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On the Lebesgue constants of Fourier-Laplace series by Riesz Means

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Abstract. An asymptotic formula for the Lesbegue constant of the Riesz means of Fourier-Laplace series on the sphere obtained in this paper.

1. INTRODUCTION

Let us define $\sigma_n^\alpha f(x)$ the Cesaro means of order α of the partial sums of Fourier-Laplace series on unite sphere S^N as

$$\sigma_n^\alpha f(x) = \int_{S^N} \Theta^\alpha(x,\ y,\ n) f(y)\, d\sigma(y),$$

where the kernel

$$\Theta^\alpha(x,y,n) = \sum_{k=0}^{n} \frac{A_{n-k}^\alpha}{A_n^\alpha} \sum_{j=1}^{a_k} Z_k(x,y),$$

$Z_k(x,y) = \sum\limits_{j=1}^{a_k} Y_j^{(k)}(x) Y_j^{(k)}(y)$, $Y_j^{(k)}(y)$ are spherical harmonics and $A_n^\alpha = \binom{n+\alpha}{\alpha}$.

Investigations on the behaviour of the Cesaro means $\sigma_n^\alpha f(x)$ can be found in works [4] - [5] and [11] - [16]. The different aspects of the convergence and summability can be also found in the book [18]. Since $\sigma_n^\alpha f(x)$ is an integral operator the precise estimation of its kernel $\Theta^\alpha(x,y,n)$ is essential for the study. First estimations of the kernel $\Theta^\alpha(x,y,n)$ obtained by Gronwall [6] for the case of Legendre polynomials and Kogbetliantz [8] for the Gegenbauer polynomials.

The Lebesgue constant is L_1 norm of the kernel above. Note, that estimations of the Lebesgue constants of the Cesaro means studied by Khocholava [8], Akhobadze [2] and Macharashvili [10]. The Lebesgue constants of multiple Fourier series studied in [1] and [9].

This article focuses on Lebesque constants related to Fourier-Laplace series of the Laplace-Beltrami operator:

$$\mathcal{L}_n^\alpha = \int\limits_{S^N} |\Theta^\alpha(x,y,n)| d\sigma(y). \tag{1.1}$$

2. MAIN RESULT.

In the present paper we consider the Riesz means instead the Cezaro means of the partial sums of Fourier-Laplace series. The Riesz means of the partial sums will also be an integral operator and its kernel can be represented by

$$\Theta^{\alpha}(x,y,n)=\sum_{k=0}^{n}\left(1-\frac{\lambda_k}{\lambda_n}\right)^{\alpha}\frac{\Gamma\left(k+\frac{N-1}{2}\right)\Gamma(k+N-1)}{\pi^{\frac{N+1}{2}}}P_k^{\frac{N-1}{2}}(\cos\gamma),$$

where $P_k^{\nu}(t)$ denote the Gegenbauer polynomials as follows

$$P_k^{\nu}(t)=\frac{(-2)^k\Gamma(k+\nu)\Gamma(k+2\nu)}{\Gamma(\nu)\Gamma(2(k+\nu))}(1-t^2)^{-(\nu-\frac{1}{2})}\frac{d^k}{dt^k}\left[(1-t^2)^{k+\nu-\frac{1}{2}}\right].$$

By this representation it is evident that $\Theta^{\alpha}(x,y,n)$ depends only on the spherical distance between x and y hence, allows the Riesz means of the spectral function to be written as $\Theta^{\alpha}(x,y,n)=\Theta_n^{\alpha}(\cos\gamma)$. The Riesz means of the kernel is studied in the works [3] and [17].

The main goal of the paper is to obtain the estimation of the Lebesque constant (L_1 norm of the kernel). Let us use the same notation $\mathcal{L}_n^{\alpha}$ for the Lebesque constant as in (1.1) . Then following theorem is valid.

