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Numerical solution of third-order Robin boundary value problems using diagonally multistep block method

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Abstract. This numerical study exclusively focused on developing a diagonally multistep block method of order five (2DDM5) to get the approximate solution of the third-order Robin boundary value problems directly. The mathematical derivation of the developed 2DDM5 method is by approximating the integrand function with Lagrange interpolation polynomial. The proposed direct integrator scheme was applied to compute numerical solution at two-point concurrently. Shooting technique adapted with the Newtons divided difference interpolation method was implemented throughout the proposed algorithm. The theoretical characteristics of the developed method including the order, consistency, zero-stable and convergence are discussed. The method are tested on four Robin boundary value problems. The numerical results signify that the computational performances of the proposed method is competitive in terms of accuracy and efficiency than the existing method.

1. Introduction

This study was conducted to directly evaluate the solution of the third-order two-point boundary value problems (BVPs) which was given by:

$$y'''(x) = f(x, y, y', y''), \quad a_1 \leq x \leq a_3 \tag{1}$$

associated with three set of boundary conditions:

$$\sum_{i=1}^{3} C_{j,i} y^{(3-i)}(a_j) = B_j, \quad \text{for } j = 1, 2, 3, \tag{2}$$

where $a_1 \leq a_2 < a_3$, $C_{j,i}$, a_j and βj are real constants. For the case of $C_{j,i} \neq 0$, the boundary conditions in (2) are referred as three sets of Robin boundary conditions. Otherwise, the boundary conditions in (2) are also referred as mixed boundary conditions. According to Keller [1] and Modebei et al. [2], the solutions for (1) to (2) are assumed to satisfy the appropriate existence and uniqueness conditions.

The direct integrator block method has been extensively used to facilitate the solution for higher-order BVPs as well as initial value problems (IVPs) which has been reported in many mathematical literatures. Studies conducted by Turki et al. [3] and Adeyeye and Omar [4] used direct block method for solving third-order IVPs and fourth-order IVPs, respectively. Meanwhile, solving third-order BVPs directly were considered in Majid and See [5] and Nasir et al. [6], whilst Ramos and Rufai [7] focused on solving the fourth-order BVPs directly. The main contribution of all these studies highlighted the advantages of the direct integrator method as being more efficient in cost computations and more accurate at tackling some of the setbacks when solving higher-order differential equations using the conventional approach.

To date, numerous studies have been focusing on solving the third-order BVPs directly for the Dirichlet and Neumann boundary conditions than the Robin type boundary conditions. One of the point because the strategy to improvise the missing condition in handling Dirichlet and Neumann cases is less challenging compared to Robin case. Due to this reason, this study aims to improve the development of the direct block method in Nasir et al. [6] by considering a higher-order block method than the previous one. The main goal of this new development block method is to increase the accuracy of the numerical results and provide a trustworthy output than the existing BVPs solver methods.

2. Derivation of the method

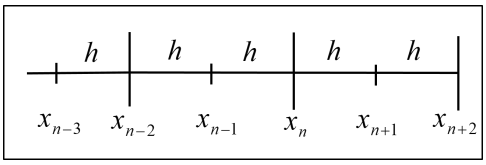


Figure 1. Two-point block method with three back values.

This section elaborates on the formulation of the two-point diagonally multistep block method to evaluate the third-order BVPs in (1). Two numerical solutions represented as y_{n+1} and y_{n+2} at x_{n+1} and x_{n+2}, respectively, were generated concurrently with constant step size h as depicted in Figure 1.

