European Journal of Molecular & Clinical Medicine
ISSN 2515-8260 Volume 07, Issue 08, 2020

Numerical Solution Of Higher Order Functional Differential Equation By Collocation Method Via Hermite Polynomials

Muhammad Bilal[1], Norhayati Rosli[2*], Norazaliza Mohd Jamil[3], Iftikhar Ahmad[4]

[1,2,3]Centre for Mathematical Sciences, College of Computing and Applied Sciences, Universiti Malaysia Pahang,
Lebuhraya Tun Razak, 26360 Gambang, Pahang, Malaysia,
[1]muhammad.bilal@math.uol.edu.pk,
[2]norhayati@ump.edu.my, [3]norazaliza@ump.edu.my*
[1]Department of Mathematics and statistics, University of Gujrat, Pakistan, muhammad.bilal@math.uol.edu.pk
[4]Department of Mathematics and statistics, University of Gujrat, Pakistan, Dr.iftikhar@uog.edu.pk

Abstract: This paper is devoted to propose the numerical solution pantograph differential equations via a new computational approach of Hermite collocation method. The convergence of the Hermite collocation method is investigated. The numerical solution of pantograph differential equations is obtained in terms of Hermite polynomial. Nonlinear pantograph differential equations are solved and compared with the exact solutions to show the validity, applicability, acceptability and accuracy of the Hermite collocation method. The approximated results show good agreement with the exact solutions, hence indicate good performance of the methods in solving the corresponding equations.

Key words : Pantograph differential equations, Hermite polynomial, Collocation method.

1. INTRODUCTION

Many phenomena in applied branches that fail to be modelled by the ordinary differential equations can be de- scribed by the functional differential equations. In recent years, many researchers have developed different numerical approaches to the generalized pantograph differential equations such as variational iteration method [1], differential transform approach [2], Taylor method [3], collocation method based via Bernoulli polynomial [4] and Bessel collocation method [5]. Yuzbasi [5] had solved pantograph differential equations via Bessel polynomial. Generalization of Jensen’s inequality and the related result corresponds to the Hermite polynomials was found in [6]. Successive interpolations method applied on pantograph differential equations was studied by [7]. Modified Runge–Kutta method was used to solve nonlinear neutral pantograph equations [8]. Jacobi rational collocation function used to solve pantograph equation [9]. The Direct operational tau method is used to solve pantograph equations [10]. While [11] had applied neural network method to solve pantograph differential equations. Neuro–heuristic computational intelligence approach used for solving nonlinear pantograph systems was investigated in [12].

In present work we exploit a new computational approach which is collocation method by using Hermite polynomial to solve higher order functional type pantograph equations. A generalization of the Hermite polynomial is given by [14]. Properties of Hermite polynomial is available in [15]. Some more works have been done in [17]-[18].

2. PROBLEM DESCRIPTION

Generalize form of higher order pantograph type functional differential equation is

$$y^{m}(t)=\sum_{j=0}^{J}\sum_{k=0}^{m-1}Q_{jk}(t)y^{(k)}(\beta_{jk}t+\eta_{jk})+g(t), \quad (1)$$

$$0\le t\le 1$$

with mixed condition

$$\sum_{k=0}^{m-1}c_{ik}y^{(k)}(t)=\beta_i, \qquad i=0,1,2,...,m-1 \quad (2)$$

where $c_{ik},\beta_i,\beta_{jk}$ are the complex and/or real coefficients, $Q_{jk}(t)$ and $f(t)$ are analytical functions which is defined in $0\le t\le 1$. The approximated result is based on the truncated Hermite expansion

$$U(t)=\sum_{n=0}^{N}c_n He_n(t) \quad (3)$$

where

$$He_n(t)=(-1)^n e^{t^2}\frac{d^n}{dt^n}\left(e^{-t^2}\right).$$

The term c_n corresponds to the Hermite coefficient, $n=0,\,1,\,2,\,....,\,N,\,N\grave{o}IN$. The derivative and their relation between the Hermite polynomial is $He_n^{'}(t)=2nHe_{n-1}(t)$. The relation between the matrices $He^{(K)}(t)$ and $He(t)$ are as follows

$$\begin{aligned}
&[He'(t)]^T = MHe^T(t)\Rightarrow He^{(1)}(t)=He(t)(M^T)\\
&He^2(t)=He'(t)M^T = He(t)(M^T)^2\\
&He^3(t)=He'(t)(M^T)^2=He(t)(M^T)^3 \qquad (4)\\
&\vdots\\
&He^k(t)=H(t)(M^T)^k, \qquad k=0,1,2,....
\end{aligned}$$