Theorem 2.1. *The Lebesgue constants for Fourier-Laplace series have the following estimations*

$$\begin{aligned}
&C' n^{\frac{N-1}{2}-\alpha}<\mathcal{L}_n^{\alpha}<C'' n^{\frac{N-1}{2}-\alpha}, \qquad \alpha<\frac{N-1}{2},\\
&C' \ln n<\mathcal{L}_n^{\alpha}<C'' \ln n, \qquad \alpha=\frac{N-1}{2},\\
&0<\mathcal{L}_n^{\alpha}<C, \qquad \alpha>\frac{N-1}{2}.
\end{aligned}$$

3. PROOF OF MAIN RESULT.

To estimate $\mathcal{L}_n^{\alpha}$, we first denote (1.1) as follows,

$$\mathcal{L}_n^{\alpha}=C\int_0^{\pi}|\Theta_n^{\alpha}(\cos\gamma)|\sin^{N-1}\gamma d\gamma \tag{3.1}$$

Let us divide the integral on the right hand side of (3.1) into three parts as follows

$$\begin{aligned}
\mathcal{L}_n^{\alpha}=&C\int_0^{\frac{1}{n+1}\frac{\pi}{2}}|\Theta_n^{\alpha}(\cos\gamma)|\sin^{N-1}\gamma d\gamma\ +\ C\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}}|\Theta_n^{\alpha}(\cos\gamma)|\sin^{N-1}\gamma d\gamma\\
&+C\int_{\pi-\frac{1}{n+1}\frac{\pi}{2}}^{\pi}|\Theta_n^{\alpha}(\cos\gamma)|\sin^{N-1}\gamma d\gamma\\
=&I_1+I_2+I_3.
\end{aligned}\tag{3.2}$$

3.1. Estimation from above

If $0 < \gamma_0 \leq \gamma \leq \pi$ and $0 \leq \gamma \leq \pi$, the kernel $\Theta_n^\alpha(\cos\gamma)$ can be easily estimated by

$$|\Theta_n^\alpha(\cos\gamma)| \leq C_4 n^{N-1-\alpha} \quad \text{and} \quad |\Theta_n^\alpha(\cos\gamma)| \leq C_5 n^N.$$

For the Riesz means of the spectral function $\Theta_n^\alpha(\cos\gamma)$ in $0 \leq \gamma \leq \pi$ we have the following estimation

$$|\Theta_n^\alpha(\cos\gamma)| < Cn^N, \quad 0 \leq \gamma \leq \pi.$$

The first integrand I_1 can be estimated as follows

$$I_1 < Cn^N \int_0^{\frac{1}{n+1}\frac{\pi}{2}} \sin^{N-1}\gamma\, d\gamma \leq Cn^N \int_0^{\frac{1}{n+1}\frac{\pi}{2}} \gamma^{N-1}\, d\gamma = \frac{Cn^N\left(\frac{\pi}{2}\right)^N}{(n+1)^N(N)} \leq \frac{C\left(\frac{\pi}{2}\right)^N}{N} = O(1). \tag{3.3}$$

Similarly, for the third term I_3 one has

$$I_3 < Cn^N \int_{\pi-\frac{1}{n+1}\frac{\pi}{2}}^{\pi} \sin^{N-1}\gamma\, d\gamma \leq Cn^N \int_{\pi-\frac{1}{n+1}\frac{\pi}{2}}^{\pi} \sin^{N-1}(\pi-\gamma)\, d\gamma.$$

Applying the change of variables $\omega = \pi - \gamma$ the last estimation gives

$$I_3 < Cn^N \int_0^{\frac{1}{n+1}\frac{\pi}{2}} \sin^{N-1}\omega\, d\omega = O(1). \tag{3.4}$$

To estimate I_2 we need the following estimation of the kernel $\Theta_n^\alpha(\cos\gamma)$ (refer [17])

$$\begin{aligned}\Theta_n^\alpha(\cos\gamma) = (N-1)n^{\frac{N-1}{2}-\alpha}\frac{\sin\left[\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\gamma-\frac{\pi}{2}\left(\frac{N-1}{2}+\alpha\right)\right]}{(2\sin\gamma)^{\frac{N-1}{2}}\left(2\sin\frac{\gamma}{2}\right)^{\alpha+1}} + \frac{\xi_n^\alpha(\gamma)(n+1)^{\frac{N-1}{2}-\alpha-1}}{(\sin\gamma)^{\frac{N+1}{2}}\left(\sin\frac{\gamma}{2}\right)^{\alpha+1}} \\ + \frac{\eta_n^\alpha(\gamma)}{(n+1)\left(\sin\frac{\gamma}{2}\right)^{N+1}}.\end{aligned} \tag{3.5}$$