Accordingly, the direct integration evaluate equation (1) t times over the limit of integration, $x_n \leq x \leq x_{n+i}$ for $i = 1, 2$. The initial integration, $(t = 1)$ of (1) yields:

$$\int_{x_n}^{x_{n+i}} y'''(x)\,dx = \int_{x_n}^{x_{n+i}} f(x, y, y', y'')\,dx, \tag{3}$$

$$y''(x_{n+i}) - y''(x_n) = \int_{x_n}^{x_{n+i}} f(x, y, y', y'')\,dx.$$

The second integration, $(t = 2)$ of (1) yields:

$$\int_{x_n}^{x_{n+i}} \int_{x_n}^{x} y'''(x)\, dxdx = \int_{x_n}^{x_{n+i}} \int_{x_n}^{x} f(x, y, y', y'')\, dxdx, \tag{4}$$
$$y'(x_{n+i}) - y'(x_n) - ihy''(x_n) = \int_{x_n}^{x_{n+i}} (x - x_n) f(x, y, y', y'')\, dx.$$

Finally, the third integration, $(t = 3)$ of (1) yields:

$$\int_{x_n}^{x_{n+i}} \int_{x_n}^{x} \int_{x_n}^{x} y'''(x)\, dxdxdx = \int_{x_n}^{x_{n+i}} \int_{x_n}^{x} \int_{x_n}^{x} f(x, y, y', y'')\, dxdxdx, \tag{5}$$
$$y(x_{n+i}) - y(x_n) - hy'(x_n) - \frac{(ih)^2}{2!} y''(x_n) = \int_{x_n}^{x_{n+i}} \frac{1}{2}(x - x_n)^2 f(x, y, y', y'')\, dx.$$

Since the formulation considers the Adam's type formulas, hence all the integrand function of $f(x, y, y', y'')$ in (3) to (5) were approximated using the Lagrange interpolation polynomial, P_k with k is the degree of the polynomial. Firstly, to develop the formulation for the first corrector formula of y_{n+1} , the set of points $\{x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_{n+1}\}$ was interpolated which resulted the Lagrange form as:

$$P_4 = \sum_{j=0}^{4} \left(\prod_{\substack{i=0 \\ i \neq j}}^{4} \frac{(x - x_{n-3+i})}{(x_{n-3+j} - x_{n-3+i})} \right) f_{n-3+j}. \tag{6}$$

Meanwhile, the set of points $\{x_{n-3}, x_{n-2}, x_{n-1}, x_n, x_{n+1}, x_{n+2}\}$ was interpolated, thus deriving the following Langrange form:

$$P_5 = \sum_{j=0}^{5} \left(\prod_{\substack{i=0 \\ i \neq j}}^{5} \frac{(x - x_{n-3+i})}{(x_{n-3+j} - x_{n-3+i})} \right) f_{n-3+j} \tag{7}$$

and it was then used to derive the second corrector formula of y_{n+2}.

This study proceeded to incorporate the variable of substitutions, $x = x_{n+i} + sh$ and $dx = hds$ into the integral part, (3) to (5). Subsequently, the simplified version using Maple for the first corrector formulas were attained:

$$y''_{n+1} = y''_n + \frac{h}{720}[-19f_{n-3} + 106f_{n-2} - 264f_{n-1} + 646f_n + 251f_{n+1}], \tag{8}$$
$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{1440}[-17f_{n-3} + 96f_{n-2} - 246f_{n-1} + 752f_n + 135f_{n+1}],$$
$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{10080}[-33f_{n-3} + 188f_{n-2} - 492f_{n-1} + 1812f_n + 205f_{n+1}].$$

Similar procedure yielded the corrector formulas for y_{n+2} as follows:

$$\begin{aligned} y''_{n+2} &= y''_n + \frac{h}{90}[f_{n-3} - 6f_{n-2} + 14f_{n-1} + 14f_n + 129f_{n+1} + 28f_{n+2}], \\ y'_{n+2} &= y'_n + 2hy''_n + \frac{h^2}{630}[-2f_{n-3} + 17f_{n-2} - 76f_{n-1} + 566f_n + 718f_{n+1} \\ &\quad + 37f_{n+2}], \\ y_{n+2} &= y_n + 2hy'_n + \frac{(2h)^2}{2!}y''_n + \frac{h^3}{630}[-4f_{n-3} + 29f_{n-2} - 104f_{n-1} + 556f_n \\ &\quad + 364f_{n+1} - f_{n+2}]. \end{aligned} \tag{9}$$

The simplified version of (8) and (9) are as follows:

$$y_{n+i}^{(3-t)} = \sum_{k=0}^{t-1} \frac{(ih)^k y_n^{(3-t+k)}}{k!} + \frac{h^t}{(t-1)!} \sum_{j=-3}^{i} \beta_{i,n+j}^t \, f_{n+j}, \quad \text{for } i = 1, 2 \tag{10}$$

where

t : refers to the number of times the integral has been evaluated;
β : represent the coefficients of the method.

