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A Generalized Triangular Intuitionistic Fuzzy Geometric Averaging Operator for Decision-Making in Engineering and Management

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Abstract: Triangular intuitionistic fuzzy number (TIFN) is a more generalized platform for expressing imprecise, incomplete, and inconsistent information when solving multi-criteria decision-making problems, as well as for expressing and reflecting the evaluation information in several dimensions. In this paper, the TIFN has been applied for solving multi-criteria decision-making (MCDM) problems, first, by defining some existing triangular intuitionistic fuzzy geometric aggregation operators, and then developing a new triangular intuitionistic fuzzy geometric aggregation operator, which is the generalized triangular intuitionistic fuzzy ordered weighted geometric averaging (GTIFOWGA) operator. Based on these operators, a new approach for solving multicriteria decision-making problems when the weight information is fixed is proposed. Finally, a numerical example is provided to show the applicability and rationality of the presented method, followed by a comparative analysis using similar existing computational approaches.

Keywords: generalized triangular intuitionistic fuzzy geometric aggregation operator; triangular intuitionistic fuzzy number; intuitionistic fuzzy set; multi-criteria decision-making; attitudinal character; flexibility

1. Introduction

In solving multi-criteria decision-making (MCDM) problems, it is often required that several criteria are considered simultaneously before selecting or ranking alternatives. Since the information required for solving the MCDM problems is often incomplete, inconsistent, and indeterminate, the manner in which it is expressed, therefore, has remained a major task and of great interest among researchers over the past several years. In handling these issues, Zadeh [1], who introduced the concept of fuzzy set theory, has outlined how the fuzzy set (FS) concept could be used for expressing such decision-making problems. However, the FS theory, which is characterized by only one function, "the membership function $\mu_A(x)$", in most cases cannot be used fully to express some kind of complex fuzzy information. "*For example, during voting, if there are ten persons voting for an issue, and three of them give the 'agree', four of them give the 'disagree', and the others abstain. Obviously, FS cannot fully express the polling information*" [2]. To solve this kind of problem, Atanassov [3] extended the fuzzy set theory by adding a new function "the non-membership function $v_A(x)$", in order to form the intuitionistic fuzzy set (IFS) theory.

The membership and non-membership functions of the IFS theory are represented by an intuitionistic fuzzy number (IFN), are more or less independent and are constrained with the conditions that the sum of the membership and non-membership must not exceed one [4]. These constraints, however, have been challenged recently by Despi [5] who defined a new IFS in which the sum of the membership and non-membership functions are more than one and their differences are either positive or negative. This new IFS has been justified and supported by Marasini et al. [6] and Li [7]. The computation of membership and non-membership function in this study will be based on the new IFS by Despi [5]. The IFS has shown its usefulness in dealing with complex engineering problems [8,9], and in providing a flexible model to elaborate uncertainty [10,11].

Extensive review of the IFS, which is represented by IFN for solving MCDM problems, has shown an increase over the past few years as revealed in the literature [8–18] with a few extensions of the IFN, such as the trapezoidal IFN (TrIFN) [19,20], the interval-valued TrIFN (IVTrIFN) [21,22], Pythagorean IFN (PIFN) [23], and the triangular IFN (TIFN) [24–26]. This study, however, will be concerned with the triangular intuitionistic fuzzy numbers (TIFNs) only, with the purpose of accounting for the attitudinal character or risk attitude of the DMs, which have not been fully studied in the reviewed literature.

The application of TIFN in MCDM is based on its ability to express decision information in several dimensions, reflect the assessment information in a more holistic manner [24], and for quantifying ill-known quantities [12]. Several research efforts have been made in the advancement of TIFN over the past few years. Among them, we can mention the characterization of membership and non-membership degrees in intuitionistic fuzzy sets (IFS) using the triangular fuzzy numbers by Shu et al. [26]. Chen and Li [27] developed a new distance measurement between two TIFNs for determining attribute weights, as well as weighted arithmetic averaging (TIFN-WAA) operators on TIFNs. Zhang and Nan [28] developed a methodology for ranking TIFNs by considering the concept of a TIFN as a special case of the IFN. Wan et al. [29], using the TIFN, extended the classical VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method for solving multi-attribute group decision-making (MAGDM) problems, while Li et al. [30] investigated the arithmetic operations and cut sets over TIFNs, and defined the values and ambiguities of the membership degree and non-membership degree for the TIFNs, as well as the value index and ambiguity index.

Other contributions to the study of TIFN are in the area of information fusion operators (aggregation), where Chen and Li [27] introduced the weighted arithmetic averaging operator on TIFNs (TIFN-WAA). Wan et al. [29] presented the triangular intuitionistic fuzzy weighted average (TIF-WA) operator for the selection of personnel. The triangular intuitionistic fuzzy weighted average (TIFWA) operator, the ordered weighted average (TIFOWA) operator, the hybrid weighted average (TIFHWA) operator, the triangular intuitionistic fuzzy generalized ordered weighted average (TIFGOWA) operator, and the generalized hybrid weighted average (TIFGHWA) operator were developed by Wan et al. [31].

Upon investigation of the different aggregation operators of TIFN, it has been revealed that the ranking of TIFNs are a bit complicated and cannot be easily compared with other TIFNs [32], as well as account for attitudinal character or risk attitude of the DMs. In order to further advance the study of aggregation operators of TIFN, simplify its comparison and application in MCDMs, and to express the risk attitude of the DMs in the decision-making process, this paper attempts to do the following:

(1) Define some triangular intuitionistic fuzzy aggregation operators, that is, the triangular intuitionistic fuzzy weighted geometric averaging (TIFWGA) operator, ordered weighted geometric averaging (TIFOWGA) operator and the hybrid weighted geometric averaging (TIFHWGA) operator;
(2) Develop a new generalized triangular intuitionistic fuzzy aggregation operator, that is, the generalized triangular intuitionistic fuzzy ordered weighted geometric averaging (GTIFOWGA) operator. This is mainly to allow for more attitudinal information to be expressed or used in accordance with the different DMs interests or preference;
(3) Propose a simple and straightforward approach for solving MCDM problems when the performance ratings are expressed in triangular intuitionistic fuzzy numbers (TIFNs).

The rest of this paper is organized as follows: in Section 2, the concepts of intuitionistic fuzzy set theory and triangular intuitionistic fuzzy sets are presented. In Section 3, some triangular intuitionistic fuzzy weighted geometric operators are defined, and the GTIFOWGA operator is developed. In Section 4, the algorithm of the proposed method is presented and applied to solving MCDM problems. Finally, in Section 5, some conclusions are presented.

2. Preliminaries

In this section, the fundamental definitions and concepts of TIFN and IFS as described by Liang et al. [32] and Despic and Simonovic [33] are presented.