By (3.5), the integrand I_2 is bounded by the sum of the following integrals

$$\begin{aligned} I_2 < & C\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{n^{\frac{N-1}{2}-\alpha}}{(\sin\gamma)^{\frac{N-1}{2}}(\sin\frac{\gamma}{2})^{1+\alpha}}\sin^{N-1}\gamma\, d\gamma + C\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{n^{\frac{N-3}{2}-\alpha}}{(\sin\gamma)^{\frac{N-1}{2}}(\sin\frac{\gamma}{2})^{1+\alpha}}\sin^{N-1}\gamma\, d\gamma \\ & + C\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{|\varepsilon_n(\gamma)|}{(\sin\frac{\gamma}{2})^{N+1}}\sin^{N-1}\gamma\, d\gamma = I_2^{'} + I_2^{''} + I_2^{'''}. \end{aligned} \tag{3.6}$$

We first estimate $I_2^{'}$:

$$\begin{aligned} I_2^{'} &= C\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{n^{\frac{N-1}{2}-\alpha}\sin^{N-1}\gamma}{(\sin\gamma)^{\frac{N-1}{2}}\left(\sin\frac{\gamma}{2}\right)^{1+\alpha}}\, d\gamma = C\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{n^{\frac{N-1}{2}-\alpha}(\sin\gamma)^{\frac{N-1}{2}}}{\left(\sin\frac{\gamma}{2}\right)^{1+\alpha}}\, d\gamma \\ &= Cn^{\frac{N-1}{2}-\alpha}\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} (\sin\gamma)^{\frac{N-1}{2}-\alpha-1}\, d\gamma \leq Cn^{\frac{N-1}{2}-\alpha}\int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \gamma^{\frac{N-1}{2}-\alpha-1}\, d\gamma. \end{aligned}$$

Consider the cases: $\alpha < \frac{N-1}{2}$, $\alpha = \frac{N-1}{2}$ and $\alpha > \frac{N-1}{2}$.

If $\alpha < \frac{N-1}{2}$, then

$$I_2' \leq \frac{C}{\frac{N-1}{2} - \alpha} \left[\left(\pi - \frac{\pi}{2(n+1)} \right)^{\frac{N-1}{2}-\alpha} - \left(\frac{\pi}{2(n+1)} \right)^{\frac{N-1}{2}-\alpha} \right] n^{\frac{N-1}{2}-\alpha} \leq C n^{\frac{N-1}{2}-\alpha}. \tag{3.7}$$

If $\alpha = \frac{N-1}{2}$,

$$I_2' \leq C \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \frac{1}{\gamma}\, d\gamma = C \ln \left[(n+1) \left(2 - \frac{1}{n+1} \right) \right] = C \ln(2n+1) \leq C \ln n. \tag{3.8}$$

If $\alpha > \frac{N-1}{2}$,

$$I_2' \leq C n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \gamma^{\frac{N-1}{2}-\alpha-1}\, d\gamma \leq \frac{C n^{\frac{N-1}{2}-\alpha}}{(\alpha - \frac{N-1}{2})(n+1)^{\frac{N-1}{2}-\alpha}} (-\frac{\pi}{2})^{\frac{N-1}{2}-\alpha} \leq C_\alpha. \tag{3.9}$$

Hence, from (3.7), (3.8) and (3.9) we obtain

$$I_2' \leq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C_\alpha, & \alpha > \frac{N-1}{2}. \end{cases} \tag{3.10}$$

