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Solution of Pantograph Differential Equations by Collocation Method using Ortho Exponential Polynomial

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Abstract

Novel matrix based on numerical technique of collocation method via truncated ortho exponential polynomial (OECM) is proposed to approximate the solution of pantograph differential equations. The applicability, reliability, and efficiency of the methodology are examined by applying the method to the pantograph differential equations. The comparison is made between the existing reported results and the present results. The proposed method shows good agreement with the existing reported method, hence indicate collocation method via truncated ortho exponential polynomial (OECM) is able to numerically simulate pantograph differential equations.

Keywords: *Orthoexponential polynomial, Collocation method, Pantograph differential equations*

1. Introduction

Pantograph differential equations is referred to the differential equations with the existence of delay argument in the state variables. It plays a vital role in explaining various types of biological and physical phenomena. Such equations evolve in industrial applications (Chelyshkov, 1987 and Slobodan Trickovic and Miomir Stankovic, 2003) and have been widely used in the economy, electrodynamics, control and biology (Chelyshkov and Liu, 2001). A polynomial type of OEP also has practical applications in the area of science and engineering, such as heat conduction problems (Ibrahim and Bokhari, 2011), electric circuits theory (Dmitriyev, 2013), thermoelasticity and thermo viscoelasticity (Mokriv and Oliyamik, 1989), diffraction problems (Yashiro, et. al, 2000)), hydrometeorology (Cizek, 1960) and vibration analysis (Yegao et. al, 2013). Pantograph differential equations belong to a specific type of delay differential equations (DDEs) (Karimi Vanani and Aminataei, 2009). The complexity arises due to the existence of delay terms that resulting to the exact solution of Pantograph differential equations is hard to be found. Recently, research on proposing numerical methods for simulating the solutions of Pantograph differential equations are amongst of the interest. The proposed methods include the differential transform method (Xie et. al, 2011), the pseudospectral methods (Ishtihaq et. al, 2009), the quadrature and interpolation procedures (Bica, 2011), the Adomian decomposition method (Evan and Raslan, 2005), the collocation methods (Brunner and Hu, 2007) and the variation iteration method (Xumei Chen and Linjun Wang, 2010). This research is aimed to propose a ortho exponential collocation method for solving Pantograph differential equations. Ortho exponential polynomial is defined as

$$d_{lg} = (-1)^{l+g} \binom{l}{g} \binom{l+g-1}{g-1} \tag{1}$$

The generalized form of the Pantograph equation is

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$$\sum_{k=0}^{m} R_k(t)u^k(t) + \sum_{i=0}^{I}\sum_{j=0}^{J} S_{ij}u^k(\zeta_i t + \mu_i) = f(t), \quad 0 \le t \le 1 \tag{2}$$

subject to the initial condition

$$\sum_{k=0}^{m-1}[a_{pk}V(0)L^k D^T + b_{pk}V(b)L^k D^T] = \lambda_p \quad p = 0,1,2,3,\cdots m-1 \tag{3}$$

where R_k, S_{ij} and $f(t)$ are functions over the interval $[0,1]$ and $\mu_i, \zeta_i, a_{pk}, b_{pk}$ are constants.

Fundamental Matrix Relation

Consider

$$D_a(t) = D_b(t) + f(t) \tag{4}$$

where

$$D_a(t) = \sum_{k=0}^{m} R_k(t)u^k(t), \qquad D_b(t) = \sum_{i=0}^{I}\sum_{j=0}^{J} S_{ij}(t)u^{(k)}(\zeta_i(t) + \mu_i)$$

The solution of (2) in OEP form is expressed as

$$u(t) \cong u_N(t) = \sum_{n=1}^{N} a_n \zeta_n(t) \tag{5}$$

The notation $u(t)$ and $u_N(t)A$ can be written as

$$u(t)A = \zeta_n(t) \tag{6}$$

where

$\zeta(t) = [\zeta_1(t)\ \zeta_2(t)\ \zeta_3(t) \ldots \zeta_N(t)]$ and $A = [a_1\ a_2\ a_3 \ldots a_N]^T$

The relationship of $\zeta_n(t)$ can be expressed as

$$\zeta(t) = V(t)D^T \tag{7}$$

where

$$V(t) = [v^{-t}\ v^{-2t}\ v^{-3t} \cdots v^{-Nt}]$$

and

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ d_{21} & d_{22} & 0 & \cdots & 0 \\ d_{31} & d_{32} & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots 0 \\ d_{N1} & d_{N2} & d_{N3} & \cdots & d_{NN} \end{bmatrix}$$