2.1. Intuitionistic Fuzzy Set (IFS)

Definition 1. **[32]** *If the IFS A in* $X = \{x\}$ *is defined fully in the form* $A = \{\langle x, \mu_A(x),\ v_A(x),\ \pi_A(x)\rangle | x \in\ X\}$, *where* $\mu_A : X\ \rightarrow [0,1]$, $v_A : X\ \rightarrow [0,1]$ *and* $\pi_A : X\ \rightarrow [0,1]$, *then the different relations and operations for the IFS are given as:*

1. $A{\cdot}B = \{\langle x, \mu_A(x){\cdot}\mu_B(x),\ v_A(x) + v_B(x) - v_A(x){\cdot}v_B(x)\rangle | x \in\ X\}$;
2. $A + B = \{\langle x, \mu_A(x) +\ \mu_B(x) -\ \mu_A(x){\cdot}\mu_B(x), v_A(x){\cdot}v_B(x)\rangle | x \in\ X\}$;
3. $\lambda A = \{\langle x, 1 - (1 - \mu_A(x))^{\lambda},\ (v_A(x))^{\lambda}\rangle | x \in\ X\},\ \lambda > 0$;
4. $A^{\lambda} = \{\langle x, (\mu_A(x))^{\lambda}, 1 - (1 - v_A(x))^{\lambda}\rangle | x \in\ X\},\ \lambda > 0$;
5. $A = B$ *if and only if* $\mu_A(x) = \mu_B(x)$ *and* $v_A(x) = v_B(x)$ *for all* $x \in\ X$;
6. $A \leq B$ *if and only if* $\mu_A(x) \leq \mu_B(x)$ *and* $v_A(x) \geq v_B(x)$ *for all* $x \in\ X$.

2.2. The Triangular Intuitionistic Fuzzy Number (TIFN)

The TIFN is basically the use of the traditional triangular fuzzy number to express the membership $\mu_A(x)$ and non-membership degree $v_A(x)$ such that the intuitionistic fuzzy number is based on the triangular fuzzy number, which is termed the triangular intuitionistic fuzzy number (TIFN). In the following, the basic concepts relating to the TIFN are introduced:

Definition 2. **[28,31,32]** *Let* α *be a TIFN, where the membership and non-membership function for* α *are defined as follows:*

Membership function:

$$\mu_\alpha(x) = \begin{cases} \frac{(x-a)\mu_\alpha}{(b-a)},\ (a \leq x < b), \\ \mu_\alpha,\ (x = b), \\ \frac{(c-x)\mu_\alpha}{c-b},\ (b < x \leq c), \\ 0, \text{otherwise}. \end{cases} \quad (1)$$

For non-membership function, it is given as:

$$v_\alpha(x) = \begin{cases} \frac{(b-x+v_\alpha(x-\acute{a}))}{(b-\acute{a})},\ (\acute{a} \leq x < b), \\ v_\alpha,\ (x = b), \\ \frac{(x-b+v_\alpha(\acute{c}-x))}{\acute{c}-b},\ (b < x \leq \acute{c}), \\ 0,\ \text{otherwise}, \end{cases} \quad (2)$$

where $0 \leq \mu_\alpha \leq 1; 0 \leq v_\alpha \leq 1; 0 \leq \mu_\alpha + v_\alpha \leq 1,\ a, b, c,\ \acute{a}, \acute{c} \in \mathbb{R}$.

The TIFN is therefore denoted as $\acute{\alpha} = \langle([a, b, c]; \mu_\alpha), ([a, b, c]; v_\alpha)\rangle$, *when* $\mu_\alpha = 1$, *and* $v_\alpha = 0$, *and* $\acute{\alpha}$ *will change into the traditional triangular fuzzy number (TFN). Generally, the TIFN* $\acute{\alpha}$ *is defined as* $\acute{\alpha} = ([a, b, c]; \mu_\alpha,\ v_\alpha)$ *for convenience. In the following, the operational rules for any two TIFNs are presented.*

Definition 3. **[32,34]** *Let* $\alpha_1 = ([a_1, b_1, c_1]; \mu_{\alpha_1}, v_{\alpha_1})$ *and* $\alpha_2 = ([a_2, b_2, c_2]; \mu_{\alpha_2}, v_{\alpha_2})$ *be two TIFNs and* $\lambda \leq 0$. *Then:*

1. $\alpha_1 + \alpha_2 = ([a_1 + a_2, b_1 + b_2, c_1 + c_2]; \mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1}v_{\alpha_2})$;

2. $\alpha_1\alpha_2 = ([a_1a_2, b_1b_2, c_1c_2]; \mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1} + v_{\alpha_2} - v_{\alpha_1}v_{\alpha_2});$
3. $\lambda\alpha = \left([\lambda a, \lambda b, \lambda c]; 1 - (1 - \mu_\alpha)^\lambda,\ (v_\alpha)\right), \lambda \geq 0;$
4. $\alpha^\lambda = \left(\left[a^\lambda, b^\lambda, c^\lambda\right]; (\mu_\alpha)^\lambda, 1 - (1 - v_\alpha)^\lambda\right), \lambda \geq 0.$

The operational results for the rules given in the Definition 3 for the two TIFNs are given in the operations:

1. $\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1;$
2. $\alpha_1 \otimes\ \alpha_2 = \alpha_2 \otimes\ \alpha_1;$
3. $\lambda(\alpha_1 + \alpha_2) = \lambda\alpha_1 + \lambda\alpha_2\ \lambda \geq 0;$
4. $\lambda_1\alpha + \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha\ \lambda_1\ \lambda_2 \geq 0;$
5. $\alpha^{\lambda_1} \otimes\ \alpha^{\lambda_2} = \alpha^{\lambda_1+\lambda_2}\ \lambda_1\ \lambda_2 \geq 0;$
6. ${\alpha_1}^\lambda \otimes\ {\alpha_2}^\lambda = (\alpha_1 \otimes\ \alpha_2)^\lambda\ \lambda \geq 0.$

Definition 4. **[31]** *Let* $\alpha_1 = ([a_1, b_1, c_1]; \mu_{\alpha_1}, v_{\alpha_1})$ *and* $\alpha_2 = ([a_2, b_2, c_2]; \mu_{\alpha_2}, v_{\alpha_2})$ *be two TIFNs, and the Hamming distance between* α_1 *and* α_2 *is given as:*

$$d(\alpha_1,\ \alpha_2) = \frac{1}{6}\left[\begin{array}{c} |(1+\mu_{\alpha_1} - v_{\alpha_1})a_1 - (1+\mu_{\alpha_2} - v_{\alpha_2})a_2| + |(1+\mu_{\alpha_1} - v_{\alpha_1})b_1 - (1+\mu_{\alpha_2} - v_{\alpha_2})b_2| \\ +|(1+\mu_{\alpha_1} - v_{\alpha_1})c_1 - (1+\mu_{\alpha_2} - v_{\alpha_2})c_2| \end{array}\right]. \tag{3}$$

Definition 5. **[31,35]** *Let* $\acute{\alpha} = ([a, b, c]; \mu_\alpha,\ v_\alpha)$ *be a TIFN. If the membership and non-membership functions are represented by the score function* $S(\acute{\alpha})$ *and accuracy function* $H(\acute{\alpha})$*, respectively, then* $\acute{\alpha}$ *can be defined as follows:*

$$S(\acute{\alpha}) = \frac{(a+2b+c)\mu_\alpha}{4}, \tag{4}$$

$$H(\acute{\alpha}) = \frac{(a+2b+c)(1-v_\alpha)}{4}. \tag{5}$$

Definition 6. **[31,35]** *Let* $\acute{\alpha}_1$ *and* $\acute{\alpha}_2$ *be two TIFNs. If* $S(\acute{\alpha}_i) = \frac{(a_i+2b_i+c_i)\mu_{\alpha_i}}{4}$ *and* $H(\acute{\alpha}) = \frac{(a_i+2b_i+c_i)\left(1-v_{\alpha_i}\right)}{4}$ *are the membership and non-membership functions of* $\acute{\alpha}$*, then:*

1. *If* $S(\acute{\alpha}_1) < S(\acute{\alpha}_2)$ *then* $\acute{\alpha}_1 < \acute{\alpha}_2$;
2. *If* $S(\acute{\alpha}_1) = S(\acute{\alpha}_2)$ *and* $H(\acute{\alpha}_1) = H(\acute{\alpha}_2)$*, then* $\acute{\alpha}_1 = \acute{\alpha}_2$;
3. *If* $S(\acute{\alpha}_1) = S(\acute{\alpha}_2)$ *and* $H(\acute{\alpha}_1) < H(\acute{\alpha}_2)$*, then* $\acute{\alpha}_1 < \acute{\alpha}_2$.