The term I_2'' is estimated by:

$$I_2'' = C \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \frac{(n+1)^{\frac{N-3}{2}-\alpha} |\eta_n(\gamma)|}{(\sin\gamma)^{(N-1)/2+1} \left(\sin\frac{\gamma}{2}\right)^{\alpha+1}} \sin^{N-1}\gamma\, d\gamma \leq C n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \frac{(\sin\gamma)^{\frac{N-3}{2}}}{\left(\sin\frac{\gamma}{2}\right)^{\alpha+1}}\, d\gamma$$

$$\leq C n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \gamma^{\frac{N-5}{2}-\alpha}\, d\gamma.$$

Let us consider the following cases:
If $\alpha < \frac{N-3}{2}$

$$I_2'' \leq C n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \gamma^{\frac{N-5}{2}-\alpha}\, d\gamma = \frac{C n^{\frac{N-3}{2}-\alpha}}{\frac{N-3}{2} - \alpha} \left[\left(\pi - \frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2}-\alpha} - \left(\frac{\pi}{2(n+1)} \right)^{\frac{N-3}{2}-\alpha} \right] \leq C n^{\frac{N-3}{2}-\alpha}. \tag{3.11}$$

If $\alpha = \frac{N-3}{2}$

$$I_2'' \leq C n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi - \frac{1}{n+1}\frac{\pi}{2}} \frac{1}{\gamma}\, d\gamma \leq C \ln n. \tag{3.12}$$

If $\alpha > \frac{N-3}{2}$

$$I_2^{''} \le Cn^{\frac{N-3}{2}-\alpha} \int\limits_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \gamma^{\frac{N-5}{2}-\alpha}\, d\gamma = \frac{Cn^{\frac{N-3}{2}-\alpha}}{\alpha - \frac{N+1}{2}} \left[\left(\frac{\pi}{2(n+1)}\right)^{\frac{N-3}{2}-\alpha} - \left(\pi - \frac{\pi}{2(n+1)}\right)^{\frac{N-3}{2}-\alpha} \right]$$
$$\le \frac{C}{\alpha - \frac{N+1}{2}} \left(\frac{\pi}{2(n+1)}\right)^{\frac{N-1}{3}-\alpha} n^{\frac{N-3}{2}-\alpha} \le C. \tag{3.13}$$

By (3.11), (3.12) and (3.13), $I_2^{'}$ is estimated by

$$I_2^{'} \le \begin{cases} Cn^{\frac{N-3}{2}-\alpha}, & \alpha < \frac{N-3}{2}, \\ C\ln n, & \alpha = \frac{N-3}{2}, \\ C, & \alpha > \frac{N-3}{2}. \end{cases} \tag{3.14}$$

Finally, to estimate I_2 from above, $I_2^{'''}$ is estimated as follows:

$$I_2^{'''} = \frac{C}{n+1} \int\limits_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{|\varepsilon_n(\gamma)|}{(\sin\frac{\gamma}{2})^{N+1}} \sin^{N-1}\gamma\, d\gamma \le \frac{C}{n+1} \int\limits_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} \frac{1}{\gamma^2}\, d\gamma \le \frac{C}{n+1}\frac{\pi(n+1)}{2} \le C, \tag{3.15}$$

Then from (3.6) taking into the note (3.10), (3.14) and (3.15) we obtain

$$I_2 \le \begin{cases} Cn^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C\ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \tag{3.16}$$

Consequently, using (3.3), (3.4) and (3.16) the Lebesgue constant $\mathcal{L}_n^\alpha$ (see (3.2)) is estimated as follows

$$\mathcal{L}_n^\alpha \le \begin{cases} Cn^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C\ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \tag{3.17}$$