The derivative $V'(t)$ is defined by

$$V'(t) = V(t)L$$

where $L = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 \\ 0 & 0 & -3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -N \end{bmatrix}$. Then, k^{th} derivative of $V(t)$ can be written as

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$$V^{(k)}(t)=V(t)L^{k} \qquad k=0,1,2,3,\ldots \tag{8}$$

By using equations (6) - (8) we have

$$u^{(k)}(t)=V(t)L^{k}D^{T}A \tag{9}$$

Equation (9) represents the non-delay part of the equation (2). It can be converted to the matrix form. For the delay part, we only need to replace $t\rightarrow\zeta_i t+\mu_i$ in the equation (9) such that

$$u^{(k)}(\zeta_i t+\mu_i)=V(\zeta_i t+\mu_i)L^{k}D^{T}A \tag{10}$$

where

$$V(\zeta_i t)=[v^{-\zeta_i t}\ \ v^{-2\zeta_i t}\ \ v^{-3\zeta_i t}\cdots v^{-N\zeta_i t}],$$

$$\mu(\mu_i)=diag[v^{-\mu_i}\ v^{-2\mu_i}\ v^{-3\mu_i}\cdots v^{-N\mu_i}],\ \ i=0,1,2,3,\cdots,I,$$

and

$$V(\zeta_i t+\mu_i)=V(\zeta_i t)\mu(\mu_i) \tag{11}$$

2. Method of the Solution

Pantograph differential equation (2) is written as

$$\sum_{k=0}^{m}R_k(t)V(t)L^{k}D^{T}A+\sum_{i=0}^{I}\sum_{j=0}^{J}S_{ij}(t)V(\zeta_i t)\mu(\mu_i)L^{k}D^{T}A=f(t),\ i=1,2,3,\cdots N \tag{12}$$

Let the collocation points are defined as

$$t_i=\frac{b(i-1)}{N-1}$$

By applying the collocation points, the fundamental pantograph equation (12) can be written in the form of

$$\sum_{k=0}^{m}R_k(t_i)V(t_i)L^{k}D^{T}A+\sum_{i=0}^{I}\sum_{j=0}^{J}S_{ij}(t_i)V(\zeta_i t_i)\mu(\mu_i)L^{k}D^{T}A=f(t_i) \tag{13}$$

The matrix involves in equation (13) are

$$S_{ij}(t_i)=\begin{bmatrix} S_{ij}(t_1) & 0 & 0 & \cdots & 0\\ 0 & S_{ij}(t_2) & 0 & \cdots & 0\\ 0 & 0 & S_{ij}(t_3) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & S_{ij}(t_N)\end{bmatrix};V(t_i)=\begin{bmatrix}V(t_1)\\ V(t_2)\\ V(t_3)\\ \vdots\\ V(t_N)\end{bmatrix}$$

$$R_i(t_i)=\begin{bmatrix} R_i(t_1) & 0 & 0 & \cdots & 0\\ 0 & R_i(t_2) & 0 & \cdots & 0\\ 0 & 0 & R_i(t_3) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & R_i(t_N)\end{bmatrix};V(\zeta_i t_i)=\begin{bmatrix}V(\zeta_i t_1)\\ V(\zeta_i t_2)\\ V(\zeta_i t_{23})\\ \vdots\\ V(\zeta_i t_N)\end{bmatrix}$$

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$F = [f(t_1)\, f(t_2)\, f(t_3) \cdots f(t_N)]$

Rewriting equation (13) yield

$$ZA = F \text{ Or } [Z; F] \tag{14}$$

where

$$Z = [z_{pq}] = \sum_{k=0}^{m} R_k(t_i)V(b)L^k D^T + \sum_{i=0}^{I}\sum_{j=0}^{J} S_{ij}(t_i)V(\zeta_i t_i)\mu(\mu_i)L^k D^T \tag{15}$$

for $q = 1, 2, 3, \cdots N$ and subject to the initial condition

$$\sum_{k=0}^{m-1}[a_{pk}V(0)L^k D^T + b_{pk}V(b)L^k D^T]A = \lambda_p \quad p = 0, 1, 2, 3, \cdots m-1$$

Write the above equation in matrix form as

$u_p A = [\lambda_p]$ or $[u_p; \lambda_p]$
(16)

where

$$u_p = [u_{p_1}\, u_{p_2}\, u_{p_3} \cdots u_{p_N}] = \sum_{k=0}^{m-1} a_{pk}V(0)L^k D^T + b_{pk}V(b)L^k D^T$$