3. Some Weighted Geometric Operators and the Generalized Ordered Weighted Geometric Operators of TIFNs

In this section, motivated by existing achievements [32,36,37], we develop some triangular intuitionistic fuzzy geometric averaging operators and discuss some of their useful properties and then introduce new generalized geometric operators for TIFNs.

3.1. Some Weighted Geometric Aggregation Operators on TIFNs

Definition 7. **[32]** *Let* $\alpha_i = ([a_i, b_i, c_i]; \mu_{\alpha_i}, v_{\alpha_i})$ *for all* $\left(i = \frac{1}{n}, \frac{1}{n},\ \ldots,\ 1/n\right)$ *be a collection of TIFNs on X. The TIFWGA operator of dimension n is a mapping TIFWGA:* $\Omega^n \to \Omega,$ *and:*

$$\text{TIFWGA}_w(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = (\alpha_1)^{w_1} \otimes (\alpha_2)^{w_2} \otimes (\alpha_3)^{w_3} \ldots \otimes (\alpha_n)^{w_n},$$

where $w = (w_1, w_2,\ w_3, \ldots, w_n)^T$ *is the weighting vector of* α_i $(i = 1,\ 2,\ 3, \ldots, n)$ *with* $w_i \in [0,\ 1]$ *and* $\sum_{i=1}^{n},\ w_i = 1$*. Furthermore:*

$$\text{TIFWGA}_w(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \prod_{i=1}^{n}(\alpha_i)^{w_i} = \left(\left[\prod_{i=1}^{n}(a_i)^{w_i}, \prod_{i=1}^{n}(b_i)^{w_i}, \prod_{i=1}^{n}(c_i)^{w_i}\right]; \prod_{i=1}^{n}(\mu_{\alpha_i})^{w_i}, 1 - \prod_{i=1}^{n}(1 - v_{\alpha_i})^{w_i}\right). \quad (6)$$

Definition 8. [36] *Let $\alpha_i = ([a_i, b_i, c_i]; \mu_{\alpha_{\sigma i}}, v_{\alpha_{\sigma i}})$ for all $\left(i = \frac{1}{n}, \frac{1}{n}, \ \ldots, \ 1/n\right)$ be a collection of TIFNs on X. The TIFOWGA operator of dimension n is a mapping TIFOWGA: $\Omega^n \to \Omega$, and:*

$$\text{TIFOWGA}_w(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = (\alpha_{\sigma 1})^{w_1} \otimes (\alpha_{\sigma 2})^{w_2} \otimes (\alpha_{\sigma 3})^{w_3} \ldots \otimes (\alpha_{\sigma n})^{w_n},$$

where $w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the exponential weighting vector of α_i $(i = 1, 2, 3, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is a permutation of X (i.e., $1, 2, 3, \ldots, n$) such that $\alpha_{\sigma 1} \leq \alpha_{\sigma 2} \leq \ldots \leq \alpha_{\sigma n}$. Furthermore:

$$\text{TIFOWGA}_w(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \prod_{i=1}^{n}(\alpha_{\alpha_{\sigma i}})^{w_i}$$
$$= \left(\left[\prod_{i=1}^{n}(a_i)^{w_i}, \prod_{i=1}^{n}(b_i)^{w_i}, \prod_{i=1}^{n}(c_i)^{w_i}\right]; \prod_{i=1}^{n}(\mu_{\alpha_{\sigma i}})^{w_i}, 1 - \prod_{i=1}^{n}(1 - v_{\alpha_{\sigma i}})^{w_i}\right). \quad (7)$$

Definition 9. [32,36] *Let $\alpha_i = \left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right)$ for all $\left(i = \frac{1}{n}, \frac{1}{n}, \ \ldots, \ 1/n\right)$ be a collection of TIFNs on X. The TIFHWGA operator of dimension n is a mapping TIFHWGA: $\Omega^n \to \Omega$, such that:*

$$\text{TIFHWGA}_{\omega w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = (\beta_{\alpha_1})^{w_1} \otimes (\beta_{\alpha_2})^{w_2} \otimes (\beta_{\alpha_3})^{w_3} \ldots \otimes (\beta_{\alpha_n})^{w_n},$$

where β_{α_i} is the ith largest of the weighted TIFN β_{α_i} $(\beta_{\alpha_i} = {\alpha_i}^{n\omega}, i = 1, 2, 3, \ldots, n)$, $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ is the exponential weighting vector of α_i $(i = 1, 2, 3, \ldots, n)$ with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} w_j = 1$, and $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is a permutation of X (i.e., $1, 2, 3, \ldots, n$) such that $\beta_{\alpha_1} \leq \beta_{\alpha_2} \leq \ldots \leq \beta_{\alpha_n}$. Furthermore:

$$\text{TIFHWGA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \left(\left[\prod_{i=1}^{n}(a_i)^{w_i}, \prod_{i=1}^{n}(b_i)^{w_i}, \prod_{i=1}^{n}(c_i)^{w_i}\right]; \prod_{i=1}^{n}(\mu_{\beta_{\sigma i}})^{w_i}, 1 - \prod_{i=1}^{n}(1 - v_{\beta_{\sigma i}})^{w_i}\right). \quad (8)$$

3.2. The Generalized Ordered Geometric Operator of TIFNs

The TIFOWGA operator is extended to develop a new generalized aggregation operator for TIFN. The new GTIFOWGA has been inspired by the work of Tan [37] and Qi et al. [38].

Definition 10. *Let $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n)$ be a collection of triangular intuitionistic fuzzy arguments and $\beta_{\alpha_i} = \left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right)$. The GTIFOWGA operator of dimension n is a mapping GTIFOWGA: $\Omega^n \to \Omega$, which has an exponential weighting vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$, $\sum_{i=1}^{n} \omega_i = 1$, and $\omega_i \in [0, 1]$, $\lambda > 0$; then:*

$$GTIFOWGA_\lambda(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) = \frac{1}{\lambda}\left(\otimes_{i=1}^{n}(\lambda \beta_{\alpha_1})^{\omega_i}\right).$$

Theorem 1. *Let $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n)$ for all $\left(i = \frac{1}{n}, \frac{1}{n}, \ \ldots, 1/n\right)$ be a collection of triangular intuitionistic fuzzy arguments and $\beta_{\alpha_i} = \left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right)$. If the exponential weighting vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$, $\sum_{i=1}^{n} \omega_i = 1$, and $\omega_i \in [0, 1]$, $\lambda > 0$, then the GTIFOWGA operator obtained is a TIFN and is given as follows:*

$$GTIFOWGA_{\lambda}(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) = (\lambda\beta_{\alpha_1})^{\omega_i} \otimes (\lambda\beta_{\alpha_2})^{\omega_2} \otimes \ldots.. \otimes (\lambda\beta_{\alpha_n})^{\omega_n}$$
$$= \left(\left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i},\right)^{1/\lambda}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i},\right)^{1/\lambda}, \left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i},\right)^{1/\lambda}\right]; 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i},\right)^{1/\lambda}, \left(1-\prod_{i=1}^{n}\left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda}\right) \tag{9}$$

Proof. Using mathematical induction on Ω:

1. For $n = 1$:

$$(\lambda\beta_{\alpha_1})^{\omega_i} =$$
$$\left[\left((a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left((b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left((c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]; 1-\left(1-\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, \left(1-\left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda}.$$

Thus, for $n = 1$, Equation (9) holds.

