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INFLUENCE OF VISCOUS DISSIPATION ON THE BOUNDARY LAYER FLOW OF $\mathrm{Cu\text{-}Al_2O_3}$ HYBRID NANOFLUID

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Abstract

This study presents the mathematical modelling of two dimensional boundary layer flow of hybrid nanofluid where the impact of viscous dissipation has been accentuated in the energy equation. The copper and aluminium oxide nanoparticles are considered in this study. The surface of the model is stretched and shrunk at certain values of

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stretching/shrinking parameter. The partial differential equations of the hybrid nanofluid are reduced to the ordinary differential equations with the utilization of the suitable similarity transformations. Then Matlab software is utilized to produce the numerical and graphical results by implementing the bvp4c function. Subsequently, dual solutions are obtained with the correct guess values. The insertion of viscous dissipation in this model tremendously lessens the rate of heat transfer. Besides, the effects of the suction and nanoparticles concentration also have been highlighted. An increment in the suction parameter and concentration of copper enhance the magnitude of the reduced skin friction coefficient while the augmentation of the aluminium oxide nanoparticles shows a different trend.

Introduction

As the continuation work from deficiency of nanofluids, the hybrid nanofluids become the new trend of the research works. Hybrid nanofluids, are also defined as composite nanofluids, since the fluids are containing two or more different types of nanoparticles $(\mathrm{TiO_2}, \mathrm{Al_2O_3}, \mathrm{CuO}, \mathrm{etc})$. Furthermore, the properties of hybrid nanofluids have gained the interest from many researchers and been studied and analyzed from time to time. They found that the hybridization of combining different nanoparticles promising in heat transfer enhancement significantly. However, the journals related to the topic of hybrid nanofluids are still limited at this moment. Hybrid nanofluids could be applied efficiently in the fields like medical treatment, electrochemical-sensors, bio-sensors, nanocatalysts, industrial and engineering applications and many others with good quality and low cost.

A hybrid material consists of different and combined substances with the physical and chemical properties simultaneously. At the same time, these provided properties are considered in a single phase (homogeneous) of fluids. Some of the researchers had determined and reviewed about the preparation of hybrid nanofluids, heat transfer and thermal properties. According to Sundar et al. [1], they concluded that the viscosity and thermal conductivity of hybrid nanofluids showed greater values than the single-nanofluids within the similar volume of nanoparticles concentration and temperature circumstance. Other than that, the coefficients of heat transfer and friction factor are higher when compared to the ordinary nanofluids. These results are also supported by Babu et al. [2] through the work of synthesis method on hybrid nanofluids. In their results, hybrid nanofluids yield higher thermal conductivity. Due to the smaller size of particles, they move even faster and caused the collision rate increased to produce more kinetic energy and eventually produced higher enhancements in the heat transfer rate. Also, the hybrid nanoparticles of cylindrical shaped have better performance in thermal conductivity than spherical shape particles. Generally, almost all of the researchers agreed that the hybrid nanofluids highly satisfied the conditions and are more efficient in the application of cooling systems like manufacturing, microelectronics, transportation and thermal power plants.

Apart from that, the effects of viscous dissipation also have been considered in the fluid flow. The viscous dissipation can be referred as the work done by the fluid on vertical plate due to the action of shear forces transformed into heat. Besides, viscous dissipation also works as the function of internal friction and dissipating the heat in fluidic materials or liquids. Mabood et al. [3] investigated the problem with magnetohydrodynamic (MHD) boundary layer flow with viscous dissipation effect. Jusoh and Nazar [4] studied the effect of viscous dissipation on the MHD stagnation point flow and heat transfer of nanofluid over a nonlinear stretching/shrinking sheet. Then the problem of boundary layer flow of nanofluids which consist of $\mathrm{Al_2O_3}$, Cu, $\mathrm{TiO_2}$ and $\mathrm{Fe_3O_4}$ nanoparticles past a bidirectional exponentially stretching/shrinking sheet with suction was studied numerically by Jusoh et al. [5].

Therefore, the purpose of this study is to investigate the influence of viscous dissipation for $\mathrm{Cu\text{-}Al_2O_3}$ hybrid nanofluid flow. The embraced methodology will reduce the governing partial differential equations into a system of ordinary differential equations and solve the problem through the implementation of the built-in solver bvp4c in Matlab. The first and second solution will be carried out as well as the explanation on the graphical results.