The symbol $(')$ is used to denote the derivative w.r.t t

$$\underbrace{\begin{bmatrix} He'_0(t) \\ He'_1(t) \\ He'_2(t) \\ He'_3(t) \\ \vdots \\ He'_{N-1}(t) \\ He'_N(t) \end{bmatrix}}_{[He'(t)]^T} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2.1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2.2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2.3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2(N-1) & 0 & 0 \\ 0 & 0 & & \cdots & 0 & 2.N & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} He_0(t) \\ He_1(t) \\ He_2(t) \\ He_3(t) \\ \vdots \\ He_{N-1}(t) \\ He_N(t) \end{bmatrix}}_{[He(t)]^T}$$

$$C = \begin{bmatrix} c_0(t) \\ c_1(t) \\ c_2(t) \\ c_3(t) \\ \vdots \\ c_{N-1}(t) \\ c_N(t) \end{bmatrix}; M = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & \cdots & 0 & 0 \\ 0 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2\cdot N & 0 \end{bmatrix}$$

$$He(t) = \left[He_0(t) \quad He_1(t) \quad \cdots \quad He_N(t) \right]$$

3. FUNDAMENTAL MATRIX RELATION

Our focus in this study is to approximate the solution of U(t) in (1) described by the truncated Hermite series in equation (3). Let

$$[y(t)] = He(t)C \qquad (5)$$

and the derivative of $y(t)$ is

$$[y(t)]^k = He^k(t)C \qquad (6)$$

By using the above relation, we obtain

$$[y(t)]^k = He(t)(M^T)^k C \qquad (7)$$

4. METHODOLOGY

Here we start to build the basic matrix equation with respect to (1). For this, we replace the matrix relation (5) and the derivatives (4) into (1) and obtain the basic matrix equation of

$$He(t)(M^T)^k C = \sum_{j=0}^{J}\sum_{k=0}^{m-1} Q_{jk}(t) He(\beta_{jk}t)(M^T)^k C + G(t) \qquad (8)$$

The first step in the solution procedure is to define the collocation points t_i as

$$t_i = \frac{i}{N}, \qquad i = 0, 1, 2, ,3 \cdots, N$$

The matrix equation can be written as

$$Het_i(M^T)^k C = \sum_{j=0}^{J} \sum_{k=0}^{m-1} Q_{jk}(t) He(\beta_{jk}t)(M^T)^k C + G(t_i)$$

$i = 0, 1, 2, \cdots, N.$

To simplify the above equation can be written as

$$\{He(t_i)(M^T)^k C - \sum_{j=0}^{J}\sum_{k=0}^{m-1} Q_{jk}(t) He(\beta_{jk} t)(M^T)^k\} C = G \qquad (9)$$

where

$$P_{jk} = \begin{bmatrix} P_{jk}(t_0) & 0 & 0 & \cdots & 0 \\ 0 & P_{jk}(t_1) & 0 & \cdots & 0 \\ 0 & 0 & P_{jk}(t_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_{jk}(t_N) \end{bmatrix}; G = \begin{bmatrix} g(t_1) \\ g(t_2) \\ g(t_3) \\ \vdots \\ g(t_N) \end{bmatrix}$$

Equation (9) corresponding to (1) will be in the form $XC = G$ or $[X; G]$. Thus, we have

$$X = [x_{ij}] = He(t_i)(M^T)^k C - \sum_{j=0}^{J}\sum_{k=0}^{m-1} Q_{jk}(t) He(\beta_{jk} t_i)(M^T)^k C = G \qquad (10)$$

In equation (10), the $(N+1)$ system of linear equations with $c_0, c_1, \cdots, c_N$ are the unknown of Hermite polynomial coefficient. The condition in (2) in the matrix forms by means of the relation (7) can be expressed as

$$\sum_{k=0}^{m-1} c_{ik} He(t)(S^T)^k C = [\beta_i],\ i = 0,1,2,\ ,3\cdots,m-1$$