3.2. Estimation from below

The next step is to obtain the lower bound of the Lebesgue constant. We proceed by first estimating I_2' from below:

$$I_2' = C \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} n^{\frac{N-1}{2}-\alpha}(N-1)\frac{\left|\sin\left[\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\gamma-\frac{\pi}{2}\left(\frac{N-1}{2}+\alpha\right)\right]\right|}{(2\sin\gamma)^{\frac{N-1}{2}}\left(2\sin\frac{\gamma}{2}\right)^{1+\alpha}}\sin^{N-1}\gamma\, d\gamma$$

$$= \frac{C}{2^\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\pi-\frac{1}{n+1}\frac{\pi}{2}} n^{\frac{N-1}{2}-\alpha}(N-1)\frac{\left|\sin\left[\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\gamma-\frac{\pi}{2}\left(\frac{N-1}{2}+\alpha\right)\right]\right|}{\left(\sin\frac{\gamma}{2}\right)^{\alpha-\frac{N-3}{2}}}\left(\cos\frac{\gamma}{2}\right)^{\frac{N-1}{2}} d\gamma$$

$$\geq Cn^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n+1}\frac{\pi}{2}}^{\frac{3}{4}\pi} n^{\frac{N-1}{2}-\alpha}(N-1)\frac{\left|\sin\left[\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\gamma-\frac{\pi}{2}\left(\frac{N-1}{2}+\alpha\right)\right]\right|}{\gamma^{\alpha-\frac{N-3}{2}}} d\gamma.$$

A new variable $\omega = (n+\frac{N}{2}+\frac{\alpha}{2})\gamma - \frac{\pi(N-1+2\alpha)}{4}$ is introduced and substituted into the chain of inequalities:

$$I_2' \geq Cn^{\frac{N-1}{2}-\alpha} \int_{\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\frac{\pi}{2(n+1)}-\frac{\pi(N-1+2\alpha)}{4}}^{\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\frac{3\pi}{2}-\frac{\pi(N-1+2\alpha)}{4}} \frac{|\sin\omega|}{\left(\frac{\omega+\frac{\pi(N-1+2\alpha)}{4}}{n+\frac{N-1}{2}+\frac{\alpha+1}{2}}\right)^{\alpha-\frac{N-3}{2}}}\frac{d\omega}{n+\frac{N-1}{2}+\frac{\alpha+1}{2}}$$

$$\geq Cn^{\frac{N-1}{2}-\alpha}\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)^{\alpha-\frac{N-1}{2}} \int_{\pi}^{\frac{n\pi}{2}} \frac{|\sin\omega|\, d\omega}{\left(\omega+\frac{\pi(N-1+2\alpha)}{4}\right)^{\alpha-\frac{N-3}{2}}}.$$

Since

$$\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\frac{\pi}{2(n+1)} - \frac{\pi(N-1+2\alpha)}{4} \leq \pi,$$

and

$$\left(n+\frac{N-1}{2}+\frac{\alpha+1}{2}\right)\frac{3\pi}{4} - \frac{\pi(N-1+2\alpha)}{4} \geq \frac{n\pi}{2},$$

we can now estimate I_2' as

$$I_2' \geq C\int_{\pi}^{\left[\frac{n}{2}\right]\pi} \frac{|\sin\omega|\, d\omega}{\left(\omega+\frac{\frac{N-1}{2}+\alpha}{2}\pi\right)^{\alpha-\frac{N-3}{2}}} = C\sum_{\tau=1}^{\left[\frac{n}{2}\right]-1}\int_{\tau\pi}^{(\tau+1)\pi} \frac{|\sin\omega|\, d\beta}{\left(\omega+\frac{\frac{N-1}{2}+\alpha}{2}\pi\right)^{\alpha-\frac{N-3}{2}}}.$$

Applying the change of variable $\beta = t+\tau\pi$, we obtain

$$I_2' \geq C\sum_{\tau=1}^{\left[\frac{n}{2}\right]-1}\int_0^\pi \frac{|\sin(t+\tau\pi)|\, dt}{\left(t+\tau\pi+\frac{\pi(N-1+2\alpha)}{4}\right)^{\alpha-\frac{N-3}{2}}} \geq C\sum_{\tau=1}^{\left[\frac{n}{2}\right]-1}\int_0^\pi \frac{\sin t\, dt}{\tau^{\alpha-\frac{N-3}{2}}} = C\sum_{\tau=1}^{\left[\frac{n}{2}\right]-1}\frac{1}{\tau^{\alpha-\frac{N-3}{2}}}$$

$$\geq C\int_1^{\left[\frac{n}{2}\right]-1} \frac{dt}{t^{\alpha-\frac{N-3}{2}}}.$$