To compute the approximate result of Pantograph differential equation (2) with the initial condition (3), we interchange the row of (16) by the last m^{th} rows of the matrix (14) such that the modified augmented matrix is obtained as

$$[\tilde{Z}; \tilde{F}] \tag{17}$$

If the

rank $\tilde{Z}$=rank[$\tilde{Z}; \tilde{F}$]= N

then we can write

$$A = (\tilde{Z})^{-1}\tilde{F}$$

Now the matrix A is uniquely determined, which implies that the Pantograph differential equation (2) with the initial condition (3) has an approximate solution by truncated OEP (5).

3. Errors Bound

The accuracy of the method can be determined by using the residual errors and error bounds. The truncated Ortho exponential polynomials is approximated solution of equation (2), when the function $u(x)$ and its derivatives are substituted in equation (2). The resulting equation must be satisfied approximately that is for $t \in [0, 1]$ we have

$$\varepsilon(t_i) = \left| u^m(t_i) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} u^{(k)}(\alpha_{jk}t_i + \beta_i) - f(t_i) \right| \cong 0, \quad i = 1, 2, 3, \ldots \tag{18}$$

where ε is referred to the absolute error and $\varepsilon(t_i) \le 10^{-k}$ (k is a positive integer). As max $10^{-k_i} = 10^{-k}$ (k is a positive integer) is defined, then the limit defined for N is increased before

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the difference in $\varepsilon(t_i)$ at each and every point becomes lesser than the recommended 10^k. The absolute error can be calculated by

$$\varepsilon_N(t) = u_N^{(m)}(t) - \sum_{j=0}^{J}\sum_{k=0}^{m-1} S_{jk}(t) u_N^{(k)}(\alpha_j(t)+\beta_j) - f(t)$$

The absolute error decreases, if $\varepsilon_N(t) \to 0$, when the size of N increases.

4. Numerical Results

Some nonlinear Pantograph differential equations are presented in this section to validate the applicability, accuracy, reliability and efficiency of the present numerical technique.

Example 1 (Feng, 2013)

Consider multi pantograph differential equation

$$u'(t) = -u(t) + (1/8)u(t/4) - (1/8)e^{t/4} \tag{19}$$

with the initial condition $u(0) = 1$. Figure 1 presents the graphical visualization of the exact and numerical solutions of Example 1. The simulated results are computed for $N = 5$ and $N = 11$ over the interval [0, 1].

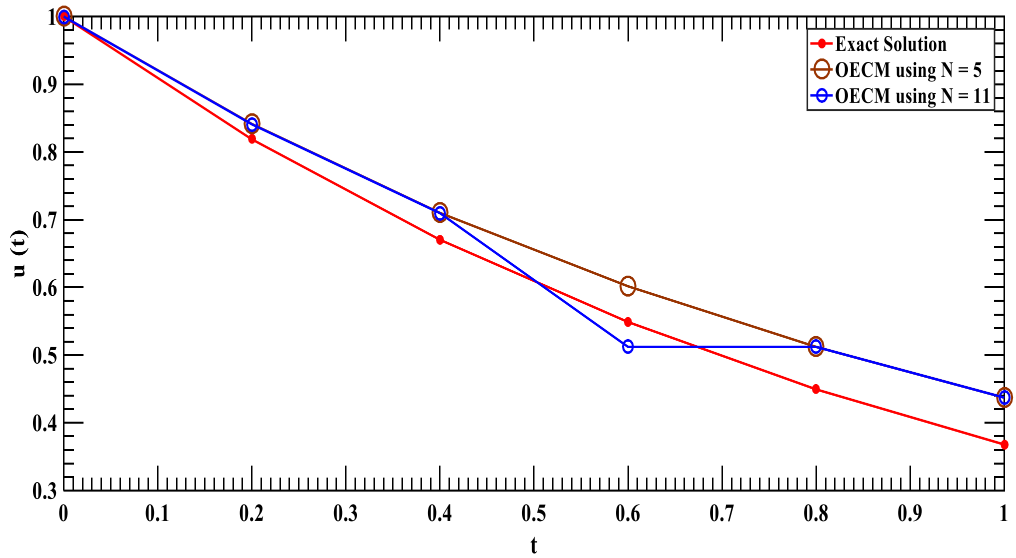

Figure 1: Exact and Numerical Results of Example 1

Table 1 shows the exact and numerical solutions of Example 1 for $N = 5$ and $N = 11$ over the interval [0, 1] with a step size of 0.2. The increasing number of N leads to good agreement of approximation solutions compared with the exact solution as indicated by low values of the absolute error.