The matrix form for condition can be written as

$$W_i C = [\beta_i] or\ [W_i; \beta_i],\ i = 0,1,2,\ ,3\cdots,m-1 \ (11)$$

where

$$W_i = \sum_{k=0}^{m-1} c_{ik} He(t)(S^T)^k = [w_{i0} \quad w_{i1} \quad u_{i2} \quad \cdots \quad w_{iN}],$$

$$i = 0,1,2,\ ,3\cdots,m-1$$

By replacing the row in equation (11) and by using the last row of equation (10), we get the new matrix of (12) that allows us to solve the equation (1) under condition (2).

$$[\hat{X}; \hat{G}] = \begin{bmatrix} x_{00} & x_{01} & x_{02} & \cdots & x_{0N} & ; & g(t_0) \\ x_{10} & x_{21} & x_{12} & \cdots & x_{1N} & ; & g(t_1) \\ x_{20} & x_{21} & x_{22} & \cdots & x_{2N} & ; & g(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ x_{(N-m)0} & x_{(N-m)1} & x_{(N-m)2} & \cdots & x_{(N-m)N} & ; & g(t_{(N-m)}) \\ w_{00} & w_{00} & w_{00} & \cdots & w_{0N} & ; & \beta_0 \\ w_{10} & w_{11} & w_{12} & \cdots & w_{0N} & ; & \beta_1 \\ w_{20} & w_{21} & w_{22} & \cdots & w_{0N} & ; & \beta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{(m-1)0} & w_{(m-1)1} & w_{(m-1)2} & \cdots & w_{(m-1)N} & ; & \beta_{m-1} \end{bmatrix} \qquad (12)$$

We always found that the rank ($\hat{X}$)=rank [$\hat{X}$; $\hat{G}$] $= N+1$, then we write

$$C = (\hat{X})^{-1} G \qquad (13)$$

Thus, the Hermite polynomial coefficients $c_i = 0,1,....,N$ are uniquely determined by (12) and (13).

5. CONVERGENCE ANALYSIS

This section is divided into two subsections. The first subsection devotes to the error bound of the proposed method. Meanwhile, in the second subsection, the convergence theory of the numerical solution is presented.

5.1 Error Bound of the Method

In this part, the error bound of the method will be provided under several mild conditions such as solution boundedness of the main neutral differential equation. Some definition and lemma are provided for clarifying the main theorem of this subsection.

5.2 Definition

A function $\zeta:[0,1]\to\Re$ belongs to the sobolev space $W^{m,p}$, if its j^{th} weak derivative, ζ^{j}, lies in $L^p[0,1]$ $\forall 0\le j\le m$ with the norm

$$\|\zeta\| W^{m,p}=\sum_{j=0}^{m}\|\zeta^{(j)}\|_{L^p} \qquad (14)$$

where $\|\zeta\|_{L^p}$ denotes the usual Lebesgue norm,

$$\|\zeta\|_{L^p}=\left(\int_0^1\|\zeta(t)\|^p dt\right)^{\frac{1}{p}} \qquad (15)$$

and $\|\zeta(t)\|$ stands for any finite dimensional norm in $\Re^n$.

5.3 Lemma

Given a function $\zeta\in W^{m,\infty}, t\in[0,1],$ there exist polynomial $u_N(t)$ of degree less than or equal to N such that

$$\|\zeta(t)-u_N(t)\|_{L^\infty}\le CC_0N^{-m},\ \forall t\in[0,1], \quad (16)$$

where C is a constant independent of \$N, m\$ is the order of smoothness of ζ ,and $C_0=\|\zeta\|_{W^{m,\infty}}$. Here, $u_N(t)$ with the smallest norm $\|\zeta(t)-u_N(t)\|_{L^\infty}$ called the N^{th} order polynomial approximation of $\zeta(t)$ in the norm of L^∞. Note that if $\zeta\in C^\infty$, then $m=\infty$. This implies that $u_N(t)$ converges to ζ at a spectral rate, i.e., it is faster than any given polynomial rate. Moreover, we denote the set of continuous functions in a linear space on $[0,T]$ by $C[0,T]$ and the uniform norm in $C[0,T]$ by

$$\|f\|_\infty=max_{0\le t\le t}|f(t)|,\ \forall f\in C[0,T] \quad (17)$$

Again, we consider equation (1) with the initial conditions (2). A similar procedure can be applied for higher values of *m*. Therefore, (1) can be written as follows

$$u'(t)=\sum_{i=0}^{I}p_i(t)(\alpha_i(t)+\beta_i)+f(t) \qquad (18)$$

Integrating both sides in the interval [0, *t*] and imposing initial condition we have

$$u(t)=u(0)+\int_0^t g(\tau d\tau) \\ +\sum_{i=0}^{I}\int_0^t p_i(\tau)u(\alpha_i(t)+\beta_i)d\tau \qquad (19)$$

Equation (19) can be written as

$$u(t)=g(t)+\sum_{i=0}^{I}\int_{0}^{t}p_i(\tau)u(\alpha_i(t)+\beta_i)d\tau \quad (20)$$

where $g(t)=u(0)+\int_0^t g(\tau d\tau)$. In the following theorem, we show that the approximate solution which was expressed in terms of Hermite polynomials converge to the exact solution under several mild conditions.