Following that, this study implemented the same derivation technique for the predictor formulas by considering the number of interpolated points being one value lesser than the corrector, which satisfies the explicit form. Table 1 and Table 2 presents the developed predictor formulas for the coefficients of y_{n+1} and y_{n+2}, respectively. It

Table 1. Coefficients,β_1 of the first point predictor formula.

t	$\beta_{1,n-3}^t$	$\beta_{1,n-2}^t$	$\beta_{1,n-1}^t$	$\beta_{1,n}^t$
1	$-9/24$	$37/24$	$-59/24$	$55/24$
2	$-38/360$	$159/360$	$-264/360$	$323/360$
3	$-17/720$	$72/7200$	$-123/720$	$188/720$

Table 2. Coefficients,β_2 of the second point predictor formula.

t	$\beta_{2,n-3}^t$	$\beta_{2,n-2}^t$	$\beta_{2,n-1}^t$	$\beta_{2,n}^t$	$\beta_{2,n+1}^t$
1	$29/90$	$-146/90$	$294/90$	$-266/90$	$269/90$
2	$5/90$	$-24/90$	$42/90$	$28/90$	$129/90$
3	-5/630	$34/630$	$-114/630$	$566/630$	$359/630$

is noticeable from the tables that all of the derived first point predictor and corrector formulas interpolated the total number of points which is one value lesser than the

respective second point for the predictor and corrector formulas. This signifies that the first point formulas have lower order than the second point formulas.

It should be noted that the proposed method in this study, specifically known as the direct two-point diagonally multistep block method of order five (2DDM5), is not a self-starting method. Therefore, 2DDM5 used the one-step method at the early stage of the algorithm to initiate the computation process for the multistep part. Additionally, the proposed numerical scheme was designed using the combination of $PE(CE)^r$ mode. In this scheme, P, C and E corresponded to the evaluation of predictor, corrector and the function, $f(x, y, y', y'')$, respectively, whilst r was the total iterations acquired in test for convergence.

3. Analysis of the block method

This section is devoted to discuss on the properties of the 2DDM5 method by providing the analysis in terms of the order, error constant, consistency, zero-stable and convergence.

The 2DDM5 formulas can be specified as a member of general linear multistep method (LMM) using the following formula:

$$\sum_{j=0}^{m}\alpha_j y_{n+j} = h\sum_{j=0}^{m}\delta_j y'_{n+j} + h^2\sum_{j=0}^{m}\gamma_j y''_{n+j} + h^3\sum_{j=0}^{m}\sigma_j f(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}). \tag{11}$$

The linear difference operator associated with (11) are given as follows:

$$L[y(x_n), h] = \sum_{j=0}^{m}\Big(\alpha_j y(x_n + jh) - h\delta_j y'(x_n + jh) - h^2\gamma y''(x_n + jh) - h^3\sigma_j y'''(x_n + jh)\Big) \tag{12}$$

with $y(x_n)$ an arbitrary function continuously differentiable on $[a_1, a_3]$. Upon expanding the function $y(x_n+jh)$ and its derivative about the x_n using Taylor's series, the following simplified version of (12) can be attained:

$$L[y(x_n), h] = C_0 y(x_n) + C_1 h y'(x_n) + \ldots + C_p h^p y^{(p)}(x_n) + \ldots \tag{13}$$

and C_p's are constant coefficients defined as follows:

$$C_p = \frac{1}{p!}\left[\sum_{j=0}^{m}\left(j^p\alpha_j - pj^{p-1}\delta_j - p(p-1)j^{p-2}\gamma_j - p(p-1)(p-2)j^{p-3}\sigma_j\right)\right] \tag{14}$$

for $p = 0, 1, 2 \ldots$

3.1. Order, error constant and consistency

Definition 1 *According to [8], the LMM have an order p if $C_0 = C_1 = \ldots = C_{p+1} = C_{p+2} = 0$ and $C_{p+3} \neq 0$, then the method is consistent whenever $p \geq 1$.*