2. For $n = 2$:

$$\begin{aligned}(\lambda\beta_{\alpha_1})^{\omega_1} \otimes (\lambda\beta_{\alpha_2})^{\omega_2} \\ &= \left[\left((a_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}, \left((b_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}, \left((c_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\right]; 1 \\ &- \left(1-\left(1-\left(1-\mu_{\beta(1)}\right)^{\lambda}\right)^{\omega_1}\right)^{\frac{1}{\lambda}}, \left(1-\left(1-v_{\beta(1)}\right)^{\omega_1}\right)^{\frac{1}{\lambda}} \\ &\otimes \left[\left((a_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}, \left((b_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}, \left((c_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}\right]; 1 \\ &- \left(1-\left(1-\left(1-\mu_{\beta(2)}\right)^{\lambda}\right)^{\omega_2}\right)^{\frac{1}{\lambda}}, \left(1-\left(1-v_{\beta(2)}\right)^{\omega_2}\right)^{\frac{1}{\lambda}}.\end{aligned}$$

Since, $\alpha_1\alpha_2 = ([a_1a_2, b_1b_2, c_1c_2]; \mu_{\alpha_1}\mu_{\alpha_2}, v_{\alpha_1} + v_{\alpha_2} - v_{\alpha_1}v_{\alpha_2})$ then:

$$(\lambda\beta_{\alpha_1})^{\omega_1} \otimes (\lambda\beta_{\alpha_2})^{\omega_2}$$
$$= \left[\left((a_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\left((a_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}, \left((b_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\left((b_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}, \left((c_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\left((c_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}\right];$$
$$1-\left(1-\left(1-\left(1-\mu_{\beta(1)}\right)^{\lambda}\right)^{\omega_1}\right)^{\frac{1}{\lambda}} \times 1-\left(1-\left(1-\left(1-\mu_{\beta(2)}\right)^{\lambda}\right)^{\omega_2}\right)^{1/\lambda}, \left(1-\left(1-v_{\beta(1)}\right)^{\omega_1}\right)^{\frac{1}{\lambda}}+$$
$$\left(1-\left(1-v_{\beta(2)}\right)^{\omega_2}\right)^{1/\lambda} - \left(1-\left(1-v_{\beta(1)}\right)^{\omega_1}\right)^{\frac{1}{\lambda}} \times \left(1-\left(1-v_{\beta(2)}\right)^{\omega_2}\right)^{1/\lambda}$$
$$= \left[\left((a_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\left((a_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}, \left((b_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\left((b_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}, \left((c_{(1)})^{\omega_1}\right)^{\frac{1}{\lambda}}\left((c_{(2)})^{\omega_2}\right)^{\frac{1}{\lambda}}\right];$$
$$1-\left(1-\left(1-\left(1-\mu_{\beta(1)}\right)^{\lambda}\right)^{\omega_1}\right)^{\frac{1}{\lambda}} \times 1-\left(1-\left(1-\left(1-\mu_{\beta(2)}\right)^{\lambda}\right)^{\omega_2}\right)^{\frac{1}{\lambda}}, \left(1-\left(1-v_{\beta(1)}\right)^{\omega_1}\right)^{\frac{1}{\lambda}}$$
$$\times\left(\left(1-v_{\beta(2)}\right)^{\omega_2}\right)^{\frac{1}{\lambda}}.$$

Thus, the result is true for $n = 2$.

3. Suppose $n = k$, then:

$$(\lambda\beta_{\alpha_1})^{\omega_i} \otimes (\lambda\beta_{\alpha_2})^{\omega_2} \otimes \ldots.. \otimes (\lambda\beta_{\alpha_n})^{\omega_n} =$$
$$\left(\left[\left(\prod_{i=1}^{k}(a_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{k}(b_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{k}(c_{(i)})^{\omega_i}\right)^{1/\lambda}\right]; 1-\left(1-\prod_{i=1}^{k}\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i},\right)^{1/\lambda}, \left(1-\prod_{i=1}^{k}\left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda}\right).$$

For $n = k + 1$, we then have:

$$\left(\lambda\beta_{\alpha_1}\right)^{\omega_i} \otimes \left(\lambda\beta_{\alpha_2}\right)^{\omega_2} \otimes \ldots . \otimes \left(\lambda\beta_{\alpha_{k+1}}\right)^{\omega_{k+1}}$$

$$= \left(\left[\left(\prod_{i=1}^{k}(a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{k}(c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}} \right]; 1 \right.$$

$$- \left(1 - \prod_{i=1}^{k}\left(1 - \left(1 - \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{i=1}^{k} \left(1 - v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}$$

$$\otimes \left[\left((a_{(k+1)})^{w_{k+1}}\right)^{\frac{1}{\lambda}}, \left((b_{(k+1)})^{w_{k+1}}\right)^{\frac{1}{\lambda}}, \left((c_{(k+1)})^{w_{k+1}}\right)^{\frac{1}{\lambda}} \right]; 1$$

$$\left. - \left(1 - \left(1 - \left(1 - \mu_{\beta(k+1)}\right)^{\lambda}\right)^{\omega_{k+1}}\right)^{\frac{1}{\lambda}}, \left(1 - \left(1 - v_{\beta(k+1)}\right)^{\omega_{k+1}}\right)^{\frac{1}{\lambda}} \right)$$

$$= \left(\left[\left(\prod_{i=1}^{k+1}(a_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{k+1}(b_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{k+1}(c_{(i)})^{\omega_i}\right)^{1/\lambda} \right]; 1 \right.$$

$$\left. - \left(1 - \prod_{i=1}^{k+1}\left(1 - \left(1 - \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, \left(1 - \prod_{i=1}^{k+1} \left(1 - v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda} \right).$$

It confirms that the result is true for $n = k + 1$, and thus it holds for all of n. Hence:

$$GTIFOWGA_{\lambda}(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) = \frac{1}{\lambda}\left(\otimes_{i=1}^{n}\left(\lambda\beta_{\alpha_1}\right)^{\omega_i}\right) = \frac{1}{\lambda}\left(\lambda\beta_{\alpha_1}\right)^{\omega_i} \otimes \left(\lambda\beta_{\alpha_2}\right)^{\omega_2} \otimes \ldots . \otimes \left(\lambda\beta_{\alpha_n}\right)^{\omega_n}$$

$$= \left(\left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i}\right)^{1/\lambda} \right]; 1 \right.$$

$$\left. - \left(1 - \prod_{i=1}^{n}\left(1 - \left(1 - \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}, \left(1 - \prod_{i=1}^{n} \left(1 - v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda} \right).$$