Once more, the following 3 cases are considered:
If $\alpha < \frac{N-1}{2}$ then

$$I_2' \geq C \int_1^{\left[\frac{n}{2}\right]-1} t^{\frac{N-1}{2}-\alpha-1} dt = \frac{C}{\frac{N-1}{2}-\alpha} \left\{ \left(\left[\frac{n}{2}\right] - 1 \right)^{\frac{N-1}{2}-\alpha} - 1 \right\} \geq C n^{\frac{N-1}{2}-\alpha}. \tag{3.18}$$

If $\alpha = \frac{N-1}{2}$ then

$$I_2' \geq C \int_1^{\left[\frac{n}{2}\right]-1} \frac{dt}{t} = C \ln \left(\left[\frac{n}{2}\right] - 1 \right) \geq C \ln n. \tag{3.19}$$

If $\alpha > \frac{N-1}{2}$ then

$$I_2' \geq C \int_1^{\left[\frac{n}{2}\right]-1} t^{\frac{N-1}{2}-\alpha-1} dt = \frac{C}{\alpha - \frac{N-1}{2}} \left\{ 1 - \left(\left[\frac{n}{2}\right] - 1 \right)^{\frac{N-1}{2}-\alpha} \right\} \geq C. \tag{3.20}$$

From (3.18),(3.19) and (3.20), we have

$$I_2' \geq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ C, & \alpha > \frac{N-1}{2}. \end{cases} \tag{3.21}$$

Applying the reverse triangle inequality, $|a-b| \geq |a| - |b|$ with the Riesz mean of the spectral function from (3.5), gives the following

$$\begin{aligned} I_2 =& C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_1^\alpha(\cos\gamma) + \Theta_2^\alpha(\cos\gamma) + \Theta_3^\alpha(\cos\gamma)| \sin^{N-1}\gamma \, d\gamma \\ \geq& C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_1^\alpha(\cos\gamma)| \sin^{N-1}\gamma \, d\gamma - C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_2^\alpha(\cos\gamma)| \sin^{N-1}\gamma \, d\gamma \\ & - C \int_{\frac{\pi}{2(n+1)}}^{\pi - \frac{\pi}{2(n+1)}} |\Theta_3^\alpha(\cos\gamma)| \sin^{N-1}\gamma \, d\gamma \\ =& I_2' - I_2'' - I_2'''. \end{aligned} \tag{3.22}$$

Given the inequalities (3.21), (3.14) and (3.15) of (3.22), I_2 is bounded as follows

$$I_2 \geq \begin{cases} C n^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ 0, & \alpha > \frac{N-1}{2}, \end{cases} \tag{3.23}$$

and by manner of (3.2), $\mathcal{L}_n^\alpha \geq I_2$. Consequently, gives the following estimates

$$\mathcal{L}_n^\alpha \geq \begin{cases} Cn^{\frac{N-1}{2}-\alpha}, & \alpha < \frac{N-1}{2}, \\ C \ln n, & \alpha = \frac{N-1}{2}, \\ 0, & \alpha > \frac{N-1}{2}. \end{cases} \tag{3.24}$$

Combination of the estimated upper bound of the Lebesgue constant in (3.17) and lower bound in (3.24), provides estimate of the Lebesgue constant, $\mathcal{L}_n^\alpha$ in form of

$$C' n^{\frac{N-1}{2}-\alpha} < \mathcal{L}_n^\alpha < C'' n^{\frac{N-1}{2}-\alpha}, \qquad \alpha < \frac{N-1}{2},$$
$$C' \ln n < \mathcal{L}_n^\alpha < C'' \ln n, \qquad \alpha = \frac{N-1}{2},$$
$$0 < \mathcal{L}_n^\alpha < C, \qquad \alpha > \frac{N-1}{2}.$$

Theorem 2.1 is proved.

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