The order and error constant of 2DDM5 method were calculated based on the main corrector formulas in (8) and (9) associated with (11). By letting $m = 5$ in (12), this implied the matrix form as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_m \\ y_{m+1} \\ y_{m+2} \\ y_{m+3} \\ y_{m+4} \\ y_{m+5} \end{bmatrix} = h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} y'_m \\ y'_{m+1} \\ y'_{m+2} \\ y'_{m+3} \\ y'_{m+4} \\ y'_{m+5} \end{bmatrix} \tag{15}$$

$$+ h^2 \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} y''_m \\ y''_{m+1} \\ y''_{m+2} \\ y''_{m+3} \\ y''_{m+4} \\ y''_{m+5} \end{bmatrix}$$

$$+ h^3 \begin{bmatrix} \text{-19/720} & 106/720 & \text{-264/720} & 646/720 & 251/720 & 0 \\ \text{-17/1440} & 96/1440 & \text{-246/1440} & 752/1440 & 135/1440 & 0 \\ \text{-33/10080} & 188/10080 & \text{-492/10080} & 1812/10080 & 205/10080 & 0 \\ 1/90 & \text{-6/90} & 14/90 & 14/90 & 129/90 & 28/90 \\ \text{-2/630} & 17/630 & \text{-76/630} & 566/630 & 718/630 & 37/630 \\ \text{-4/630} & 29/630 & \text{-104/630} & 556/630 & 364/630 & \text{-1/630} \end{bmatrix} \begin{bmatrix} f_m \\ f_{m+1} \\ f_{m+2} \\ f_{m+3} \\ f_{m+4} \\ f_{m+5} \end{bmatrix}$$

where $m = n - 3$. Then, by simplifying the set of coefficients from (14), the following was obtained:

$$C_0 = C_1 = \ldots = C_7 = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \tag{16}$$

with the vector of error constant given by:

$$C_8 = \frac{1}{8!} \sum_{j=0}^{5} \left(j^8 \alpha_j - 8j^7 \delta_j - 56j^6 \gamma_j - 336j^5 \sigma_j \right) \tag{17}$$

$$= \begin{bmatrix} -3/160 & -41/5040 & -89/40320 & 0 & 0 & 0 \end{bmatrix}^T.$$

This gives the order of the block method to be of order five, $p = 5$. Therefore, 2DDM5 method is consistent since the method possesses an order of at least one.

3.2. Zero-stability of the block method

Definition 2 *According to [8], the LMM is said to be zero-stable provided that the first characteristic polynomial, $\rho(\xi)$ specified as $\rho(\xi) = det \left| \sum_{j=0}^{k} A^{(j)} \xi^{(k-j)} \right| = 0$ having roots*

such that $|\xi_j| \leq 1$ and if $|\xi_j| = 1$, the multiplicity must not exceed three. The roots are defined as ξ_j for $j = 1, \ldots, k$.

Therefore, to analyze the zero-stability for the 2DDM5 method, the roots of the first characteristic polynomial can be specified as:

$$\begin{aligned}\rho(\xi) &= det\left(\xi A^0 - A^1\right) \\ &= det\begin{bmatrix}\xi & 0 & 0 & -1 & 0 & 0\\ 0 & \xi & 0 & 0 & -1 & 0\\ 0 & 0 & \xi & 0 & 0 & -1\\ 0 & 0 & 0 & \xi-1 & 0 & 0\\ 0 & 0 & 0 & 0 & \xi-1 & 0\\ 0 & 0 & 0 & 0 & 0 & \xi-1\end{bmatrix} \\ &= \xi^3(\xi-1)^3\end{aligned} \tag{18}$$

with A^0 is an identity matrix with dimension 6×6. From (18), $\rho(\xi) > 0$ yields roots of $\xi = \{0, 0, 0, 1, 1, 1\}$. Hence, it is concluded that 2DDM5 is zero-stable since $\rho(\xi)$ has roots satisfying $|\xi_j| \leq 1$.