The theorem is true for any number of TIFN, which completes the proof. □

3.3. Some Useful Properties of the GTIFOWGA Operator

Theorem 2. Commutative property. *Let* $\beta_{\alpha_i} = \left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right)$ *and* $\widetilde{\beta}_{\alpha_i} = \left([\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i]; \widetilde{\mu}_{\beta_{\alpha_i}}, \widetilde{v}_{\beta_{\alpha_i}}\right)$ $(i = 1, 2, 3, \ldots, n)$ *be two TIFNs. If the exponential weighting vector* $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$, $\sum_{i=1}^{n} \omega_i = 1$, *and* $\omega_i \in [0, 1]$, $\lambda > 0$, *then the GTIFOWGA operator obtained is a TIFN. Its commutative property for the TIFN is given as follows:*

$$\beta_{\alpha_i} = \widetilde{\beta}_{\alpha_i} \; (\forall i = 1, 2, 3, \ldots, n).$$

Proof. If $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n)$ is a permutation of $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3, \ldots, \widetilde{\beta}_n)$, then we have:

$$\beta_{\alpha_i} = \widetilde{\beta}_{\alpha_i} (i = 1,\ 2,\ 3, \ldots, n),$$

that is:

$$\left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right) = \left([\widetilde{a}_i, \widetilde{b}, \widetilde{c}_i]; \widetilde{\mu}_{\beta_{\alpha_i}}, \widetilde{v}_{\beta_{\alpha_i}}\right).$$

Then:

$$GTIFOWGA_\lambda(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) = GTIFOWGA_\lambda\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3, \ldots, \widetilde{\beta}_n\right).$$

Theorem 3. Idempotent property. *Let* $\beta_{\alpha_i} = \left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right)$ *and* $\widetilde{\beta}_\alpha = \left([\widetilde{a}, \widetilde{b}, \widetilde{c}]; \widetilde{\mu}_{\beta_\alpha}, \widetilde{v}_{\beta_\alpha}\right)$ $(i = 1,\ 2,\ 3, \ldots, n)$ *be two TIFNs. If the exponential weighting vector* $\omega = (\omega_1, \omega_2,\ \omega_3, \ldots, \omega_n)^T$, $\sum_{i=1}^n \omega_i = 1$, *and* $\omega_i \in [0,\ 1]$, $\lambda > 0$, *then the GTIFOWGA operator obtained is a TIFN. Its idempotent property for the TIFN is given as follows:*

$$\beta_{\alpha_i} = \beta_\alpha (\forall i = 1,\ 2,\ 3, \ldots, n).$$

Proof: Since $\beta_{\alpha_i} = \beta_\alpha$, then we have:

$$GTIFOWGA_\lambda(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) = \frac{1}{\lambda}\left(\otimes_{i=1}^n (\lambda\beta_{\alpha_1})^{\omega_i}\right) = \frac{1}{\lambda}\left((\lambda\beta_{\alpha_1})^{\omega_i} \otimes (\lambda\beta_{\alpha_2})^{\omega_2} \otimes \ldots.. \otimes (\lambda\beta_{\alpha_n})^{\omega_n}\right)$$

$$= \frac{1}{\lambda}((\lambda\beta_\alpha)^{\omega_i} \otimes (\lambda\beta_\alpha)^{\omega_2} \otimes \ldots.. \otimes (\lambda\beta_\alpha)^{\omega_n}) = \frac{1}{\lambda}((\lambda\beta_\alpha)^{\omega_1+\omega_1+\omega_1+\cdots+\omega_n})$$

$$= \frac{1}{\lambda}(\lambda\beta_\alpha) = \beta_\alpha.$$

□

Theorem 4. Monotonicity property. *Let* $\beta_{\alpha_i} = \left([a_i, b_i, c_i]; \mu_{\beta_{\alpha_i}}, v_{\beta_{\alpha_i}}\right)$ *and* $\widetilde{\beta}_{\alpha_i} = \left([\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i]; \widetilde{\mu}_{\beta_{\alpha_i}}, \widetilde{v}_{\beta_{\alpha_i}}\right)$ $(i = 1,\ 2,\ 3, \ldots, n)$ *be two TIFNs. If the exponential weighting vector* $\omega = (\omega_1, \omega_2,\ \omega_3, \ldots, \omega_n)^T$, $\sum_{i=1}^n \omega_i = 1$, *and* $\omega_i \in [0,\ 1]$, $\lambda > 0$, *then the GTIFOWGA operator obtained is a TIFN and is given as follows for the monotonicity property:*

$$\beta_{\alpha_i} \le \widetilde{\beta}_{\alpha_i} (\forall i = 1,\ 2,\ 3, \ldots, n).$$

Proof. Since $\beta_{\alpha_i} \le \widetilde{\beta}_{\alpha_i}$, then:

$$GTIFOWGA_\lambda(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \le GTIFOWGA_\lambda\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3, \ldots, \widetilde{\beta}_n\right),$$

where $[a_i, b_i, c_i] \le [\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i]$, $\mu_{\beta_{\alpha_i}} \le \widetilde{\mu}_{\beta_{\alpha_i}}$, and $v_{\beta_{\alpha_i}} \ge \widetilde{v}_{\beta_{\alpha_i}}$. □

It follows that:

$$\left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}},\left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}},\left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]$$
$$\leq\left[\left(\prod_{i=1}^{n}(\tilde{a}_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}},\left(\prod_{i=1}^{n}(\tilde{b}_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}},\left(\prod_{i=1}^{n}(\tilde{c}_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right],$$

$$1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\leq 1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\tilde{\mu}_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}},$$

$$\left(1-\prod_{i=1}^{n}\left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda}\geq\left(1-\prod_{i=1}^{n}\left(1-\tilde{v}_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}.$$

According to Definition 2, number 6, we can conclude that $GTIFOWGA_{\lambda}(\beta_1,\beta_2,\beta_3,\ldots,\beta_n)\leq GTIFOWGA_{\lambda}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{\beta}_3,\ldots,\widetilde{\beta}_n\right)$.

Therefore:

$$\beta_{\alpha_i}\leq\widetilde{\beta}_{\alpha_i}(\forall i=1,\ 2,\ 3,\ldots,n).$$

In addition, let $(\beta_1,\beta_2,\beta_3,\ldots,\beta_n)$ be a collection of triangular intuitionistic fuzzy arguments and $\beta_{\alpha_i}=\left([a_i,b_i,c_i];\mu_{\beta_{\alpha_i}},v_{\beta_{\alpha_i}}\right)$. If the exponential weighting vector $\omega=(\omega_1,\omega_2,\ \omega_3,\ldots,\omega_n)^T$, $\sum_{i=1}^{n}\omega_i=1$, and $\omega_i\in[0,\ 1]$, $\lambda>0$, then the GTIFOWGA operator obtained is a TIFN and is given as follows for the monotonicity property:

If:

$$\beta_{\alpha}{}^{+}=\left([min_i(a_i),min_i(b_i),min_i(c_i)];min_i(\mu_{\beta_{\alpha i}}),max_i(v_{\beta_{\alpha i}})\right),$$

and:

$$\beta_{\alpha}{}^{-}=\left([max_i(a_i),max_i(b_i),max_i(c_i)];max_i(\mu_{\beta_{\alpha i}}),min_i(v_{\beta_{\alpha i}})\right).$$

Then:

$$\beta_{\alpha}{}^{+}\leq GTIFOWGA_{\delta}(\beta_1,\beta_2,\beta_3,\ldots,\beta_n)\leq\beta_{\alpha}{}^{-}.$$

Proof. If:

$$\beta_{\alpha i}=\left([a_i,b_i,c_i];\mu_{\beta_{\alpha i}},v_{\beta_{\alpha i}}\right)$$

for all:

$$(i=1,\ 2,\ 3,\ldots,n),$$

then:

$$\alpha^{+}=\left([min_i(a_i),min_i(b_i),min_i(c_i)];min_i(\mu_{\beta_{\alpha i}}),max_i(v_{\beta_{\alpha i}})\right),$$

and:

$$\alpha^{-}=\left([max_i(a_i),max_i(b_i),max_i(c_i)];max_i(\mu_{\beta_{\alpha i}}),min_i(v_{\beta_{\alpha i}})\right)$$

are TIFNs.

