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Two Point Diagonally Block Method for Solving Boundary Value Problems with Robin Boundary Conditions

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ABSTRACT

This numerical study emphasizes on fifth order multistep block method for solving second order boundary value problems(BVPs) imposing Robin boundary conditions.The shooting technique will be utilized to compute the approximate solutions at two point simultaneously. The implementation of predictor-corrector scheme follows the $PE(CE)^r$ mode. Numerical results are presented to give a clear view of the performances for the proposed method. The order and stability of the method are also discussed.

Keywords: Block method, predictor-corrector, Robin boundary conditions, shooting technique.

1. Introduction

A wide range of science and engineering applications when it comes to produce either analytical or numerical solutions can be represented as boundary value problems (BVPs) subject to Dirichlet, Neumann or Robin boundary conditions. The present study concern on solving two point BVPs as follows

$$y''(x) = f(x, y, y') \quad \text{for} \quad a \leq x \leq b \tag{1}$$

with Robin boundary conditions

$$c_1 y'(a) + c_2 y(a) = \alpha \quad \text{and} \quad c_3 y'(b) + c_4 y(b) = \beta \tag{2}$$

where $a, b, c_1, c_2, c_3, c_4, \alpha$ and β are all constants and c_1, c_2, c_3 and c_4 all nonzero. Such condition play an essential role in the study of diffusion equation occur in biology and chemistry field [Lawley and Keener (2015)]. Numerous techniques were revealed for solving two point BVPs with Robin boundary conditions including bernoulli polynomials [Islam and Shirin (2011)], Adomian decomposition method [Duan et al. (2013)], cubic Hermite collocation method [Ganaie et al. (2014)] and Gegenbaur integration matrices [Elgindy and Smith-Miles (2013)]. Meanwhile, our interest is to explore on the direct method for solving higher order differential equations that has been discussed in detail by Majid (2004).

The approach will be adapted with the combination of shooting technique and Newton divided difference interpolation method as the iterative formula for predicting the new initial guessing.

2. Formulation of the Method

In this study, the interval $x \in [a, b]$ is divided into series of blocks with each block provide two numerical solutions as depicted in Figure 1. Two approximates values, y_{n+1} at x_{n+1} and y_{n+2} at x_{n+2} will be computed simultaneously until end of the interval. The formulas for y_{n+1} and y_{n+2} are derived by integrating (1) as follows

$$\int_{x_n}^{x_{n+v}} y''(x)\,dx = \int_{x_n}^{x_{n+v}} f(x, y, y')\,dx$$
$$\int_{x_n}^{x_{n+v}} \int_{x_n}^{x} y''(x)\,dxdx = \int_{x_n}^{x_{n+v}} \int_{x_n}^{x} f(x, y, y')\,dxdx \tag{3}$$

where the point, $v = 1, 2$. The derivation proceeds by approximating the function $f(x, y, y')$ with the Lagrange interpolating polynomial, P_m. Define $P_m(x)$

as follows

$$
\begin{aligned}
P_m(x) &= L_{m,0}(x)f_{n+v} + L_{m,1}(x)f_{n+v-1} + L_{m,2}(x)f_{n+v-2} \\
&\quad + \ldots + L_{m,m}(x)f_{n+v-m} \\
&= \sum_{j=0}^{m} L_{m,j}(x)f_{n+v-j}, \text{ for } v = 1, 2
\end{aligned}
$$

where

$$
L_{m,j}(x) = \prod_{\substack{i=0 \\ i\neq j}}^{m} \frac{(x - x_{n+v-i})}{(x_{n+v-j} - x_{n+v-i})}.
$$