3.3. Convergence of the block method

Theorem 1 *The LMM associated with* (11) *is convergent iff it is consistent and zero-stable [9].*

Since the consistency and zero-stable of the method have been fulfilled, it is concluded that the proposed 2DDM5 is convergent.

4. Implementation

In order to implement 2DDM5 method on solving the considered problems in (1), the constructing algorithm involves the shooting technique via Newton's divided difference interpolation method. Our developed BVPs solver takes

$$y_1(a_1) = G_1 \quad \text{and} \quad y_2(a_1) = G_2$$

where $G_1 = 0$ and $G_2 = 1$ as first and second guesses, respectively. The guessing criteria have a sole strategy of imposing suitable starting values to initiate the shooting process based on the consideration in [10].

In this study, the underlying iterative strategy throughout the numerical experiments was adopted from [6]. The first iterative process was stopped once the stopping condition had been met using

$$\left|F(y_1(a_3, G_1), y_1'(a_3, G_1), y_1''(a_3, G_1)) - B_3\right| \leq TOL$$

where

$$F(y_1(a_3, G_1), y_1'(a_3, G_1), y_1''(a_3, G_1)) = C_{3,1}y''(a_3, G_1) + C_{3,2}y'(a_3, G_1) + C_{3,3}y(a_3, G_1).$$

Algorithm: The 2DDM5 method

Step 1:	**INPUT**
	Total number of subintervals: N;
	Initial and terminal value of integration:$[x_0, x_N]$;
	Tolerance: TOL=10(−5);
	End boundary value:B_3.
Step 2:	For $i = 1$, set the initial guess, G_j.
Step 3:	For $k = 1$ to 3, set $x_k = x_0 + kh$.
	Evaluate $y_k^{(j)}, j = 0, 1, 2$ and f_k.
Step 4:	For $j = 1$ to 2, set $x_{j+3} = x_3 + jh$.
	Evaluate $y_{j+3}, y'_{j+3}, y''_{j+3}$ and f_{j+3} using 2DDM5 method.
Step 5:	At x_N, verify the stopping condition. If fulfilled, then go to Step 7.
	Else, continue Step 6.
Step 6:	Set $i = i + 1$. Generate new G_j. Go to Step 3.
Step 7:	**OUTPUT**
	Numerical values of the problems in (1) at specified grid points. Complete.

If vice-versa, then the guessing value was revised and continued with the next shooting process. Details of the proposed algorithm is outlined as follows.
Several essential formulas were also involved throughout the development of the algorithm including calculation of numerical error using

$$\left| \frac{(y(x_i)) - (y_i)}{A + B\,(y(x_i))} \right|, \tag{19}$$

with y_i and $y(x_i)$ are numerical results and exact solution, respectively. Meanwhile, for a better accuracy result, a convergence test as given by

$$\left| \frac{\left(y_{n+1}^r\right) - \left(y_{n+1}^{r-1}\right)}{A + B\left(y_{n+1}^r\right)} \right| < 0.1 \times TOL \tag{20}$$

was set at the corrector part of 2DDM5 method. The values of $A = B = 1$ had been assigned in (19) and (20) which was denoted as mixed test. The complete C source code for our computational procedure were computed using Code::Blocks 16.01 platform.

