Journal of Physics: Conference Series

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IOP Conf. Series: Journal of Physics: Conf. Series **890** (2017) 012028 doi:10.1088/1742-6596/890/1/012028

The uniform convergence of the eigenfunctions expansions of the biharmonic operator in closed domain

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Abstract. The mathematical models of the various vibrating systems are partial differential equations and finding the solutions of such equations are obtained by developing the theory of eigenfunction expansions of differential operators. The biharmonic equation which is fourth order differential equation is encountered in plane problems of elasticity. It is also used to describe slow flows of viscous incompressible fluids. Many physical process taking place in real space can be described using the spectral theory of differentiable operators, particularly biharmonic operator. In this paper, the problems on the uniform convergence of eigenfunction expansions of the functions from Nikolskii classes corresponding to the biharmonic operator are investigated.

1. Introduction

Let $\Omega \subset R^2$ be a domain with smooth boundary $\partial\Omega$. We denote by $L_p(\Omega)$ a class of the measurable functions which are p-integrable over Ω. We say that a function $f(x,y) \in L_p(\Omega)$ belongs to the $H_p^a(\Omega)$, if for any $h=(h,k) \in R^2$ and for all integers α, β satisfying $\alpha+\beta=l$:

$$\left|\partial_x^\alpha \partial_y^\beta f(x+h,y+k) - 2\partial_x^\alpha \partial_y^\beta f(x,y) + \partial_x^\alpha \partial_y^\beta f(x-h,y-k)\right|_{L_p(\Omega_{\sqrt{h^2+k^2}})} \le C\left(h^2+k^2\right)^{\frac{\sigma}{2}}.$$

where a is written as $a=l+\sigma$, $l-$positive integer and $0<\sigma\le 1$. Using the notation $\Delta_{h,k}^2 f(x,y) = f(x+h,y+k) - 2f(x,y) + f(x-h,y-k)$ we define a norm in $H_p^\alpha(\Omega)$ by the following

$$\|f\|_{p,\alpha} = \|f\|_{L_p(\Omega)} + \sum_{\alpha+\beta=l} \sup_{h,k} \left(h^2+k^2\right)^{-\frac{\sigma}{2}} \left\|\Delta_{h,k}^2 \partial_x^\alpha \partial_y^\beta f(x,y)\right\|_{L_p(\Omega_{\sqrt{h^2+k^2}})}.$$

The closure of the space $C_0^\infty(\Omega)$ in the norm of $H_p^a(\Omega)$ denoted by $\dot{H}_p^a(\Omega)$.

We consider the biharmonic operator Δ^2 with the domain $D_{\Delta^2} = \{ u \in C^4\Omega): u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \}$, where Δ denotes well known Laplace operator. The biharmonic operator $(\Delta)^2$ with the domain of definition D_{Δ^2} is symmetric

$$(\Delta^2 u, v) = (u, \Delta^2 v), \forall u, v \in D_{\Delta^2}.$$

and nonnegative

$$(\Delta^2 u, u) = (\Delta u, \Delta u) = \|\Delta u\|^2_{L_2(\Omega)} \geq 0, \forall u \in D_{\Delta^2}.$$

It follows from Fredrichs Theorem (see [1]) that such defined biharmonic operator can be extended to as self-adjoint operator, which we denote by A. It is well known that self-adjoint operator A has a set of eigenfunctions $\{u_{nm}(x,y)\}$, which is complete in $L_2(\Omega)$. We denote by $\{\lambda_{nm}\}$ the set of eigenvalues of biharmonic operator in Ω:

$$\Delta^2 u_{nm}(x,y) - \lambda_{nm} u_{nm}(x,y) = 0, \qquad x \in \Omega.$$

Let E_λ be the spectral resolution of unity corresponding to self adjoint operator A. Then for any $f \in L_2(\Omega)$ we have

$$E_\lambda f(x,y) = \sum\sum_{\lambda_{nm}<\lambda} f_{nm} u_{nm}(x,y),$$

where f_{nm} is the Fourier coefficients of the function f:

$$f_{nm} = \iint_\Omega f(x,y)\, u_{nm}(x,y)\, dx\, dy, \qquad n,m = 1,2,...$$

In the present paper we study the spectral resolutions of $E_\lambda f$ and their Riesz means

$$E_\lambda^s f(x,y) = \sum\sum_{\lambda_{nm}<\lambda} \left(1 - \frac{\lambda_{nm}}{\lambda}\right)^s f_{nm} u_{nm}(x,y), \ s \geq 0,$$

for the functions from the classes of Nikolskii $H_p^\alpha(\Omega)$.