5.4 Theorem

Consider equation (20) and assume that $u(t)$ and $u_N(t)$ are the exact and approximate solutions of (20), respectively. Consider equation (20) such that $p_i(t)=p_{i,k}(t)$ and assuming

$\forall t\in[0,1], \| u(t)\|_\infty \le \rho, \| K_i(t)\|_\infty$

$=\| p_i\dfrac{(t-\beta_i)}{\alpha_i}\|_\infty \le \Lambda_i,\ i=1,2,\cdots i$

and $\sum_{i=1}^{I}(\Lambda_i+e_{k_i})\neq 1$, then

$\| u(t)-u_N(t)\|_\infty \le \dfrac{e_f+\rho\sum_{i=1}^{I}e_{k_i}}{1-\sum_{i=1}^{I}(\Lambda_i+e_{k_i})}$, where

$e_{k_i}=\| K_i(t)-K_{i,N}(t)\|_\infty, i=1,2,3,\cdots,I$ and

$e_f=\| f(t)-f_N(t)\|_\infty$

5.5 Proof

According to the assumptions above, equation (20) becomes

$$u(t)=g(t)+\sum_{i=0}^{I}\int_{0}^{t}p_i(\tau)u(\alpha_i(t)+\beta_i)d\tau \quad (21)$$

Letting $z=\alpha_i(t)+\beta_i$, we get,

$$u(t)=g(t)+\sum_{i=0}^{I}\int_{\beta_i}^{\alpha_i t+\beta_i}\frac{1}{\alpha_i}p_i\left(\frac{z-\beta_i}{\alpha_i}\right)u(z)dz \quad (22)$$

If we assume $K_i(z)=p_i\left(\dfrac{z-\beta_i}{\alpha_i}\right)$, then equation (22) can be written as

$$u(t)=g(t)+\sum_{i=0}^{I}\int_{\beta_i}^{\alpha_i t+\beta_i}\frac{1}{\alpha_i}K_i(z)u(z)dz \quad (23)$$

Now, suppose that the functions $K_i(z)$ and $g(t)$ are expanded in terms of Hermite polynomials, then the approximated solution $u_N(t)$ is also in terms of Hermite polynomials. Our aim here is to find an upper bound for the associated error between the exact solution $u(t)$ and the approximated solution $u_N(t)$ for equation (1). By the above assumptions, we have

$$\| u(t)-u_N(t)\|_\infty=\| g(t)-g_N(t) + \sum_{i=0}^{I}\int_{\beta_i}^{\alpha_i t+\beta_i}\frac{1}{\alpha_i}(K_i(z)u(z)-K_{i,N}(z)u_N(z))dz\|_\infty \quad (24)$$

By the properties of norm, the equation (24) is then expressed as

$$\| u(t)-u_N(t)\|_\infty = \| g(t)-g_N(t)\|_\infty + \sum_{i=0}^{I}\int_{\beta_i}^{\alpha_i t+\beta_i} |\frac{1}{\alpha_i}| \| K_i(z)u(z) - K_{i,N}(z)u_N(z)\|_\infty \, dz \qquad (25)$$

Write

$$\begin{aligned} &\| K_i(z)u(z)-K_{i,N}(z)u_N(z)\|_\infty \\ &\le \| K_i(z)\|_\infty \| u(z)-u_N(z)\|_\infty \\ &+\| K_i - K_{i,N}(z)\| \\ &(\| u(z)-u_N(z)\|_\infty + \| u(z)\|_\infty) \end{aligned} \qquad (26)$$

Thus, putting (27) into (25) we get

$$\| u(t)-u_N(t)\|_\infty \le e_f + \sum_{i=0}^{I}(\Lambda_i + e_{K_i})\| u(t)-u_N(z)\|_\infty + \rho\sum_{i=0}^{I} e_{K_i} \qquad (27)$$

Hence, $\| u(t)-u_N(t)\|_\infty \le \dfrac{e_f+\rho\sum_{i=0}^{I} e_{K_i}}{1-\sum_{i=0}^{I}(\Lambda_i+e_{K_i})}$. This complete the proof.