Table 1: Exact Solution, Numerical Solution and Absolute Error of Example 1

x	Exact Solution, $u(t)$	Numerical Solution		Absolute Error	
		$N = 5$	$N = 11$	ε_{N5}	ε_{N11}

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0	1	1	1	0	0
0.2	0.818730	0.841406	0.840880	0.022767	0.022157
0.4	0.670320	0.710016	0.709671	0.039690	0.039351
0.6	0.548812	0.601630	0.512100	0.052820	0.052520
0.8	0.449329	0.512100	0.512100	0.062771	0.062462
1	0.367879	0.437182	0.437182	0.069300	0.069300

Example 2 (Xing, 2006)

Let consider

$$u'(t) + \frac{5}{4} e^{-\frac{t}{4}} u\left(\frac{4t}{5}\right) = 0 \quad (20)$$

subject to the initial condition $u(0) = 1$ and $u(t) = e^{-1.25t}$ is the exact solution of equation (20). Table 2 shows the comparison results of Example 2 that has been simulated by using OECM and Legender and Haar Wavelet methods. The simulated results are obtained over the interval [0, 1] with a step size of 0.125. Based on Table 2, it can be concluded that the solution obtained using OECM are in good agreement with the exact solution. The proposed method has similar performance with Legender method and perform better than Haar Wavelet method.

Table 2: Exact and Numerical Solutions obtained via OECM, Legender Method and Haar Wavelet Method of Example 2

x	**Exact Solution,** $u(t)$	**Legender (Xing, 2006)**	**Haar Wavelets (Xing, 2006)**	**OEP Collocation Method**
0	1	1	1	1
0.125	0.855345	0.855345	0.855345	0.855345
0.25	0.731616	0.731616	0.731612	0.731616
0.375	0.625784	0.625784	0.625778	0.6257840
0.5	0.535261	0.535261	0.535255	0.535261
0.625	0.457833	0.457833	0.457824	0.457833
0.75	0.391606	0.391606	0.391597	0.391606
0.875	0.334958	0.334958	0.334949	0.334958
1	0.286505	0.286505	0.286496	0.286505

Graphical visualization of the absolute error of Example 2 is illustrated in Figure 2. It can be seen that OECM show low values of the absolute error, hence the prediction quality of the numerical solution obtained via OECM is confirmed.

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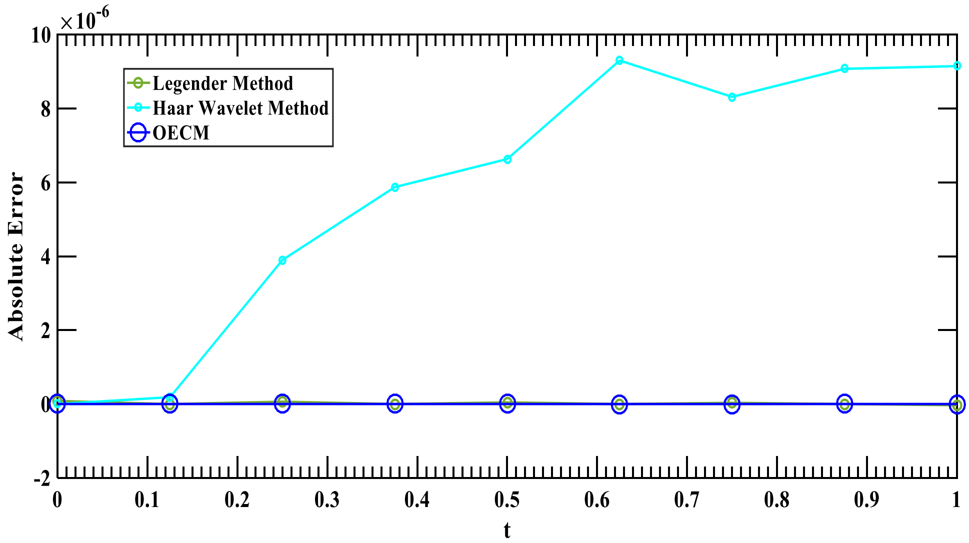

Figure 2: Absolute Error of the Numerical Solution of Example 2 via Legender, Haar Wavelet and OECM.

