (7) with the function $F(x) = D_{2m}^{\alpha}[f](x)$ and with the additional condition $w(0) = 0$.

4. Uniqueness and existence of solution of the main problem

In this section we investigate the uniqueness and existence of the solution of the problem equation (1) and (2). The following proposition is true.

Theorem 3. Let $\mu > 0, (-a)^l \neq 1, 0 < \lambda < 1, 0 < \alpha \leq 1$. If $f(x) \in C^{\lambda+1}(\bar{\Omega})$, and $g_k(x) \in C^{\lambda+2m-k}(\partial\Omega), k = 0, 1, ..., m-1$, then

1) in the case $\mu > 0$ there exists a unique solution of problem (1) and (2);

2) in the case $\mu = 0$ the necessary and sufficient condition for solvability of the problem equation (1) and (2) is

$$\frac{1}{1+a}\int\limits_{\partial\Omega} \left(1-|x|^2\right)^{m-1} J_{2m}^{1-\alpha}[f](x)dx = \sum_{k=0}^{m-1} c_{k,m} \int\limits_{\partial\Omega} g_k(x)dS_x, \tag{10}$$

subject to the condition $a \neq -1$.

If there exists a solution of the problem, then it is unique up to a constant term.

3) If there exists a solution of the problem, then it belongs to the class $C^{\lambda+2m}(\bar{\Omega})$, and can be represented in the form

$$u(x) = G_a[v](x), \tag{11}$$

where $v(x)$ is the solution of problem (5) and (6) with the functions $\psi_k(x) = (1 + aI_S)g_k(x), k = 0, 1, ..., m$.

Proof Let $u(x)$ be a solution of problem equation (1) and (2). If $(-a)^l \neq 1$ then the operator G_a exists. We apply the operator $1 + aI_S$ to the function and set $v(x) = (1 + aI_S)u(x)$. Let find conditions, which a function $v(x)$ satisfy. By application of the operator $D_{\mu}^{\alpha+k}, k = 0, 1, ..., m-1$ and using the boundary conditions of (1) and (2), we obtain following relations for the function $v(x)$

$$D_{\mu}^{\alpha+k}v(x) = D_{\mu}^{\alpha+k}(1 + aI_S)u(x) = (1 + aI_S)D_{\mu}^{\alpha+k}u(x) = (1 + aI_S)g_k(x) \equiv \psi_k(x), x \in \partial\Omega.$$

Next, we apply the operator $(-\Delta)^m$ to the relation $v(x) = (1 + aI_S)u(x)$ and use relation (1), then we obtain

$$(-\Delta)^m v(x) = (-\Delta)^m(1 + aI_S)u(x) = (-\Delta)^m u(x) + a(-\Delta)^m u(Sx) = f(x), x \in \Omega.$$

We have thereby shown that if $u(x)$ is a solution of problem (1) and (2), then the function $v(x) = (1 + aI_S)u(x)$ is a solution of problem (5) and (6) with the functions $\psi_k(x) = (1 + aI_S)g_k(x), k = 0, 1, ..., m$.

Moreover by application of the operator G_a to the relation $v(x) = (1 + aI_S)u(x)$ and using relation (4), we obtain a representation (11) for the solution of problem (1) and (2).

Conversely, let $v(x)$ be a solution of problem (5) and (6) with the functions $\psi_k(x) = (1 + aI_S)g_k(x), k = 0, 1, ..., m$.

Next we show that the function $u(x) = G_a v(x)$ satisfies all assumptions of problem (1) and (2). Indeed, if $g_k(x) \in C^{\lambda+2m-k}(\partial\Omega), k = 0, 1, ..., m-1$, then $\psi_k(x) = (1 + aI_S)g_k(x) \in C^{\lambda+2m-k}(\partial\Omega), k = 0, 1, ..., m.$

Let $\mu > 0$. In this case, by Theorem 1, problem (5) and (6) corresponding to these functions has a unique solution, which belongs to the class $C^{\lambda+2m}(\bar{\Omega})$, that the function $G_a v(x)$ belongs to that class as well.