6. NUMERICAL RESULT

Now, we take some test examples to examine our method. The numerical computations are carried out in MAT- LAB R2015a, with 10 GB RAM, processor Intel core i7 and hard drive of 1000 GB.

6.1 Example 1

Let us consider the following 2^{nd} order pantograph differential equation

$$\frac{d^2y}{dt^2}(t) = \left(\frac{2}{3}\right)y(t) + \left(\frac{1}{3}\right)e^{\frac{t}{2}}y\left(\frac{t}{2}\right)$$

with the initial condition $y(0)=1,\quad y\left(\frac{1}{2}\right)=\sqrt{e}.$

The first step in the solution procedure is to define the collocation points which are

$$\left\{t_1=0, t_2=\frac{1}{3}, t_3=\frac{2}{3}, t_4=1\right\}$$

Matrix form of the 2^{nd} order pantograph differential equation is

$$\left\{He(t)M^2 + Q_{00}He(\frac{t}{2}) + Q_{10}He(t)\right\}C = G$$

where $N=3,\ Q_{00}=\frac{-1}{3}e^{\frac{t}{2}}$ and $Q_{10}=\frac{-2}{3}$. The approximated solution will be obtained from

$$y(t)=\sum_{n=0}^{N} a_n He_n(t)$$

By applying the methodology in Section 4, yield

$$Q_{10} = \begin{bmatrix} -\frac{2}{3} & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -\frac{2}{3} \end{bmatrix},$$

$$Q_{00} = \begin{bmatrix} -0.3333 & 0 & 0 & 0 \\ 0 & -0.39379 & 0 & 0 \\ 0 & 0 & -0.4652 & 0 \\ 0 & 0 & 0 & -0.54957 \end{bmatrix},$$

$$He(t) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0.66667 & -1.55556 & -3.7037 \\ 1 & 1.33333 & -0.22222 & -5.62963 \\ 1 & 2 & 2 & -4 \end{bmatrix},$$

$$He\left(\tfrac{1}{2}\right) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0.333333 & -1.88889 & -1.96296 \\ 1 & 0.666667 & -1.55556 & -3.7037 \\ 1 & 1 & -1 & -5 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix for this fundamental matrix equations

$$[X;G] = \begin{bmatrix} -1 & 0 & 10 & 0 & ; & 0 \\ -1.06045 & -0.57571 & 9.780857 & 19.24212 & ; & 0 \\ -1.13187 & -1.19902 & 8.871799 & 37.47606 & ; & 0 \\ -1.21624 & -1.88291 & 7.21624 & 53.41454 & ; & 0 \end{bmatrix}$$

After solving linearly, the above matrix, we will get C as follows

$$C = \begin{bmatrix} \frac{5}{4} & \frac{1215}{1877} & \frac{1}{8} & \frac{116}{4693} \end{bmatrix}^T.$$

Table 1 present the results produced for Example 1 by Hermite collocation method in the interval [0, 1] with step size of 0.2. Also Table 1 presents the results for three values of N = 3, 6, 19, respectively, to show the improvement of the results by the increasing size of N. The exact solution was compared along with the absolute error for Example 1 for N = 3, 6, 19. By increasing the size of N, the algorithm gives the improvement in the results. Table 2 discusses the error analysis between the result obtained in [18] and the present method in this paper.

Table 1: Results of Example 1 for different values of *N*.

i	*t*	**Exact Method**	**Present Method**					
			N_3		N_6		N_{19}	
		$y(t_i)=\sqrt{t_i}$	$y(t_i)$	$E(t_i)$	$y(t_i)$	$E(t_i)$	$y(t_i)$	$E(t_i)$
1	0	1	1	0	1	0	1	0
2	0.2	1.221402758	1.221183358	0.00021940	1.221402766	7.9330E-09	1.221402758	1.0E-11
3	0.4	1.490824698	1.491858332	3.363390E-0.5	1.491824707	9.37300E-09	1.491824698	2.70E-11
4	0.6	1.822118800	1.821516447	0.0006023530	1.822118794	6.3905700E-09	1.8221188	3.90570E-11
5	0.8	2.225540928	2.21969303	0.0058478990	2.2255408	1.2849200E-07	2.225540928	4.600760E-13
6	1	2.718281828	2.695748479	0.022533350	2.71827379	8.038460E-06	2.718281828	4.041210E-14