Example 3 (Xing, 2008)

Consider the following Pantograph differential equation

$$u'(t)+(5/6)u(t)-4u(t/2)-9u(t/3)=t^2-1 \tag{21}$$

with the initial condition $u(0)=1$ and the exact solution is $u(t)=1+\frac{67t}{6}+\frac{1675t^2}{72}+\frac{12857t^3}{1296}$.

Figure 3 shows the graphical visualization of the exact, Homotopy Perturbation Method and OECM. It can be concluded that the approximated result shows good agreement with the exact solution compare than the Homotopy Perturbation Method result.

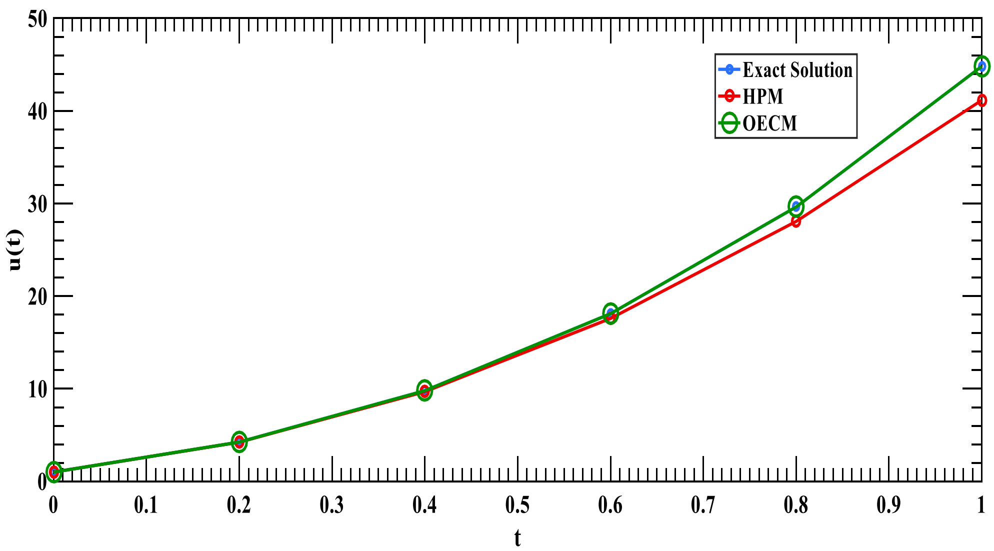

Figure 3: Comparison of HPM and OECM result of Example 3.

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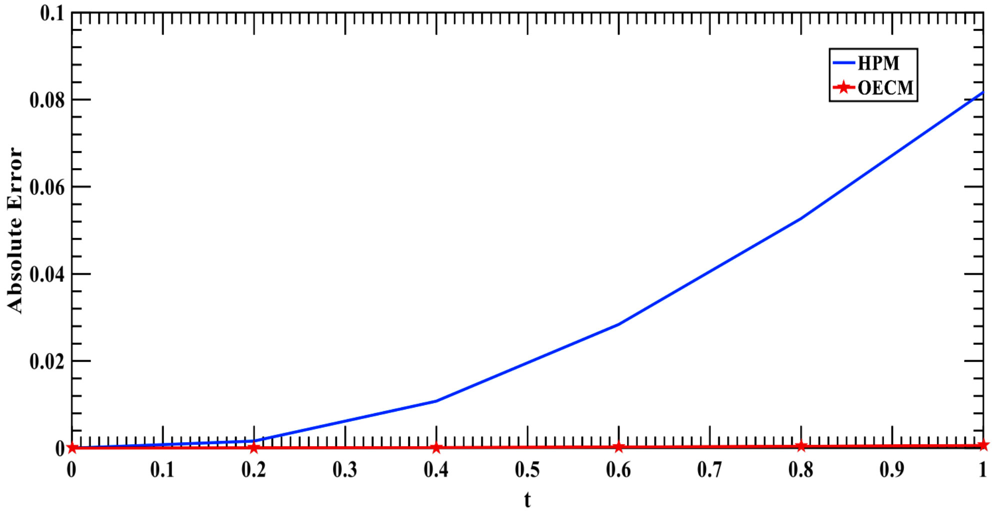

5. Figure 4: Absolute Error of the Numerical Solution of Example 3 via Homotopy Perturbation Method and OECM.

The absolute error of the present result shows low values of error hence indicate the solution obtained via OECM has better performance than the HPM results.