Next, we apply the operator $(-\Delta)^m$ to the relation $u(x) = G_a v(x)$. Then we have

$$(-\Delta)^m u(x) = G_a(-\Delta)^m v(x) = G_a f(x) \Longrightarrow a(-\Delta)^m u(Sx) = aG_a(-\Delta)^m v(Sx) = aG_a I_S f(x)$$

Hence

$$(-\Delta)^m u(x) + a(-\Delta)^m u(Sx) = (1 + aI_S)\,(-\Delta)^m u(x) = G_a\,(1 + aI_S)\,f(x) = f(x), x \in \Omega,$$

i.e., the function $u(x) = G_a v(x)$ satisfies equation (1). In addition, we apply the operator $D_\mu^{\alpha+k}$ a to the function $u(x) = G_a v(x)$ and take into account relations (6) and (4), then for $\mu > 0$ and $x \in \partial\Omega$ we have

$$D_a^{\alpha+k} u(x) = G_a D_\mu^{\alpha+k} v(x) = G_a \psi_k(x) = G_a\,(1 + aI_S)\,g_k(x) = g_k, k = 0, 1, ..., m-1,$$

i.e., the boundary conditions of problem (1) and (2) are satisfied as well. The fact of uniqueness of the solution of the boundary problem (1) and (2) is the consequence from the result that the problems (5) and (6) have unique solution.

Let $\mu = 0$. In this case, by Theorem 2, the necessary and sufficient solvability condition of the problem (5) and (6) is the integral equality

$$\int\limits_{\partial\Omega} \left(1 - |x|^2\right)^{m-1} J_{2m}^{1-\alpha}[f](x)dx = \sum_{k=0}^{m-1} c_{k,m} \int\limits_{\partial\Omega} \psi_k(x)dS_x, \tag{12}$$

where $\psi_k(x) = (1 + aI_S)g_k(x), k = 0, 1, ..., m$, the numbers $c_{k,m}$ are defined in (9).

The following is giving the method for transformation of the integral on the right hand side of (12).

Lemma 3. Let S be an orthogonal matrix, then for any continuous function $\varphi(x)$ on $\partial\Omega$ a following equality holds:

$$\int\limits_{\partial\Omega} \varphi(Sx)\,dS_x = \int\limits_{\partial\Omega} \varphi(x)\,dS_x.$$

Proof. Let a function $w(x)$ be a solution of the Dirichlet problem for the Laplace equation in Ω with condition $w(x)|_{\partial\Omega} = \varphi(x)$, $x \in \partial\Omega$. Then the function $w(Sx)$ is a solution of the Dirichlet problem for the Laplace equation in Ω with the condition $w(Sx)|_{\partial\Omega} = \varphi(Sx)$, $x \in \partial\Omega$. Therefore, due to the Poisson's formula, we have

$$\int\limits_{\partial\Omega} \varphi(Sx)\,dS_x = \int\limits_{\partial\Omega} w(Sx)\,dS_x = \omega_n w(0) = \int\limits_{\partial\Omega} \varphi(x)\,dS_x,$$

where ω_n is the area of the unit sphere. The proof of Lemma 3 is established.

Using the proved Lemma 3, the condition $\alpha \neq -1$, and taking into account that the natural degree of the orthogonal matrix is an orthogonal matrix as well, we find

$$\int\limits_{\partial\Omega} \psi_k(x)dS_x = \int\limits_{\partial\Omega} (1 + aI_S)\,g_k(x)dS_x = \int\limits_{\partial\Omega} g_k(x)dS_x + a\int\limits_{\partial\Omega} g_k(Sx)dS_x = (1 + a)\int\limits_{\partial\Omega} g_k(x)dS_x.$$

This implies that condition (12) can be transformed to the form (10).

The fact that a function $u(x) = G_a v(x)$ satisfies all assumptions of problem (1) and (2) can be verified just as in the case $\mu > 0$. Theorem is proved.

Remark. Note that the relation $J_{2m}^{0}[f](x) \equiv f(x)$ holds for $\alpha = 1$; therefore, the solvability condition for problem (1) and (2) can be represented in the form

$$\frac{1}{1+a} \int\limits_{\partial\Omega} \left(1 - |x|^2\right)^{m-1} f(x)dx = \sum_{k=0}^{m-1} c_{k,m} \int\limits_{\partial\Omega} g_k(x)dS_x.$$

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