5. Numerical results and discussion

This section discussed the performances of 2DDM5 block method by considering the four numerical tested problems. The accuracy and efficiency of 2DDM5 method were assessed by comparing it with the existing methods including bvp5c, 2P3BVS, 2DDM4 and BVM3 methods. Below are the indicated notations used in the tables.

h	: Step size.
MAXE	: Maximum error.
AVE	: Average error.
TS	: Total steps at the last iteration.
FCN	: Total function calls.
TG	: Total iteration of guesses.
TIME	: Time computation in second.
bvp5c	: Matlab solver with fifth order collocation method.
2P3BVS	: Fourth order block method proposed in [11].
BVM3	: Boundary value methods of order four in [8].
2DDM4	: Direct two-point diagonally multistep block method of order four in [6]
2DDM5	: Direct two-point diagonally multistep block method of order five proposed in this study.

Problem 1 Consider the third-order linear differential equation in [12]

$$y'''(x) = xy(x) + (x^3 - 2x^2 - 5x - 3)\exp(x) \tag{21}$$

with Robin boundary conditions

$$y''(0) - y'(0) - y(0) = -1,\ y''(0) - y(0) = 0,\ y''(10) + y'(10) + y(10) = -329\exp(10).$$

Exact solution : $y(x) = x(1-x)\exp(x)$.

Problem 2 Consider the third-order nonlinear differential equation in [6]

$$y'''(x) = -\exp(-2y(x))\big(y'(x) + xy''(x) - 2x(y'(x))^2\big) \tag{22}$$

with Robin boundary conditions

$$y''(1) + y'(1) + y(1) = 0,\ y''(1) + y(1) = -1,\ y''(2) + y'(2) + y(2) = \ln(2) + \frac{1}{4}.$$

Exact solution: $y(x) = \ln(x)$.

Problem 3 Consider the third-order nonlinear differential equation in [13]

$$y'''(x) = -2\exp(-3y(x)) + 4(1+x)^{-3} \tag{23}$$

with Robin boundary conditions

$$y''(0) - y'(0) - y(0) = -2,\ y''(0) + y(0) = -1,\ y''(1) + y'(1) + y(1) = \ln(2) + \frac{1}{4}.$$

Exact solution : $y(x) = \ln(1+x)$.

Problem 4 Consider the third-order linear differential equation in [8]

$$y'''(x) = \frac{1}{\sqrt{1+x}}y'(x) - 2y(x) + 3 - \frac{3x^2}{2\sqrt{1+x}} + x^3 \tag{24}$$

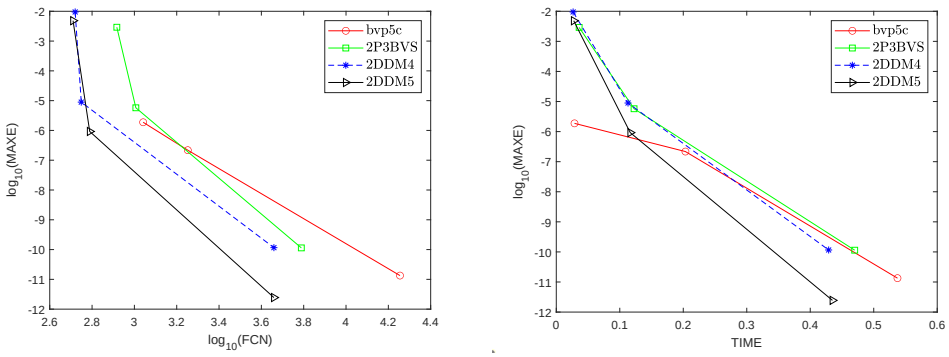
with mixed boundary conditions

$$y(0) = y'(0) = 0, \;\; 3y(1) - y'(1) = 0.$$

Exact solution : $y(x) = \frac{1}{2}x^3$.

Table 3. Comparison of the numerical results for solving Problem 1.