6. Conclusion

A numerical technique of collocation method with ortho exponential polynomial is presented to approximate the numerical solution of Pantograph differential equations. Numerical examples show that the numerical solution obtained via OECM produce good agreement of the results, hence demonstrate efficiency of this method in approximating the solution of Pantograph differential equations.

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References

[1] Bica. A. M. (2011). About a Numerical Method of Successive Interpolations for Two-Point Boundary Value Problems with Deviating Argument. Applied Mathematics and Computation, 217(19), 7772–7789.

[2] Chelyshkov, V. S. (1987). Sequences of Exponential Polynomials Orthogonal on the Semi-axis. Dopovidi Akademii Nauk Ukrainskoi Rsr Seriya A-Fiziko-Matematichni Ta Technichni Nauki, 1, 14–17.

[3] Chelyshkov, V. S. and Liu, C. (2001). Direct Numerical Simulation Of Fully Developed Flow Near A Flat Plate. Technical Report, New Orleans Univesrity, Dept of Mathematics.

[4] Cizek, V. (1960). Methods of Time Domain Synthesis Research Report z-44. Czechoslovak Academy of Sciences, Institute of Radioelektronics, Praha, 1960.

[5] Dmitriyev, A. A. (1973). Orthogonal Exponential Functions in Hydrometeorology. Gidrometeoizdat, Leningrad, https://doi.org/10.1016/j.euromechsol.2013.07.006.

[6] Evans, D. J. and Raslan, K. R. (2005). The Adomian Decomposition Method for Solving Delay Differential Equation. International Journal of Computer Mathematics, 82(1), 49–54.

[7] Feng, X. (2013). An Analytic Study on the Multi-Pantograph Delay Equations with Variable Coefficient. Bulletin Mathematique De La Societe Des Sciences Mathematiques De Roumanie,56, 205-215.

[8] Brunner, H. and Hu, Q. (2007). Optimal Superconvergence Results for Delay Integro-Differential Equations of Pantograph Type. SIAM Journal on Numerical Analysis, 45(3), 986-1004.

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[9] Ishtiaq Ali, Brunner, H. and Tang, T. (2009). A Spectral Method for Pantograph-Type Delay Differential Equations and Its Convergence Analysis. Journal of Computational Mathematics, 27(2-3), 254–265.

[10] Ibrahim, S. S. and Bokhari, M. A. (2011). Optimal Boundary Control of Heat Conduction Problems on an Infinite Time Domain by Control Parameterization. Journal of the Franklin Institute, 348(7), 1656-1667.

[11] Karimi Vanani, S. and Aminataei, A. (2009). On the Numerical Solution of Functional Differential Equations. Journal of Concrete & Applicable Mathematics, 7(1), 54-63.

[12] Mokrik, R. I. and Oliyarnik, I. V. (1989). Use of an Orthoexponential Transformation in the Study of the Non-Stationary Equations of Thermoelasticity and Thermoviscoelasticity. Journal of Applied Mathematics and Mechanics, 53(6), 791–797.

[13] Slobodan Trickovic and Miomir StankoviC. (2003). On the Orthogonality of Classical Orthogonal Polynomials. Integral Transforms and Special Functions, 14(3), 271–280.

[14] Xie, H., Zhang, R. and Brunner, H. (2011). Collocation Methods for General Volterra Functional Integral Equations with Vanishing Delays. SIAM Journal on Scientific Computing, 33(6), 3303–3332.

[15] Xing, T. W. (2006). Numerical Solutions of Time-Varying Generalized Delay Systems via General Legendre Wavelets. Int. J. Innovative Comput. Inf. Control, 2, 1355–1363.

[16] Xing, T. W. (2008). Numerical Solution of Time-Varying Systems with a Stretch by General Legendre Wavelets. Applied Mathematics and Computation, 198(2), 613–620.

[17] Xumei Chen and Linjun Wang (2010). The Variational Iteration Method for Solving a Neutral Functional-Differential Equation with Proportional Delays. Computers & Mathematics with Applications, 59(8), 2696–2702.

[18] Yashiro K., Guan, N. and Ohkawa, S. (2000). Application of Orthogonal Polynomials to Diffraction Problems. Telecommunications and Radio Engineering, 54(3), 57-65.

[19] Yegao Qu, Guoqing Yuan, Shihao Wu and Guang Meng (2013). Three-Dimensional Elasticity Solution for Vibration Analysis of Composite Rectangular Parallelepipeds. European Journal of Mechanics-A/Solids, 42, 376–394.

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