The main result of the paper is the following

Theorem 1. *If* $f(x,y) \in H_p^a(\Omega), \Omega \subset R^2$, *and numbers* $a, p,$ *and* s *are related as follows:*

$$a + s \geq \frac{1}{2},\ pa > 2,\ p \geq 1,$$

then the Riesz means $E_\lambda^s f(x,y)$ *uniformly converges to* $f(x,y)$ *on the closure* $\overline{\Omega}$ *of the domain* Ω*.*

A sufficient condition for the uniform convergence of $E_\lambda^s f$ on any compact set from $\Omega \subset R^N$ to the function from $H_p^\alpha(\Omega), p \geq 1, s > 0, l > 0$, is that the following conditions be satisfied: $l + s \geq \frac{N-1}{2},\ p \cdot \alpha > N$, which were first found in the work of Ilin [2] for uniform convergence of the $E_\lambda f$ in the classes Sobolev $W_p^\alpha(\Omega)$. Later the uniform convergence of Riesz means $E_\lambda^s f$ is established in [3] for the functions from Nikol'skii classes $H_p^\alpha(\Omega)$, which is broader than Sobolev classes $W_p^\alpha(\Omega)$. II'in proved that the conditions $l + s \geq \frac{N-1}{2},\ p \cdot \alpha > N$ are best possible for uniform convergence. Namely, if $l + s < \frac{N-1}{2}$, then there exists a function having all partial derivatives in Ω through order

l for which the means $E_\lambda^s f$ are unbounded at some point. As for the condition $p \cdot \alpha > N$, if the opposite inequality $p \cdot \alpha \le N$ is satisfied, then there exists an unbounded function $f \in W_p^\alpha(\Omega)$ whose Riesz mean clearly cannot converge to it uniformly, since the function in question is not continuous. The convergence of the spectral decompositions of the Laplace operator on closed domain firstly investigated by Il'in and continued by Moiseev [4] and he proved uniformly convergence of the eigenfunction expansions of the functions from $W_p^{(\frac{N}{2}+1)}(\Omega)$ on closed domain $\overline{\Omega}$. In the work [5] Eskin considered the 2*m* order elliptic differential operator with Lopatinsky boundary condition and proved uniformly convergent of the spectral expansions of the functions from $W_p^{(\frac{N-1}{2}+\varepsilon)}(\Omega), \varepsilon > 0$ on closed domain $\overline{\Omega}$. The uniform convergence of the eigenfunction expansions of the Laplace operator in closed domain are considered by Rakhimov [6], where he showed that the conditions $l + s \ge \dfrac{N-1}{2},\ p \cdot \alpha > N$ guarantee uniform convergence of the expansions in closed domain for the functions from Nikolskii classes $H_p^\alpha(\Omega)$. The estimation in closed domain for eigenfunctions of the biharmonic operator, which gurantees the uniform convergence of the eigenfunction expansions of continuos functions in closed domain, is obtained in [9]. In the current research we investigate sufficient conditions for uniform convergence of the eigenfunction expansions corresponding to the biharmonic operator. Using the estimations of the paper [9] we prove uniform convergence of the mentioned eigenfunction expansions from Nikolskii classes $H_p^\alpha(\Omega)$ in closed domain $\overline{\Omega}$.

2. Preliminaries

Here we formulate the results of the paper [9] in the form which is convenient for our current investigation of the convergence of the eigenfunction expansions.

Lemma 1. *For the eigenfunctions* $u_{nm}(x,y)$ *and eigenvalues* λ_{nm} *of biharmonic operator with the domain of definition* D_{Δ^2} *we have:*

$$\sum_{|\sqrt[4]{\lambda_{nm}}-\lambda|\le 1}\sum u_{nm}^2(x,y) = O\left(\lambda \ln^2 \lambda\right), \qquad \lambda > 1, \tag{1}$$

uniformly for all $(x,y) \in \overline{\Omega} = \Omega \cup \partial\Omega$.