Let $m = 4$ and $m = 5$ for the first point ($v = 1$) and second point ($v = 2$) respectively. Hence, introduce the variable substitution $x = x_{n+v} + sh$ for $v = 1, 2$ and $dx = hds$. Finally, by changing the limit of integration and evaluate these integrals using MAPLE will yields the corrector formulas for first and second point as follows

$$
\begin{aligned}
y'_{n+1} &= y'_n + \frac{h}{720}[-19f_{n-3} + 106f_{n-2} - 264f_{n-1} + 646f_n + 251f_{n+1}] \\
y_{n+1} &= y_n + hy'_n + \frac{h^2}{1440}[-17f_{n-3} + 96f_{n-2} - 246f_{n-1} + 752f_n + 135f_{n+1}]
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
y'_{n+2} &= y'_n + \frac{h}{90}[28f_{n+2} + 129f_{n+1} + 14f_n + 14f_{n-1} - 6f_{n-2} + f_{n-3}] \\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{630}[37f_{n+2} + 718f_{n+1} + 566f_n - 76f_{n-1} + 17f_{n-2} - 2f_{n-3}].
\end{aligned}
\tag{5}
$$

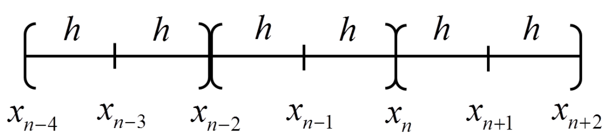

Figure 1: Two - point block method

The derivation of the predictor formula follows the same procedure but the number of interpolation points is one less than the corrector formula. This is because we want to reduce the computation of the function evaluation. The systematic procedure of the shooting technique and Newton divided difference interpolation approach will be employed to give the new initial guess. The proposed multistep methods is not a self-starting method. Therefore, it is necessary to use one step method at the beginning of the calculation in order to compute the starting values. Then, the entire calculation will be solved directly using the proposed predictor-corrector formula until the end of the interval.

2.1 Order of the method

The proposed multistep formulas can be specified as a member of the Linear Multistep Method (LMM). In general form, (4) and (5) can be represented as

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j y'_{n+j} + h^2\sum_{j=0}^{k}\gamma_j y''_{n+j} \tag{6}$$

or in the form of linear difference operator as

$$L[y(x),h] = \sum_{j=0}^{k}\left[\alpha_j y(x+jh) - h\beta_j y'(x+jh) - h^2\gamma_j y''(x+jh)\right]. \tag{7}$$

Assuming that $y(x)$ is sufficiently differentiable, so that expanding the terms in (7) using Taylor's series about the point x will give the following simplified expression as

$$L[y(x),h] = C_0 y(x) + C_1 h y'(x) + \ldots + C_p h^p y^{(p)}(x) + \ldots$$

where

$$C_0 = \sum_{j=0}^{k}\alpha_j = \alpha_0 + \alpha_1 + \ldots + \alpha_k \tag{8}$$

$$C_1 = \sum_{j=0}^{k}(j\alpha_j - \beta_j) = (\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k) - (\beta_0 + \beta_1 + \ldots + \beta_k)$$

$$\vdots$$

$$C_p = \sum_{j=0}^{k}\left(\frac{j^p}{p!}\alpha_j - \frac{j^{p-1}}{(p-1)!}\beta_j - \frac{j^{p-2}}{(p-2)!}\gamma_j\right),\ p = 2, 3, \ldots.$$

According to Fatunla (1995) and Lambert (1973), the proposed method satisfies order p and error constant C_{p+2} if

$$C_0 = C_1 = C_2 = \ldots = C_{p+1} = 0, \quad \text{and} \quad C_{p+2} \neq 0.$$

This concept was used to determine the order p and error constant of the proposed method known as 2PDD5. Now, transform the corrector formulae in matrix difference form and by choosing $k = 5$, the calculation from (8) gives

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = [0,0,0,0]^T$$

and

$$\begin{aligned} C_7 &= \sum_{j=0}^{5} \left(\frac{j^7}{7!}\alpha_j - \frac{j^6}{6!}\beta_j - \frac{j^5}{5!}\gamma_j\right) \\ &= \begin{bmatrix} -\frac{3}{160} \\ -\frac{41}{5040} \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, 2PDD5 has order $p = 5$ and the error constant is

$$C_{p+2} = C_7 = \left[-\frac{3}{160}, -\frac{41}{5040}, 0, 0\right]^T.$$

2.2 Consistency of the method

Definition 1: The linear multistep method is said to be consistent if it possesses an order $p \geq 1$ [Lambert (1973)].