Table 2: Comparison between the approximated result by collocation method via Hermite polynomial which is N = 19 and the reference [18]

i	*t*	N_{19}	E_{19}	Result-[18]	Error-[18]
1	0	1	0	1	0
2	0.2	1.221402758	1.00E-11	1.221403	7.738440E-008
3	0.4	1.491824698	2.70E-11	1.491825	5.521239E-008

6.2 Example 2

Let us consider the following 3^{rd} order pantograph differential equation

$$\frac{d^3y}{dt^3}-t\frac{d^2y}{dt^2}\left(\frac{t}{3}-1\right)+t\frac{dy}{dt}\left(\frac{t}{4}+1\right)$$
$$+y(t)-t\left(e^{\left(\frac{t}{3}+1\right)}+e^{\left(\frac{t}{4}-1\right)}\right)=0$$

with the initial conditions $y(0)=1,\frac{dy}{dt}(0)=-1,\frac{d^2y}{dt^2}(0)=1$. We will find the solution of Example 2 by

$$y(t)=\sum_{n=0}^{6}a_nH_n(t)$$

where $N=6$, and $Q_{00}=-t,\ P_{10}=t,\ g(t)=-t\left(e^{\left(\frac{t}{3}+1\right)}+e^{\left(\frac{t}{4}-1\right)}\right)$. The first step in the solution procedure is to define the collocation points which are

$$\left\{t_1=0,t_2=\frac{1}{6},t_3=\frac{1}{3},t_4=\frac{1}{2},t_5=\frac{2}{3},t_6=\frac{5}{6},t_7=1\right\}.$$

The fundamental equation for the previous equation is

$$\left\{\begin{array}{l}He(t)M^3+Q_{00}He(\frac{t}{3}-1)M^2\\+Q_{10}He(\frac{t}{4}+1)M+He(t)\end{array}\right\}C=G$$

where

$$Q_{00}=\begin{bmatrix}0&0&0&0&0&0&0\\0&-\frac{1}{6}&0&0&0&0&0\\0&0&-\frac{1}{3}&0&0&0&0\\0&0&0&-\frac{1}{2}&0&0&0\\0&0&0&0&-\frac{2}{3}&0&0\\0&0&0&0&0&-\frac{5}{6}&0\\0&0&0&0&0&0&-1\end{bmatrix};Q_{10}=\begin{bmatrix}0&0&0&0&0&0&0\\0&\frac{1}{6}&0&0&0&0&0\\0&0&\frac{1}{3}&0&0&0&0\\0&0&0&\frac{1}{2}&0&0&0\\0&0&0&0&\frac{2}{3}&0&0\\0&0&0&0&0&\frac{5}{6}&0\\0&0&0&0&0&0&1\end{bmatrix}$$

$$He(t)=\begin{bmatrix}1&0&-2&0&12&0&-120\\1&0.333333&-1.88889&-1.96296&10.67901&19.26337&-100.369\\1&0.666667&-1.55556&-3.7037&6.864198&34.20576&-45.8381\\1&1&-1&-5&1&41&31\\1&1.333333&-0.222222&-5.62963&-6.17284&36.80658&110.8038\\1&1.666667&0.777778&-5.37037&-13.6173&20.26749&169.952\\1&2&2&-4&-20&-8&184\end{bmatrix}$$

$$He\left(\frac{1}{4}+1\right)=\begin{bmatrix}1&2&2&-4&-20&-8&184\\1&2.083333&2.340278&-3.45775&-21.2453&-19.5991&177.8719\\1&2.166667&2.694444&-2.8287&-22.2955&-25.6773&167.321\\1&2.25&3.0625&-2.10938&-23.1211&-35.1475&152.1292\\1&2.333333&3.444444&-1.2963&-23.6914&-44.9095&132.1248\\1&2.416667&3.840278&-0.386&-23.9745&-54.8504&107.1898\\1&2.5&4.25&0.625&-23.9375&-64.8438&77.26563\end{bmatrix}$$