The proof of the Lemma 1 is obtained in [9] by transforming the biharmonic equation to the system of equations, where each of them are Laplace equations. From the estimation (1) we derive that for any $\delta > 0$ and uniformly for all $(x,y) \in \overline{\Omega} = \Omega \cup \partial\Omega$

$$\sum_{1<\lambda_{nm}\le\lambda}\sum u_{nm}^2(x,y)\, \lambda_{nm}^{\delta-1/2} = O\,(\lambda^{\delta} \ln^2\lambda), \tag{2}$$

$$\sum_{\lambda_{nm}\ge\lambda}\sum u_{nm}^2(x,y)\, \lambda_{nm}^{-\delta-1/2} = O\,(\lambda^{-\delta} \ln^2\lambda).$$

Obtained estimations for the eigenfunctions in closed domain for the eigenfunctions of the Laplace operator (see [5]) is applied to extend the similar estimations for the eigenfunctions of the biharmonic operator in closed domain by transforming the biharmonic equation into Laplace equations.

It is well known that the Riesz means of the eigenfunction expansions of the Biharmonic operator is can be represented by the integrals of the Bessel function. Let recall a definition of the Bessel function of order ν

$$J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma\left(\frac{2\nu+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its}(1-s^2)^{\frac{2\nu-1}{2}} ds.$$

The Bessel function has a trivial estimation near the zero: $J_\nu(t) \approx Ct^\nu$ as $t \to 0$. But for sufficiently large values of t we have

$$J_\nu(t) = \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(t^{-\frac{3}{2}}\right),$$

as $t \to \infty$. Using these well-known estimations for Bessel function we obtain

Lemma 2. *Let* $R > 0$. *Then*

$$\left|\int_R^\infty J_{1+s}(t\lambda) J_0(t\lambda_{nm}) t^{-s} dt\right| < A\lambda^{-\frac{1}{2}} \lambda_{nm}^{-\frac{1}{2}},$$

for all positive values of μ *and* μ_{nm}. *Futhermore we have*

$$\left|\int_R^\infty J_{1+s}(r\lambda) J_0(r\lambda_{nm}) r^{-s} dr\right| < \begin{cases} \dfrac{A\lambda^{-\frac{3}{2}}\lambda_{nm}^{\frac{1}{2}}}{\lambda - \lambda_{nm}} + A\lambda^{-\frac{3}{2}}\lambda_{nm}^{-\frac{1}{2}}, & \lambda_{nm} < \lambda, \\ \dfrac{A\lambda^{\frac{1}{2}}\lambda_{nm}^{-\frac{3}{2}}}{\lambda_{nm} - \lambda} + A\lambda^{-\frac{1}{2}}\lambda_{nm}^{-\frac{3}{2}}, & \lambda_{nm} > \lambda. \end{cases} \quad (3)$$

For the proof of the estimation (3) we refer the readers to the paper [10]. The estimation of (3) can be written in one as follows:

$$\left|\int_R^\infty J_{1+s}(t\lambda) J_0(t\lambda_{nm}) t^{-s} dt\right| \le C(\lambda\lambda_{nm})^{-\frac{1}{2}} |\lambda_{nm} - \lambda|^{-1}, \qquad \lambda_{nm} \ne \lambda. \quad (4)$$

3. Uniform convergence of eigenfunction expansions from Nikolskii classes

In this section we prove the statement of the Theorem 1. First we establish facts on the estimation of the Riesz means. Using the following formula

$$\Gamma(s+1).2^s \lambda^{\frac{1-s}{2}} \int_R^\infty t^{-s} J_{1+s}(t\lambda) J_0(t\lambda_{nm}) dt = \begin{cases} \left(1 - \dfrac{\lambda_{nm}}{\lambda}\right)^s & , if \quad \lambda_{nm} < \lambda, \\ 0, & , if \quad \lambda_{nm} \ge \lambda. \end{cases}$$

and the definition of Riesz means we have

$$E_\lambda^s f(x,y) = 2^s (2\pi)^{-1} \Gamma(s+1) \lambda^{\frac{1-s}{2}} \iint_{r<R} f(x,y) r^{-1-s} J_{s+1}(r\sqrt{\lambda}) \, dx \, dy$$

$$+ 2^s \Gamma(s+1) \lambda^{\frac{1}{4}-\frac{s}{2}} \sum_{n=1}^\infty \sum_{m=1}^\infty \lambda_{nm}^{-\frac{1}{4}} f_{nm} u_{nm}(x,y) I^s(\lambda, \lambda_{nm}),$$

where

$$I^s(\lambda, \lambda_{nm}) = (\lambda \cdot \lambda_{nm})^{\frac{1}{4}} \int_R^\infty r^{-s} J_{s+1}(r\sqrt{\lambda}) J_0(r\sqrt{\lambda}_{nm}) \, dr.$$