Since the order of the proposed method is $p = 5 \geq 1$, therefore the method is consistent.

2.3 Stability analysis

Definition 2: According to Lambert (1973), a linear multistep method is zero-stable provided that the root $\xi_j, j = 0(1)k$ of the first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = det\left|\sum_{j=0}^{k} A^{(j)}\xi^{(k-j)}\right| = 0$ satisfies $|\xi_j| \leq 1$ and for those roots with $|\xi_j| = 1$, the multiplicity must not exceed two. Rewrite (4) and (5) into matrix form as follows

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{n+1} \\ y_{n+1} \\ y'_{n+2} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y'_{n-1} \\ y_{n-1} \\ y'_{n} \\ y_{n} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} y'_{n-1} \\ y_{n-1} \\ y'_{n} \\ y_{n} \end{bmatrix} \tag{9}$$

$$+ h \begin{bmatrix} -\frac{19}{720} & \frac{106}{720} & -\frac{264}{720} & \frac{646}{720} \\ 0 & 0 & 0 & 0 \\ \frac{1}{90} & -\frac{6}{90} & \frac{14}{90} & \frac{14}{90} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{bmatrix} + h \begin{bmatrix} \frac{251}{720} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{129}{90} & \frac{28}{90} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}$$

$$+ h^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{17}{1440} & \frac{96}{1440} & -\frac{246}{1440} & \frac{752}{1440} \\ 0 & 0 & 0 & 0 \\ -\frac{2}{630} & \frac{17}{630} & -\frac{76}{630} & \frac{566}{630} \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{135}{1440} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{718}{630} & \frac{37}{630} & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}.$$

From (9), by taking $p(\xi) = det[\xi A^{(0)} - A^{(1)}] = 0$, the first characteristic polynomial can be written as

$$p(\xi) = det \begin{bmatrix} \xi & 0 & -1 & 0 \\ 0 & \xi & 0 & -1 \\ 0 & 0 & \xi - 1 & 0 \\ 0 & 0 & 0 & \xi - 1 \end{bmatrix} \tag{10}$$
$$0 = \xi^2(\xi - 1)^2,\ \xi = 0, 0, 1, 1.$$

According to Definition 2 and all the roots obtained in (10), the diagonally two point block method is concluded as zero stable. The test equation applied to obtain the stability polynomial of the two point block is as follows

$$y'' = f = \theta y' + \lambda y. \tag{11}$$

The stability polynomial for 2PDD5 by taking

$$det[t^2 A^{(0)} - t(A^{(1)} + hB^{(1)} + h^2 C^{(1)}) - (hB^{(2)} + h^2 C^{(2)})] = 0$$

are as follows

$$
\begin{aligned}
&t^8\left(1+\frac{22517}{453600}H1H2-\frac{1537}{10080}H2-\frac{95}{144}H1+\frac{37}{6720}H2^2+\frac{1757}{16200}H1^2\right)+ \\
&t^7\left(-2-\frac{8743}{6480}H1H2-\frac{1853}{504}H2-\frac{43}{180}H1-\frac{223333}{302400}H2^2-\frac{11717}{8100}H1^2\right)+ \\
&t^6\left(1-\frac{967}{5040}H2+\frac{89}{180}H1-\frac{9179}{151200}H2^2-\frac{392729}{226800}H1H2+\frac{14627}{16200}H1^2\right)+ \\
&t^5\left(\frac{59}{2520}H2+\frac{11}{30}H1-\frac{1111}{151200}H2^2-\frac{3599}{11340}H1H2+\frac{341}{810}H1^2\right)+ \\
&t^4\left(-\frac{5}{2016}H2+\frac{3}{80}H1-\frac{121}{10080}H2^2-\frac{4379}{453600}H1H2+\frac{29}{2025}H1^2\right)+ \\
&t^3\left(\frac{29}{100800}H2^2+\frac{101}{226800}H1H2-\frac{1}{8100}H1^2\right)=0 \quad \text{where} \quad H1=h\theta,\ H2=h^2\lambda.
\end{aligned} \tag{12}
$$