$$He\left(\frac{1}{3}-1\right)=\begin{bmatrix}1&-2&2&4&-20&8&184\\1&-1.88889&1.567901&4.593964&-18.0849&-2.59136&185.7437\\1&-1.77778&1.160494&5.048011&-15.9372&-12.0513&180.7965\\1&-1.66667&0.777778&5.37037&-13.6173&-20.2675&169.952\\1&-1.55556&0.419753&5.569273&-11.1818&-27.1602&154.0676\\1&-1.44444&0.08642&5.652949&-8.68389&-32.6802&134.0436\\1&-1.33333&-0.22222&5.62963&-6.17284&-36.8066&110.8038\end{bmatrix}$$

$$M=\begin{bmatrix}0&2&0&0&0&0&0\\0&0&4&0&0&0&0\\0&0&0&6&0&0&0\\0&0&0&0&8&0&0\\0&0&0&0&0&10&0\\0&0&0&0&0&0&12\\0&0&0&0&0&0&0\end{bmatrix}$$

The augmented matrix for this fundamental matrix equations is

$$[X;G]=\begin{bmatrix}1&0&-2&48&12&-960&-120&;&0\\1&0.666667&-1.83333&55.93287&57.5246&-984.065&1656.31&;&-0.54285\\1&1.333333&-1.33333&63.90741&108.7531&-921.393&-3066.61&;&-1.14586\\1&2&-0.5&72.1875&165.8958&-796.42&-4162.85&;&-1.81407\\1&2.666667&0.666667&81.03704&229.4815&-524.83&-4758.37&;&-2.55288\\1&3.333333&2.166667&90.71991&300.3526&-183.05&-4665.72&;&-3.36812\\1&4&4&101.5&379.6667&262.2546&-3693.38&;&-4.26603\end{bmatrix}$$

Fundamental matrix for Example 2 is

$W_jC=[\beta_j]$ *or* $[W_j;\beta_j];\ j=0,1,2,3$

or clearly

$$[W_0;\beta_0]=\begin{bmatrix}1&0&-2&0&12&0&-120\end{bmatrix}$$

$$[W_1;\beta_1]=\begin{bmatrix}0&2&0&-12&0&120&0\end{bmatrix}$$

$$[W_2;\beta_2]=\begin{bmatrix}0&0&8&0&-96&0&1440\end{bmatrix}$$

The new augmented matrix based on conditions can be obtained as

$$[\hat{X};\hat{G}] = \left[\begin{array}{ccccccccc} 1 & 0 & -2 & 48 & 12 & -960 & -120 & ; & 0 \\ 1 & 0.666667 & -1.83333 & 55.93287 & 57.52546 & -984.065 & -1656.31 & ; & -0.542850452 \\ 1 & 1.333333 & -1.33333 & 63.90741 & 108.7531 & -921.393 & -3066.61 & ; & -1.145860477 \\ 1 & 2 & -0.5 & 72.1875 & 165.8958 & -769.42 & -4162.85 & ; & -1.814066281 \\ 1 & 0 & -2 & 0 & 12 & 0 & -120 & ; & 1 \\ 0 & 2 & 0 & -12 & 0 & 120 & 0 & ; & -1 \\ 0 & 0 & 8 & 0 & -96 & 0 & 1440 & ; & 1 \end{array}\right]$$

After Solving $[\hat{X};\hat{G}]$ we will get the matrix C.

Table 3: Approximated Results and the absolute errors (AE) of present method via Hermite Polynomials for different *N* values of Example 2.

i	t	**Exact**	**Present Method**					
		$y(t_i)=e^{t_i}$	N_7		N_{11}		N_{48}	
			$y(t_i)$	$E(t_i)$	$y(t_i)$	$E(t_i)$	$y(t_i)$	$E(t_i)$
1	0	1	1	0	1	0	0.999999	1.00E-06
2	0.2	0.818730	0.819467	0.0006735	0.818917	0.000187	0.819395	0.000665
3	0.4	0.670320	0.674134	0.0038148	0.670319	1.50E-06	0.674042	0.003722
4	0.6	0.548811	0.556411	0.0076319	0.559976	0.011159	0.555972	0.007161
5	0.8	0.449328	0.454835	0.0055066	0.476254	0.026926	0.453427	0.004099
6	1	0.367879	0.353658	0.0142213	0.367688	0.000199	0.350466	0.017412

Table 3 illustrates the approximated result produced by Hermite collocation method for Example 2, which is a third order pantograph differential equation. Exact result of Example 2 is $u(t)=e^{t}$, and the absolute error are obtained using this exact solution on $N=7,11,48$ for interval [0,1] for 0.2 step size. The results of Hermite collocation method have sufficient accuracy along with the whole interval.