Method	h	MAXE	AVE	TS	FCN	TG	TIME
	1.0	1.88E-06	7.77E-07	10	1100	-	0.0287
bvp5c	0.1	2.17E-07	9.30E-09	100	1788	-	0.2034
	0.01	1.34E-11	9.91E-14	1000	17988	-	0.5374
	1.0	2.91E-03	9.04E-04	7	826	5	0.0360
2P3BVS	0.1	5.76E-06	5.21E-07	52	1016	4	0.1225
	0.01	1.14E-10	1.28E-11	502	6150	3	0.4700
	1.0	9.49E-03	2.61E-03	6	525	5	0.0265
2DDM4	0.1	8.92E-06	8.83E-07	51	561	3	0.1130
	0.01	1.16E-10	1.29E-11	501	4563	3	0.4290
	1.0	4.84E-03	1.45E-03	7	513	4	0.0275
2DDM5	0.1	9.24E-07	8.84E-08	52	615	3	0.1165
	0.01	2.45E-12	2.96E-13	502	4653	3	0.4350

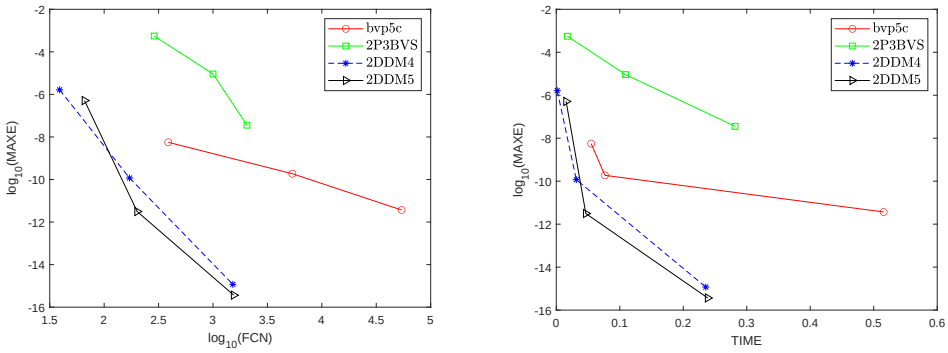


(a) $\log_{10}$(FCN) vs. $\log_{10}$(MAXE) (b) TIME vs. $\log_{10}$(MAXE)

Figure 2. Performance graphs of numerical results for Problem 1

Table 4. Comparison of the numerical results for solving Problem 2.

Method	h	MAXE	AVE	TS	FCN	TG	TIME
	0.1	5.60E-09	3.69E-09	10	388	-	0.0549
bvp5c	0.01	1.87E-10	1.83E-10	100	5358	-	0.0766
	0.001	3.69E-12	2.33E-12	1000	53958	-	0.5156
	0.1	5.56E-04	2.24E-04	7	288	4	0.0175
2P3BVS	0.01	9.14E-06	6.13E-06	52	1000	4	0.1090
	0.001	3.58E-08	1.13E-08	502	2050	1	0.2815
	0.1	1.64E-06	7.44E-07	6	39	1	0.0015
2DDM4	0.01	1.18E-10	4.25E-11	51	171	1	0.0315
	0.001	1.18E-15	3.03E-16	501	1521	1	0.2355
	0.1	5.15E-07	1.40E-07	7	66	1	0.0155
2DDM5	0.01	3.13E-12	1.02E-12	52	201	1	0.0465
	0.001	3.67E-16	1.64E-16	502	1551	1	0.2385

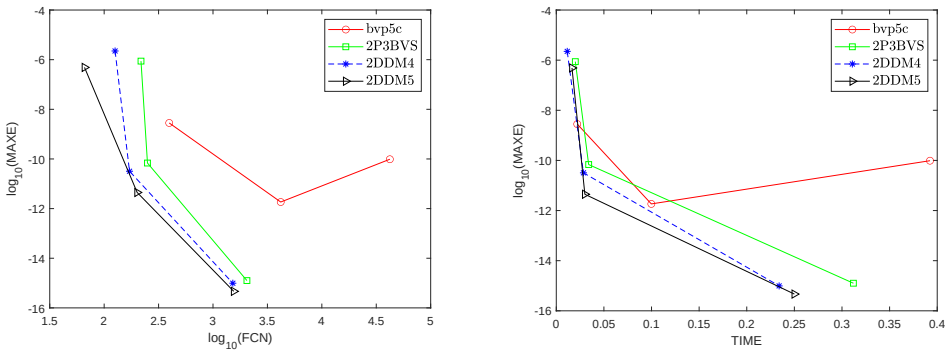


(a) $\log_{10}$(FCN) vs. $\log_{10}$(MAXE) (b) TIME vs. $\log_{10}$(MAXE)

Figure 3. Performance graphs of numerical results for Problem 2.