Lemma 3. *Let* $f(x,y) \in \dot{H}_2^a$ *and* $a = \ell + \kappa$, *where* ℓ *is a nonnegative integer and* $0 < \kappa \le 1$. *Then uniformly for all* $(x,y) \in \overline{\Omega}$ *we have*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{nm}\, u_{nm}(x,y)\, \lambda_{nm}^{\ell/2-1/4}\, I_\ell(\lambda,\lambda_{nm}) = O(\lambda^{-\kappa/2}\ln\lambda)\,\|f\|_{H_2^\alpha}.$$

Proof. For $a > 0$ and $\lambda > 1$, we have

$$\sum\sum_{\lambda<\lambda_{nm}<4\lambda} f_{nm}^2\, \lambda_{nm}^a \le C\,\|f\|_{H_2^a}^2. \tag{5}$$

for the value of $\lambda \ge 1$, we have

$$|I^s(\lambda,\lambda_{nm})| \le \frac{C}{1+|\sqrt{\lambda}-\sqrt{\lambda}_{mn}|}. \tag{6}$$

$$\sum\sum_{\lambda_{nm}>1} f_{nm}^2\, \lambda_{nm}^{\alpha-\varepsilon} \le C\,\|f\|_{H_2^\alpha}^2.$$

Next we estimate the first sum

$$\left|\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} f_{nm}\, u_{nm}(x,y)\, \lambda_{nm}^{\ell/2-1/4}\, I_\ell(\lambda,\lambda_{nm})\right| \le$$

$$C\left(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} f_{nm}^2\, \lambda_{nm}^{\alpha}\right)^{1/2}\left(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} \lambda_{nm}^{\frac{2\ell-1}{2}-\alpha}\, u_{nm}^2(x,y)\, I_\ell^2(\lambda,\lambda_{nm})\right)^{1/2}$$

$$\le C\,\|f\|_{H_2^\alpha}\left(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} \lambda_{nm}^{-\frac{1}{2}-\kappa}\, u_{nm}^2(x,y)\, I_\ell^2(\lambda,\lambda_{nm})\right)^{1/2}$$

By applying (6) to the above, we obtain

$$\left(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} \lambda_{nm}^{-\frac{1}{2}-\kappa}\, u_{nm}^2(x,y)\, I_\ell^2(\lambda,\lambda_{nm})\right)^{1/2} \le \left(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} \lambda_{nm}^{-\frac{1}{2}-\kappa}\, u_{nm}^2(x,y)\,|\sqrt{\lambda}-\sqrt{\lambda}_{nm}|^{-2}\right)^{1/2}$$

$$\le C\,\lambda^{-1/2-\kappa}\,\Big(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} u_n^2(x,y)\,|\sqrt{\lambda}-\sqrt{\lambda}_{nm}|^{-2}\Big)^{1/2} \le C\,\lambda^{-1/2-\kappa}\,\Big(\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} u_{nm}^2(x,y)\,\lambda_{nm}^{-1}\Big)^{1/2}$$

Finally, by using (2)

$$\left|\sum\sum_{1<\lambda_{nm}\le\frac{1}{4}\lambda} f_{nm}\, u_{nm}(x,y)\, \lambda_{nm}^{\ell/2-1/4}\, I_\ell(\lambda,\lambda_{nm})\right| \le C\lambda^{-1/2-\kappa/2}\ln\lambda\;\|f\|_{H_2^\alpha}.$$

The second term can be estimated by using (1), (2) and (5) as follows:

$$\Big|\sum\sum_{\frac{1}{4}\lambda<\lambda_{nm}\le\frac{9}{4}\lambda} f_{nm}\, u_{nm}(x,y)\, \lambda_{nm}^{\ell/2-1/4}\, I_\ell(\lambda,\lambda_{nm})$$

$$\le C\,(\sum_{\frac{1}{4}\lambda\le\lambda_{nm}<\frac{9}{4}\lambda}\sum f_{nm}^2\,\lambda_{nm}^{a})^{1/2}(\sum_{\frac{1}{4}\lambda\le\lambda_{nm}<\frac{9}{4}\lambda}\sum \lambda_{nm}^{(2\ell-1)/2-a}\,u_{nm}^2(x,y)\,I_\ell^2(\lambda,\lambda_{nm}))^{1/2}$$