The boundary of the absolute stability region in $H1-H2$ plane is determined by substituting t in the stability polynomial with $1,-1$ and $e^{i\theta}$ for $0\leq\theta\leq 2\pi$. This is done by using MAPLE. The shaded region in Figure 2 illustrate the region of the absolute stability for the two point block method that lies inside the boundary and it is obtained by tracing the values of $H1$ and $H2$.

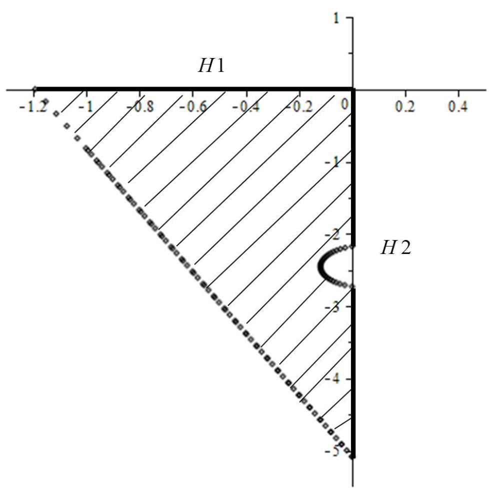

Figure 2: Stability region of 2PDD5

2.4 Convergence of the method

The linear multistep method is convergent if and only if it is consistent and zero stable [Lambert (1973)]. Since the consistency and zero stable of the method have been achieved, as a conclusion the proposed two point block method is convergent.

3. Implementation of the Method

In this study, the Newton divided difference interpolation formula has been used to replace the common root seeking procedure applied in the iterative part of the shooting method. This approach can overcome the difficulties occur during the calculation of derivative value involved in the denominator part when using Newton's Raphson method as the strategy for estimating the new guessing values while performing nonlinear shooting method.
The algorithm works is as follows.

1. Set TOL and $y_0(a) = s_0, y_0'(a) = V_1 - C_1 y_0(a)$ where $V_1 = \frac{\alpha}{c_1}$ and $C_1 = \frac{c_2}{c_1}$.
2. Set $x_n = x_0 + nh$ and calculate the approximate values, y_{n+1} and y_{n+2} using (4) and (5) with $PE(CE)^r$ mode.
3. Correct the corrector values of y', y and f in Step 2 and iterate until $\left|\left(y^c_{n+1,r}\right)_t - \left(y^c_{n+1,r-1}\right)_t\right| < 0.1 \times TOL$ and $r = 1, 2, \ldots$.
4. If $x_n < b$, then repeat Step 2. If $x_n = b$, then go to Step 5.
5. If fulfill the stop condition $\left|h(y_j(b), y_j'(b)) - \beta\right| \le TOL$, go to Step 6, else choose the second guessing values, $y_1(a) = s_1$ and $y_1'(a) = V_1 - C_1 y_1(a)$. Repeat Step 2-4. For third guessing onwards, update the new estimate $y_j(a) = s_j$ and $y_j'(a) = V_1 - C_1 y_j(a)$ for $j = 2, 3, \ldots N$ using Newton divided difference interpolation formula.
6. Exit the program and execute the result.