5. CONCLUSION

In this paper, we have presented the collocation method with Hermite polynomial for solving the higher order pantograph equations. It can be seen from the numerical examples, the approximated results have good agreement with the exact solutions as indicated by low values of the absolute error. The comparison between the present method and the reported methods show that collocation method with Hermite polynomial is efficient in solving pantograph types differential equations. A considerable advantage of using collocation method with Hermite polynomial is that the approximate solutions can be found at low computational cost by using computer program.

6. ACKNOWLEDGEMENT

We would like to thank the Research and Innovation Department, University Malaysia Pahang (UMP) for their financial supports through UMP research grant RDU180301 and FRGS grant FRGS/1/2019/STG06/UMP/02/2.

7. REFERENCES

[1] Saadatmandi, A. and Dehghan, **Variational iteration method for solving a generalized pantograph equation. Computers and Mathematics with Applications**, 2009, pp.2190-2196.

[2] Keskin, Y., Kurnaz, A., Kiris., E. and Oturanc, G. **Approximate solutions of generalized pantograph equations by the differential transform method. International Journal of Nonlinear Sciences and Numerical Simulation**, 2007, pp.159-164.

[3] Sezer, M. and Akyz-Dascioglu, A., **A Taylor method for numerical solution of generalized pantograph equations with linear functional argument**, Journal of Computational and Applied Mathematics, 2007, pp.217-225.

[4] Tohidi, E., Bhrawy, A.H. and Erfani, K., **A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Applied Mathematical Modelling**, 2013, pp.4283-4294.

[5] Yzbasi, S., Sahin, N. and Sezer, M., **A Bessel collocation method for numerical solution of generalized pantograph equations. Numerical Methods for Partial Differential Equations**, 2012, pp.1105-1123.

[6] Aras-Gazic, G., Culjak, V., Pecaric, J. and Vukelic, A., **Generalization of JensenS Inequality by Hermite Polynomials and Related Results. Mathematical reports**, 2015, pp.201-223.

[7] Bica, A.M., Curila, M. and Curila, S., **About a numerical method of successive interpolations for two-point boundary value problems with deviating argument. Applied Mathematics and Computation**, 2011, pp.7772-7789.

[8] Ma, S.F., Yang, Z.W. and Liu, M.Z., **Ha-stability of modified RungeKutta methods for nonlinear neutral pantograph equations. Journal of mathematical analysis and applications**, 2007, pp.1128-1142.

[9] Doha, E.H., Bhrawy, A.H., Baleanu, D. and Hafez, R.M., **A new Jacobi rationalGauss collocation method for numerical solution of generalized pantograph equations. Applied Numerical Mathematics**, 2014**,** pp.43-54.

[10] Trif, D., **Direct operatorial tau method for pantograph-type equations. Applied Mathematics and Computation**, 2012, pp.2194-2203.

[11] Ahmad, I. and Mukhtar, A., **Stochastic approach for the solution of multi-pantograph differential equation arising in cell-growth model. Applied Mathematics and Computation**, 2015, pp.360-372.

[12] Raja, M.A.Z., Ahmad, I., Khan, I., Syam, M.I. and Wazwaz, A.M., **Neuro-heuristic computational intelligence for solving nonlinear pantograph systems. Frontiers of Information Technology Electronic Engineering**, 2017, pp.464-484.

[13] Skorokhod, A.V. and Stepakhno, V.I., **Generalization of Hermite polynomials. Ukrainian Mathematical Journal**, 1990, pp.1366-1370.

[14] Demenin, A.N., **Properties of functional Hermitian polynomials**, *Ukrainian Mathematical Journal*, 1969, pp.320-324.

[15] Terry, R.D., **An application of Lyapunov's direct method to the study of oscillations of a delay differential equation of even order,** *Journal of the Australian Mathematical Society (Series A),* 1978, pp.201-209.

[16] Terry, R.D., **Iterative criteria for bounds on the growth of positive solutions of a delay differential equation**, *Journal of the Australian Mathematical Society (Series A),* 1978, pp.195-200.

[17] Terry, R.D., **Oscillatory and asymptotic properties of homogeneous and nonhomogeneous delay differential equations of even order**. *Journal of the Australian Mathematical Society (Series A)*, 1976, pp.282-304.

[18] Bica, A.M., Curila, M. and Curila, S., **About a numerical method of successive interpolations for two-point boundary value problems with deviating argument. Applied Mathematics and Computation**, 2011, pp.7772-7789.