Let denote the least number k such that $2^k\ge\sqrt{\lambda}/2$. Then the interval $[1,\sqrt{\lambda}/2]\subset\bigcup_{p=1}^{k}[2^{p-1},2^p]$ where $p=1,2,...,k$. By applying (4) to the above, we obtain

$$\left|\sum_{\frac{1}{4}\lambda<\lambda_{nm}\le\frac{9}{4}\lambda}\sum f_{nm}\,u_{nm}(x,y)\,\lambda_{nm}^{\ell/2-1/4}\,I_\ell(\lambda,\lambda_{nm})\right|\le$$

$$C\,\|f\|_{H_2^\alpha}(\sum_{p=1}^{k}[\sum_{2^{p-1}\le|\sqrt{\lambda}_{nm}-\sqrt{\lambda}|<2^p}\sum \lambda_{nm}^{-1/2-\kappa}\,u_{nm}^2(x,y)\,|\sqrt{\lambda}-\sqrt{\lambda}_{nm}|^{-2}])^{1/2}$$

Since $\lambda_{nm}<9\lambda/4$, we have

$$\left|\sum_{\frac{1}{4}\lambda<\lambda_{nm}\le\frac{9}{4}\lambda}\sum f_{nm}\,u_{nm}(x,y)\,\lambda_{nm}^{\ell/2-1/4}\,I_\ell(\lambda,\lambda_{nm})\right|$$

$$\le C\,\lambda^{-1/2-\kappa}\,\|f\|_{H_2^\alpha}(\sum_{p=1}^{k}[\sum_{2^{p-1}\le|\sqrt{\lambda}_{nm}-\sqrt{\lambda}|<2^p}\sum u_{nm}^2(x,y)\,|\sqrt{\lambda}-\sqrt{\lambda}_{nm}|^{-2}])^{1/2}$$

$$\le C\,\lambda^{-1/2-\kappa}\,\|f\|_{H_2^\alpha}(\sum_{p=1}^{k}4^{1-p}[\sum_{2^{p-1}\le|\sqrt{\lambda}_{nm}-\sqrt{\lambda}|<2^p}\sum u_{nm}^2(x,y)])^{1/2}$$

By using Lemma 3, we obtain

$$\left|\sum_{\frac{1}{4}\lambda<\lambda_{nm}\le\frac{9}{4}\lambda}\sum f_{nm}\,u_{nm}(x,y)\,\lambda_{nm}^{\ell/2-1/4}\,I_\ell(\lambda,\lambda_{nm})\right|\le C\,\lambda^{-1/2-\kappa}\,\|f\|_{H_2^\alpha}(\lambda^{1/2}\sum_{p=1}^{k}4^{1-p}\,2^p\,\ln^2\lambda)^{1/2}$$

$$\le C\lambda^{-\kappa/2}\ln\lambda\,\|f\|_{H_2^\alpha}(\sum_{p=1}^{k}2^{-p})^{1/2}.$$

Taking into account that the series $\sum_{\ell=1}^{\infty}2^{-\ell}$ is converge to 1, we now complete the proof of as follows

$$\left|\sum_{\frac{1}{4}\lambda<\lambda_{nm}\le\frac{9}{4}\lambda}\sum f_{nm}\,u_{nm}(x,y)\,\lambda_{nm}^{\ell/2-1/4}\,I_\ell(\lambda,\lambda_{nm})\right|\le C\lambda^{-\kappa/2}\ln\lambda\,\|f\|_{H_2^\alpha}.$$

The condition for $\kappa>0$ implies that the right side of the latter is finite as $\lambda\to\infty$. The proof of the Lemma 3 is completed. After the statement of Lemma 3 is obtained we can apply the properties of the functions from Nikolskii classes and obtain:

Lemma 4. *Let a function* $f(x,y)$ *belongs to the set* $\dot{H}_2^a(\Omega)\cap H_p^a(\overline{\Omega})$. *The numbers* a, p *and* s *be related by the following conditions:* $p\ge1,\ a\ge\frac{1}{2}-s,\ s>0.$ *Then uniformly for all* $(x,y)\in\overline{\Omega}$ *the Riesz means of order* s *can be estimated as follows*

$$\left|E_{\lambda}^{s} f(x,y)\right| \le C\left(\|f\|_{\dot{H}_2^a(\Omega)} + \|f\|_{H_p^a(\Omega)}\right).$$

After that statement of the Lemma 4 is established the proof of the Theorem 1 can be obtained using the density of $C_0^{\infty}(\Omega)$ in the classes of Nikolskii as in the paper [7].

Acknowledgments
The current research is supported by Universiti Malaysia Pahang under Universiti Research Grant Scheme RDU170364 and Universiti Putra Malaysia under Fundamental Research Grant Scheme (FRGS) 01-01-16-1868 SR 5524975.

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