4. Numerical Results

In this section, we have applied the algorithm of 2PDD5 to four tested problems to illustrate its accuracy and efficiency. All the tested problems used absolute error test at tolerance, $TOL = 10^{-6}$ throughout the calculation for obtaining the required result. The following notations are used in the following result.

MAXE	: Maximum absolute error
AVE	: Average absolute error
h	: step size
TS	: Total step at last iteration
FCN	: Total function call
ITN	: Total iteration of guess
2PDD5	: Direct two point diagonally block method of order five proposed in this study
2PDAM5	: Direct two step Adams Moulton block method of order five as in Phang et al. (2011)
DAM5	: Direct Adams Moulton method of order five as in Majid et al. (2011)
BP	: Bernoulli polynomials as in Islam and Shirin (2011)
n	: Order of Bernoulli polynomials

Problem 1. Given linear second order differential equation

$$y''(x) = y(x) - 4xe^x, \quad y'(0) - y(0) = 1 \quad \text{and} \quad y'(1) + y(1) = -e^1.$$

Exact solution : $y(x) = x(1-x)e^x$.
Source: Usmani (1972).

Problem 2. Given nonlinear second order differential equation

$$y''(x) = \frac{1}{2}\left(e^{2y} + (y')^2\right), \quad y'(0) - y(0) = -1 \quad \text{and} \quad y'(1) + y(1) = -log(2) - \frac{1}{2}.$$

Exact solution : $y(x) = log\left(\frac{1}{1+x}\right)$.
Source: Chawla (1978).

Problem 3. Given nonlinear second order differential equation

$$y''(x) = \frac{1}{2}(1 + x + y(x))^3 \quad y'(0) - y(0) = -\frac{1}{2} \quad \text{and} y'(1) + y(1) = 1.$$

Exact solution : $y(x) = \frac{2}{2-x} - x - 1$.
Source: Islam and Shirin (2011).

Problem 4. (*Nonlinear physical applications*)
A nonlinear BVPs with diffusion applications as discussed in Agarwal and

O'Regan (2008)

$$y''(x) = be^{ay(x)}, \quad \text{with} \quad y(0) = y(1) = 0. \tag{13}$$

Table 1: Description on diffusion applications in (13)

	Diffusion of heat generated by positive temperature-dependent sources	Frictional heating
$a = 1$	analysis of Joule losses in electrically conducting solids	analysis of Joule losses in electrically conducting solids
b	the square of the constant current	square of the constant shear stress
$e^{y(x)}$	temperature dependent resistance	temperature dependent fluidity

Given nonlinear second order differential equation as in (13):

$$y''(x) = \pi^2 e^{y(x)} \quad 2y'(0) + y(0) = -2\pi \quad \text{and} - y'(1) + 2y(1) = -\pi.$$

Exact solution : $y(x) = -2ln\left(cos(\frac{\pi}{2}x - \frac{\pi}{4})\right) - ln2.$
Source : Lang and Xu (2012).

Table 2: Comparison of the numerical result for solving Problem 1

h	Method	TS	MAXE	AVE	FCN	ITN
0.10	DAM5	10	1.1639E-03	7.7139E-04	130	3
	2PDAM5	7	1.5723E-06	1.3044E-06	76	1
	2PDD5	7	1.5723E-06	1.3044E-06	40	1
0.05	DAM5	20	7.3265E-04	4.4426E-04	174	3
	2PDAM5	12	4.3574E-08	3.7751E-08	132	1
	2PDD5	12	4.3574E-08	3.7751E-08	46	1
0.01	DAM5	100	1.7402E-04	9.7868E-05	649	3
	2PDAM5	52	1.2764E-11	1.1267E-11	416	1
	2PDD5	52	1.2764E-11	1.1267E-11	122	1