Since:

$$min_i[a_i,b_i,c_i]\leq[a_i,b_i,c_i]\leq max_i[a_i,b_i,c_i],min_i\mu_{\beta_{\alpha i}}\leq\mu_{\beta_{\alpha i}}\leq max_i\mu_{\beta_{\alpha i}}$$

and:

$$min_iv_{\beta_{\alpha i}}\leq v_{\beta_{\alpha i}}\leq max_iv_{\beta_{\alpha i}},$$

then we have:

$$\left[\left(\prod_{i=1}^{n} min(a_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{n} min(b_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{n} min(c_{(i)})^{\omega_i}\right)^{1/\lambda}\right]$$
$$\leq \left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]$$
$$\leq \left[\left(\prod_{i=1}^{n}\max(a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}\max(b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}\max(c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right],$$

$$\left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\min\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right) \leq \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right)$$
$$\leq \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-max\ \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right),$$

$$\left(1-\prod_{i=1}^{n}\ \left(1-min\ v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda} \leq \left(1-\prod_{i=1}^{n}\ \left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}} \leq \left(1-\prod_{i=1}^{n}\ \left(1-max\ v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}.$$

That is:

$$\left[min\left((a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, min\ \left((b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, min\left((c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]$$
$$\leq \left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]$$
$$\leq \left[max\left((a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, max\ \left((b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, max\left((c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]$$

$$\left(1-\left(1-\left(1-\left(1-\min\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{1/\lambda}\right) \leq \left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right)$$
$$\leq \left(1-\left(1-\left(1-\left(1-max\ \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right),$$
$$\left(1-\left(1-min\ v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda} \leq \left(1-\prod_{i=1}^{n}\ \left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}} \leq \left(1-\left(1-max\ v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}.$$

According to Definition 2, number 6, the above equation can be rewritten as:

$$\left[min\left((a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, min\ \left((b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, min\left((c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right]; \left(1-\left(1-\left(1-\left(1-\min\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right),$$
$$\left(1-\left(1-max\ v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}} \leq \left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, \left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right];$$
$$\left(1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right), \left(1-\prod_{i=1}^{n}\ \left(1-v_{\beta(i)}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}$$
$$\leq \left[max\left((a_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, max\ \left((b_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}, max\left((c_{(i)})^{\omega_i}\right)^{\frac{1}{\lambda}}\right];$$
$$\left(1-\left(1-\left(1-\left(1-max\ \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i}\right)^{\frac{1}{\lambda}}\right), \left(1-\left(1-min\ v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda}.$$

That is, $\beta_\alpha{}^+ \leq GTIFOWGA_\lambda(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \leq \beta_\alpha{}^-$. $\square$

4. Multi-Criteria Decision Making (MCDM) with the Generalized Geometric Operators for TIFNs

In this section, the GTIFOWGA operator is utilized for solving MCDM problems in which the performance ratings of the alternatives with respect to a given criteria are expressed in TIFN.

Consider an MCDM problem in which the alternatives $A = \{A_1, A_2, A_3, \ldots, A_m\}$, are assessed with respect to the criteria $C = \{C_1, C_2, C_3, \ldots, C_m\}$. The motivation here is to select the best alternative according to the intuitionistic fuzzy decision matrix given by the DM(s) $R^k(\alpha_{ij})$ $(k = 1,\ 2,\ 3, \ldots,\ l)$ when the criteria weights information is fixed. In collecting the DMs preference information for the alternatives with respect to the given criteria, a linguistic scale has been introduced which comprises of some linguistic variables which can be presented to the DMs and the TIFNs which are used for the evaluation proper. The new linguistic and TIFNs scale is given in Table 1.

Table 1. Linguistic and TIFNs scale.

Linguistic Terms	TIFNs
Low (L)	([0.10, 0.90, 0.2]; 0.4, 0.4)
Medium (M)	([0.20, 0.80, 0.2]; 0.4, 0.1)
Good (G)	([0.30, 0.60, 0.1]; 0.4, 0.3)
Very Good (VG)	([0.60, 0.30, 0.1]; 0.5, 0.2)
High (H)	([0.80, 0.10, 0.1]; 0.6, 0.1)
Very High (VH)	([0.90, 0.10, 0.2]; 0.7, 0.1)

4.1. Algorithm of the Proposed Approach for Solving the MCDM Problems

The algorithm of proposed approach for solving MCDM problems is given in the following steps:

Step 1: Use the decision information given by the DMs R^k, aggregate all the decision matrices $R^k(k = 1,\ 2,\ 3, \ldots,\ l)$ into a collective decision matrix $R = (r_{ij})_{mxn}$ using either the TIFOWGA operator, the TIFWGA operator or the TIFHWGA operator, where $w = (w_1, w_2,\ w_3, \ldots, w_n)^T$ is the weighting vector of the DMs (see Definitions 7–9).

$$R = (r_{ij})_{mxn} = \begin{bmatrix} ([a_{11}, b_{11}, c_{11}]; \mu_{11}\ ,\ v_{11}) & \ldots & \ldots & ([a_{1n}, b_{1n}, c_{1n}]; \mu_{1n}\ ,\ v_{1n}) \\ ([a_{21}, b_{21}, c_{21}]; \mu_{21}\ ,\ v_{21}\ ) & \ldots & \cdots & ([a_{2n}, b_{2n}, c_{2n}]; \mu_{2n}\ ,\ v_{2n}\ ) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ ([a_{m1}, b_{m1}, c_{m1}]; \mu_{m1}\ ,\ v_{m1}\ ) & \ldots & \cdots & ([a_{mn}, b_{mn}, c_{mn}]; \mu_{mn}\ ,\ v_{mn}\ ) \end{bmatrix}.$$

where $[a_{ii}, b_{ii}, c_{ii}]; \mu_{ii}\ ,\ v_{ii}$ is the TIFN which contain both the membership and non-membership degree.

Step 2: Use the decision information given in matrix $R = (r_{ij})_{mxn}$, utilizing the GTIFOWGA operator to derive the overall preference values r_i $(i = 1,\ 2,\ 3, \ldots, l)$, which is the collective comprehensive value r_i of alternative A_i :

$$r_i = \left([r^k{}_i, r^k{}_i, r^k{}_i]; \mu_{r^k{}_i}, v_{r^k{}_i}\right) =$$

$$GTIFOWGA_\lambda(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) = (\lambda\beta_{\alpha_1})^{\omega_i} \otimes (\lambda\beta_{\alpha_2})^{\omega_2} \otimes \ldots . \otimes (\lambda\beta_{\alpha_n})^{\omega_n}$$

$$= \left(\left[\left(\prod_{i=1}^{n}(a_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{n}(b_{(i)})^{\omega_i}\right)^{1/\lambda}, \left(\prod_{i=1}^{n}(c_{(i)})^{\omega_i}\right)^{1/\lambda}\right]; 1 - \left(1 - \prod_{i=1}^{n}\left(1 - \left(1 - \mu_{\beta(i)}\right)^{\lambda}\right)^{\omega_i},\right)^{1/\lambda}, \left(1 - \prod_{i=1}^{n}\left(1 - v_{\beta(i)}\right)^{\omega_i}\right)^{1/\lambda}\right),$$

where $w = (w_1, w_2, w_3, \ldots, w_n)^T$ is the weighting vector of the criteria.