Table 5. Comparison of the numerical results for solving Problem 3.

Method	h	MAXE	AVE	TS	FCN	TG	TIME
	0.1	2.82E-09	1.17E-09	10	396	-	0.0220
bvp5c	0.01	1.84E-12	5.83E-13	100	4191	-	0.0998
	0.001	9.74E-11	3.80E-11	1000	42171	-	0.3924
	0.1	8.74E-07	4.63E-07	7	218	3	0.0200
2P3BVS	0.01	6.85E-11	2.14E-11	52	250	1	0.0340
	0.001	1.26E-15	4.23E-16	502	2050	1	0.3120
	0.1	2.24E-06	1.46E-06	6	126	3	0.0115
2DDM4	0.01	3.20E-11	1.46E-11	51	171	1	0.0285
	0.001	9.86E-16	3.45E-16	501	1521	1	0.2340
	0.1	4.88E-07	1.22E-07	7	66	1	0.0165
2DDM5	0.01	4.47E-12	1.44E-12	52	201	1	0.0300
	0.001	4.65E-16	2.01E-16	502	1551	1	0.2500



(a) $\log_{10}$(FCN) vs. $\log_{10}$(MAXE) (b) TIME vs. $\log_{10}$(MAXE)

Figure 4. Performance graphs of numerical results for Problem 3.

Table 6. Comparison between BVM3, 2DDM4 and 2DDM5 for Problem 4.

N	BVM3		2DDM4		2DDM5	
	MAXE	FCN	MAXE	FCN	MAXE	FCN
10	1.388E-15	30	3.9328E-07	36	3.9662E-09	66
20	4.441E-16	60	1.5860E-08	51	6.2766E-11	81
40	2.887E-14	120	5.5247E-10	81	9.9743E-13	111
80	6.856E-14	240	1.8164E-11	141	1.5932E-14	171
160	2.449E-14	480	5.8181E-13	261	2.1649E-15	291
320	5.228E-14	960	1.8404E-14	501	4.2744E-15	531

The computational performances of 2DDM5 for solving Problem 1 to Problem 3 were compared with bvp5c and 2P3BVS methods of order 5 as well as 2DDM4 method of order 4. Table 3 demonstrates that the accuracy acquired by the 2DDM5 method is comparable with bvp5c but superior to 2P3BVS and 2DDM4 methods as the step size decreases.

Table 4 and Table 5 present the computational comparison of all methods for solving nonlinear problems. The tabulated data clearly highlight the numerical results acquired by 2DDM5 to have closer true value than other methods which reflects the smallest errors attained by 2DDM5 method. The respective data also signify that 2DDM5 are less expensive in terms of total function calls and faster in timing than bvp5c and 2P3BVS. In contrast, 2DDM5 acquired extra computational cost in both elements than the 2DDM4 method due to the behavior of the higher order method. These efficient performances demonstrated by 2DDM5 are visualized in Figure 2 to Figure 4.

In Problem 4, we tested a mixed boundary conditions involving Dirichlet, Neumann and Robin conditions. This problem were also solved using BVM3 and 2DDM4 methods. From Table 6, it is observed that 2DDM5 dominated other methods in terms of accuracy as the N increase. Meanwhile, 2DDM5 cost lesser for its function calls than the to BVP3.

6. Conclusion

This study remarks that the proposed two-point diagonally multistep block method of order five with constant step size manage efficiently to solve numerical tested problems at economic in computational cost. The 2DDM5 is also reliable in measuring approximate solutions for the third-order boundary value problems associated with two-point Robin boundary conditions as it able to solve these problems directly.

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