Table 3: Comparison of the numerical result for solving Problem 2

h	Method	TS	MAXE	AVE	FCN	ITN
0.10	DAM5	10	2.0981E-04	1.2338E-04	161	4
	2PDAM5	7	1.2241E-06	1.0095E-06	72	1
	2PDD5	7	1.2479E-06	7.5034E-07	164	4
0.05	DAM5	20	1.1985E-04	5.7861E-05	232	4
	2PDAM5	12	4.4053E-08	2.8134E-08	108	1
	2PDD5	12	6.3072E-08	4.9714E-08	44	1
0.01	DAM5	100	5.7861E-05	1.0135E-05	872	4
	2PDAM5	52	1.4293E-10	5.7353E-11	416	1
	2PDD5	52	2.2669E-11	1.9128E-11	122	1

Table 4: Comparison of the numerical result for solving Problem 3

h	Method	TS	MAXE	AVE	FCN	ITN
0.10	DAM5	10	4.6144E-04	2.5339E-04	183	4
	2PDAM5	7	5.9071E-06	2.4065E-06	360	4
	2PDD5	7	2.0108E-05	1.3211E-05	180	4
0.05	DAM5	20	2.6533E-04	1.4303E-04	253	4
	2PDAM5	12	1.9592E-07	3.5965E-08	132	1
	2PDD5	12	5.8808E-07	3.5094E-07	222	4
0.01	DAM5	100	6.2018E-05	3.0122E-05	872	4
	2PDAM5	52	1.1374E-10	1.8234E-11	416	1
	2PDD5	52	4.0342E-12	3.1147E-12	122	1
Method	n	Iteration	MAXE			
BP	8	8	5.5229E-07			
	10	8	1.5084E-08			

Table 5: Comparison of the numerical result for solving Problem 4

h	Method	TS	MAXE	AVE	FCN	ITN
0.10	DAM5	10	9.4637E-03	3.4543E-03	180	4
	2PDAM5	7	6.3115E-04	2.9366E-04	368	4
	2PDD5	7	9.1109E-04	4.7876E-04	194	4
0.05	DAM5	20	4.9048E-03	1.6735E-03	300	4
	2PDAM5	12	7.5037E-06	3.3335E-06	552	4
	2PDD5	12	3.6585E-05	1.8147E-05	242	4
0.01	DAM5	100	1.0530E-03	3.4138E-04	872	4
	2PDAM5	52	2.7232E-09	1.3145E-09	416	1
	2PDD5	52	1.3672E-08	5.0528E-09	122	1

The numerical results in Tables 2 - 5 demonstrates the ability of 2PDD5 method to achieve the same accuracy in terms of maximum and average error as obtained by 2PDAM5 for all tested problems. It is observed that the performance of 2PDD5 is much better than DAM5 because the numerical result computed by 2PDD5 is more accurate than DAM5. However, 2PDD5 need extra guessing values than 2PDAM5 in order to achieve the required solution when solving Problems 2 and 3 at $h = 0.10$ and $h = 0.05$ respectively. As shown in Table 4, 2PDD5 manage to give an excellent accuracy result as compared to 2PDAM5 when the step size is reduced to $h = 0.01$. At the same time, 2PDD5 required less number of iterations than BP method which result in better accuracy. In solving problem 4, both 2PDD5 and 2PDAM5 need the same number of guessing values in order to achieve the solution and to attain the comparable accuracy. The 2PDD5 need less total function call than other methods at $h = 0.05$ and $h = 0.01$ as presented in Table 5. The summarized result also showed that the accuracy significantly better as the step size is reduced. It is also proved that, two point method represent by 2PDD5 manage to reduce the number of total step to almost half compared to DAM5 since 2PDD5 compute two values at a time.

5. Conclusion

This study reports an efficient result given by the two point diagonally block method of order five. The proposed method has its own advantages in solving second order BVPs with Robin boundary conditions directly. In addition, Newton divided difference interpolation method can be a suitable alternative in improvised the estimating values when employed the shooting technique. The achievement obtained exhibit that the proposed method can give a good competitive algorithm.

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