Step 3: Calculate the scores function $S(r_i)$ $(i = 1, 2, \ldots, n)$ and accuracy function $H(r_i)$ $(i = 1, 2, \ldots, n)$ for the membership and non-membership functions:

$$S(\acute{\alpha}) = \frac{(a + 2b + c)\mu_\alpha}{4}, H(\acute{\alpha}) = \frac{(a + 2b + c)(1 - v_\alpha)}{4}.$$

Step 4: Rank the alternatives by virtue of Definition 6.

4.2. Numerical Example

Suppose the product development team of a design company "*X*" has generated four new design alternatives (A_1, A_2, A_3, and A_4) for a new crane machine during the conceptual design phase. If a group of experts (E_1, E_2, E_3, and E_4), within the company are assigned to evaluate the designs. If their aggregated assessment (using the TIFOWGA operator, the TIFWGA operator, or the TIFHWGA operator) of the design alternatives with respect to the criteria: expected mechanical safety C_1, amount of wear C_2, operating and maintenance cost C_3, and mass and size C_4, whose weight vectors are given as $\omega = \{0.15, 0.25, 0.32, 0.28\}$ are given in Table 2. We select the best alternative design using the proposed algorithm.

Table 2. Aggregation of all the experts' assessments (group intuitionistic fuzzy decision matrix).

A_i	C_1	C_2	C_3	C_4
A_1	([0.28, 0.46, 0.65]; 0.7, 0.2)	([0.57, 0.76, 0.96]; 0.6, 0.3)	([0.47, 0.62, 0.77]; 0.6, 0.2)	([0.59, 0.80, 1.00]; 0.6, 0.3)
A_2	([0.52, 0.62, 0.71]; 0.6, 0.3)	([0.74, 0.87, 1.00]; 0.8, 0.1)	([0.48, 0.74, 1.00]; 0.8, 0.2)	([0.47, 0.57, 0.67]; 0.7, 0.3)
A_3	([0.40, 0.54, 0.68]; 0.6, 0.4)	([0.59, 0.65, 0.72]; 0.6, 0.3)	([0.46, 0.68, 0.90]; 0.5, 0.5)	([0.55, 0.68, 0.82]; 0.8, 0.1)
A_4	([0.54, 0.77, 1.00]; 0.8, 0.2)	([0.60, 0.76, 0.92]; 0.6, 0.2)	([0.37, 0.56, 0.74]; 0.8, 0.2)	([0.73, 0.80, 0.86]; 0.7, 0.1)

Since the experts' assessments have already been aggregated, we jump to Step 2 in the algorithm to derive the overall preference values. Using the GTIFOWGA operator when the criteria weighting vector is given as $\omega = \{0.15, 0.25, 0.32, 0.28\}$, the comprehensive evaluation for the four design alternatives are shown in Table 3.

By applying Definitions 6, we can obtain the ranking of all the design alternatives as shown in Table 4. In addition, from Tables 3 and 4, we can see that, when the value of λ changes, the rankings of the design alternatives also change. Furthermore, if we decide to use any of the operators (TIFOWGA, TIFWGA, or TIFHWGA) in Step 2, the ranking of the design alternatives will, therefore, be in the order $A_2 > A_4 > A_1 > A_3$, where the best alternative is A_2.

Table 3. The overall preference values (comprehensive evaluations for four alternatives).

A_i	λ=1	λ=2	λ=3	λ=4
A_1	([0.486354, 0.66996, 0.853457]; 0.614035, 0.25466)	([0.030397, 0.041872, 0.053341]; 0.925073, 0.12733)	([0.006004, 0.008271, 0.010537]; 0.980486, 0.084887)	([0.0019, 0.002617, 0.003334]; 0.994251, 0.063665)
A_2	([0.538137, 0.69749, 0.84916]; 0.738094, 0.222068)	([0.033634, 0.043593, 0.053073]; 0.963488, 0.111034)	([0.006644, 0.008611, 0.010483]; 0.992694, 0.074023)	([0.002102, 0.002725, 0.003317]; 0.998236, 0.055517)
A_3	([0.503972, 0.64952, 0.795124]; 0.613474, 0.341045)	([0.031498, 0.040595, 0.049695]; 0.920472, 0.170523)	([0.006222, 0.008019, 0.009816]; 0.977019, 0.113682)	([0.001969, 0.002537, 0.003106]; 0.992254, 0.085261)
A_4	([0.534505, 0.700587, 0.85263]; 0.717162, 0.173177)	([0.033407, 0.043787, 0.053289]; 0.95734, 0.086588)	([0.006599, 0.008649, 0.010526]; 0.990805, 0.057726)	([0.002088, 0.002737, 0.003331]; 0.997633, 0.043294)
	λ=5	λ=9	λ=10	λ=50
A_1	([0.000778, 0.001072, 0.001366]; 0.998186, 0.050932)	([7.41×10^{-5}, 0.000102, 0.00013]; 0.999975, 0.028296)	([4.86×10^{-5}, 6.7×10^{-5}, 8.53×10^{-5}]; 0.999991, 0.025466)	([7.78×10^{-8}, 1.07×10^{-7}, 1.37×10^{-7}]; 1.0000, 0.005093)
A_2	([0.000861, 0.001116, 0.001359]; 0.999519, 0.044414)	([8.2×10^{-5}, 0.000106, 0.000129]; 0.999995, 0.024674)	([5.38×10^{-5}, 6.97×10^{-5}, 8.49×10^{-5}]; 0.999998, 0.022207)	([8.61×10^{-8}, 1.12×10^{-7}, 1.36×10^{-7}]; 1.0000, 0.004441)
A_3	([0.000806, 0.001039, 0.001272]; 0.997147, 0.068209)	([7.68×10^{-5}, 9.9×10^{-5}, 0.000121]; 0.999919, 0.037894)	([5.04×10^{-5}, 6.5×10^{-5}, 7.95×10^{-5}]; 0.999965, 0.034105)	([8.06×10^{-8}, 1.04×10^{-7}, 1.27×10^{-7}]; 1.0000, 0.006821)
A_4	([0.000855, 0.001121, 0.001364]; 0.99932, 0.034635)	([8.15×10^{-5}, 0.000107, 0.00013]; 0.999992, 0.019242)	([5.35×10^{-5}, 7.01×10^{-5}, 8.53×10^{-5}]; 0.999997, 0.017318)	([8.55×10^{-8}, 1.12×10^{-7}, 1.36×10^{-7}]; 1.0000, 0.003464)

Table 4. The rankings of all design alternatives.

λ	Ranking	Best Design Alternative
1	$A_2 > A_4 > A_1 > A_3$	A_2
2	$A_2 > A_4 > A_1 > A_3$	A_2
3	$A_4 > A_2 > A_1 > A_3$	A_4
4	$A_4 > A_2 > A_1 > A_3$	A_4
9	$A_4 > A_2 > A_1 > A_3$	A_4
10	$A_4 > A_2 > A_1 > A_3$	A_4
20	$A_4 > A_2 > A_1 > A_3$	A_4
50	$A_4 > A_2 > A_1 > A_3$	A_4

4.3. Comparison Analysis and Discussion

To verify the effectiveness and the feasibility of the proposed MCDM approach based on the GTIFOWGA operator, a comparative study has been conducted between the proposed MCDM approach, the triangular intuitionistic fuzzy aggregation operator proposed by Li [35] and the extended VIKOR method of TIFNs by Wan et al. [29] using the same numerical example above.

4.3.1. The Triangular Intuitionistic Fuzzy Aggregation Operator by Li [35]

Since the TIFN decision matrix has been normalized already, we calculate the weighted comprehensive values S_i for the alternative A_i:

$$S_i = \sum_{j=1}^{l} w_j r_{ij} = \langle \left(\sum_{j=1}^{l} w_j r_{ij}, \sum_{j=1}^{l} w_j r_{ij}, \sum_{j=1}^{l} w_j r_{ij} \right); \min\{\mu_{ij}\}, \max\{v_{ij}\} \rangle = ([a_1, b_1, c_1]; \mu_{\alpha_1}, v_{\alpha_1}).$$

In applying the weighted comprehensive values S_i for the alternatives A_i $(i = 1, 2, 3, 4)$, we have:

$$S_1 = \langle (0.50, 0.68, 0.86); 0.6, 0.3 \rangle, \; S_2 = \langle (0.55, 0.71, 0.86); 0.6, 0.3 \rangle;$$

$$S_3 = \langle (0.51, 0.65, 0.80); 0.5, 0.5 \rangle, \; S_4 = \langle (0.55, 0.71, 0.86); 0.6, 0.2 \rangle.$$

To rank the alternatives we have, $Z(S_i, \lambda) = \frac{V(S_i,\lambda)}{1+A(S_i,\lambda)}$, where $\lambda \in [0, 1]$ is the weight representing the DM preference information. If $\lambda \in \left[0, \frac{1}{2}\right]$, then it indicates that the DM prefers uncertainty or negative feelings, while if $\lambda \in \left[\frac{1}{2}, 1\right]$, it means that the DM prefers certainty or positive feelings. Finally, if $\lambda = \left[\frac{1}{2}\right]$, the DM can be said to be indifferent between positive feelings and negative feelings.

In this case, since there are no indications of the feeling of the experts or DMs in the question, we assume $\lambda = \left[\frac{1}{2}\right]$. In calculating the values of $Z\left(S_i, \frac{1}{2}\right)$ from the comprehensive values S_i, we have:

$$Z\left(S_1, \frac{1}{2}\right) = 0.416,\ Z\left(S_2, \frac{1}{2}\right) = 0.432,\ Z\left(S_3, \frac{1}{2}\right) = 0.311, \text{and } Z\left(S_4, \frac{1}{2}\right) = 0.462.$$

The ranking of the design alternatives are, therefore, in the order $A_4 > A_2 > A_1 > A_3$, where A_4 is the best design alternative.

4.3.2. The Extended VIKOR Method of TIFNs by Wan et al. [29]

Using the extended VIKOR method of TIFNs to solve the design selection problem above, we have the following for the group utility values and individual regret values:

$$S(A_1) = 0.413, R(A_1) = 0.141,\ S(A_2) = 0.284,\ R(A_2) = 0.126,$$

$$S(A_3) = 0.428, R(A_3) = 0.151, S(A_4) = 0.307,\ R(A_4) = 0.104.$$

Thus, $S^- = 0.428,\ S^+ = 0.284$, and $R^- = 0.151,\ R^+ = 0.104$.

The comprehensive values of each of the alternatives and the ranking orders, which are in increasing order with the different coefficients of decision mechanism λ, are obtained as follows:

$$Q(A_i) = \lambda \frac{S(A_i) - S^+}{S^- - S^+} + (1 - \lambda) \frac{R(A_i) - R^+}{R^- - R^+}.$$

When there is variation in the coefficients of decision mechanism λ, the rankings of the alternatives change, as shown in Table 5.

Table 5. The rankings of all design alternatives.

λ	Ranking	Best Design Alternative
0.1	$A_4 > A_2 > A_1 > A_3$	A_4
0.2	$A_4 > A_2 > A_1 > A_3$	A_4
0.3	$A_4 > A_2 > A_1 > A_3$	A_4
0.4	$A_4 > A_2 > A_1 > A_3$	A_4
0.5	$A_4 > A_2 > A_1 > A_3$	A_4
0.6	$A_4 > A_2 > A_1 > A_3$	A_4
0.7	$A_4 > A_2 > A_1 > A_3$	A_4
0.8	$A_2 > A_4 > A_1 > A_3$	A_2
0.9	$A_2 > A_4 > A_1 > A_3$	A_2
1.0	$A_2 > A_4 > A_1 > A_3$	A_2

From the comparison analysis, we can conclude that the proposed approach is effective, feasible, and rational, as both the results of the triangular intuitionistic fuzzy aggregation operator proposed by Li [35] and the extended VIKOR method of TIFNs by Wan et al. [29] are in agreement with our proposed method. The main advantage of the proposed new method is that it is straightforward and with less computational steps and formulae, unlike the other methods. The GTIFOWGA operator allows for more attitudinal information and flexibility of the DMs to be expressed and is used in accordance with the different interests or preferences in solving MCDM problems, since it is a multi-measure of neutralism, pessimism, and the optimistic characteristics of DMs rather than one single measure as shown in Table 4.

5. Conclusions

Triangular intuitionistic fuzzy numbers (TIFNs), which are a more generalized platform for expressing imprecise, incomplete, and inconsistent information when solving multi-criteria decision-making problems and for reflecting the evaluation information in different dimensions, has been applied in this study by developing a new triangular intuitionistic fuzzy geometric aggregation operator that is the generalized triangular intuitionistic fuzzy ordered weighted geometric averaging (GTIFOWGA) operator and defining some triangular intuitionistic fuzzy geometric aggregation operators, including the triangular intuitionistic fuzzy weighted geometric averaging (TIFWGA) operator, ordered weighted geometric averaging (TIFOWGA) operator and the hybrid geometric averaging (TIFHWGA) operator.

Based on these operators, a new approach has been proposed for effectively solving multicriteria decision-making problems when the weight information are fixed and the performance rating information are expressed in TIFNs. From the perspective of the aggregation operators, the generalized aggregation (GTIFOWGA) operator developed in this study allows the values of coefficients of decision mechanism λ to be variables (parameters) rather than fixed numbers, and allows for more attitudinal information and flexibility of the DMs to be expressed and used in accordance with the different interests or preferences in solving MAGDM problems, since it is a multi-measure of neutralism, pessimism and the optimistic characteristics of DMs rather than one single measure.

Finally, an illustrative example was provided to show the applicability and rationality of the presented method and it was followed by a comparative analysis using similar existing computational approaches. The results shows that, the proposed method is effective, feasible, and rational, as it was in total agreement with both the results of triangular intuitionistic fuzzy aggregation operator proposed by Li [35] and the extended VIKOR method of TIFNs proposed by Wan et al. [29]. In the future, we will consider other applications of the new method, specifically for real-life case studies, where the computational cost of the new aggregation operators will be determined using the computational complexity analysis approach.

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