Problem Formulation

Assume a two dimensional, steady and incompressible viscous flow of a Cu-Al_2O_3 hybrid nanofluid flow on a stretching/shrinking surface with the fixed origin. The direction of the sheet motion and its perpendicular line are defined as *x* and *y* axes, respectively. The wall is considered permeable with $v = -v_w$. With these assumptions, the steady boundary layer equations are given as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu_{hnf}}{\rho_{hnf}}\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_{hnf}\frac{\partial^2 T}{\partial y^2} + \frac{\mu_{hnf}}{(\rho C_p)_{hnf}}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

and tally with the boundary conditions:

$$u = \lambda u_w(x) = \lambda a x, \quad v = -v_w, \quad T = T_w \text{ at } y = 0,$$

$$u \to 0, \quad T \to T_\infty \text{ as } y \to \infty, \tag{4}$$

where velocity components in the *x* and *y* axes are *u* and *v*, respectively; λ denotes the shrinking/stretching parameter, *a* is the constant, *T* represents the temperature of the fluid, T_w is the wall temperature, and T_∞ denotes the constant fluid temperature far away from the sheet. Besides, some important parameters also involve like μ_{hnf} (viscosity of hybrid nanofluid), ρ_{hnf} (density of hybrid nanofluid), α_{hnf} (thermal diffusivity of hybrid nanofluid) and $(\rho C_p)_{hnf}$ (effective heat capacity of hybrid nanofluid). Table 1 lists the formulas for the thermophysical properties of nanofluid and hybrid nanofluid. In addition, Table 2 gives the values of thermophysical properties for the fluid and nanoparticles.

Table 1. Thermophysical properties for nanofluid and hybrid nanofluid

Properties	Nanofluid	Hybrid nanofluid
Density	$\rho_{nf} = (1-\phi_1)\rho_f + \phi_1\rho_s$	$\rho_{hnf} = (1-\phi_2)[(1-\phi_1)\rho_f + \phi_1\rho_{s1}] + \phi_2\rho_{s2}$
Heat capacity	$(\rho C_p)_{nf} = (1-\phi_1)(\rho C_p)_f + \phi_1(\rho C_p)_{s1}$	$(\rho C_p)_{hnf} = (1-\phi_2)\begin{bmatrix}(1-\phi_1)(\rho C_p)_f \\ +\phi_1(\rho C_p)_{s1}\end{bmatrix} + \phi_2(\rho C_p)_{s2}$
Dynamic viscosity	$\mu_{nf} = \frac{\mu_f}{(1-\phi_1)^{2.5}}$	$\mu_{hnf} = \frac{\mu_f}{(1-\phi_1)^{2.5}(1-\phi_2)^{2.5}}$
Thermal conductivity	$k_{nf} = \frac{k_{s1} + 2k_f - 2\phi_1(k_f - k_{s1})}{k_{s1} + 2k_f + \phi_1(k_f - k_{s1})} \times k_f$	$k_{hnf} = \frac{k_{s2} + 2k_{nf} - 2\phi_2(k_{nf} - k_{s2})}{k_{s2} + 2k_{nf} + \phi_2(k_{nf} - k_{s2})} \times k_{nf},$ where $k_{nf} = \frac{k_{s1} + 2k_f - 2\phi_1(k_f - k_{s1})}{k_{s1} + 2k_f + \phi_1(k_f - k_{s1})} \times k_f$

Table 2. Thermophysical properties of the fluid and nanoparticles

Properties	Water (f)	Al_2O_3 $(s1)$	Cu $(s2)$
ρ (kg/m^3)	997.0	3970	8933
C_p (J/kgK)	4180	765	385
k (W/mK)	0.6071	40	400

The following similarity transformations are used to solve equations (1)-(4):

$$u = axf'(\eta), \quad v = -\sqrt{a\nu_f} \cdot f(\eta), \quad \eta = y\sqrt{\frac{a}{\nu_f}}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{5}$$

where primes represents the differentiation with respect to η. It means that, we can take

$$v_w = \sqrt{a\nu_f}\, s. \tag{6}$$

After the implication, equations (2) and (3) could be reduced to

$$\frac{\mu_{hnf}/\mu_f}{\rho_{hnf}/\rho_f} f''' + ff'' - f'^2 = 0, \tag{7}$$

$$\frac{k_{hnf}/k_f}{Pr(\rho C_p)_{hnf}/(\rho C_p)_f}\theta''$$

$$+ f\theta' + \frac{Ec}{[(1-\phi_1)^{2.5}(1-\phi_2)^{2.5}(\rho C_p)_{hnf}]/(\rho C_p)_f} f''^2 = 0, \tag{8}$$

and the boundary conditions become

$$f(0) = s, \quad f'(0) = \lambda, \quad \theta(0) = 1,$$

$$f'(\eta) \to 0, \quad \theta(\eta) \to 0 \text{ as } \eta \to \infty, \tag{9}$$

where $Pr = \frac{v_f}{\alpha_f}$ is the Prandtl number, $s = \frac{v_w}{\sqrt{av_f}}$ is the suction parameter and $Ec = \frac{u_w^2}{(C_p)_f(T_w - T_\infty)}$ is the Eckert number. The physical quantities of interest are the skin friction coefficient and the Nusselt number. The formulas are given by

$$C_f = \frac{\mu_{hnf}\left(\frac{\partial u}{\partial y}\right)_{y=0}}{\rho_f u_w^2}, \quad Nu_x = -\frac{xk_{hnf}\left(\frac{\partial T}{\partial y}\right)_{y=0}}{k_f(T_w - T_\infty)}. \tag{10}$$

Then, with the substitution of equation (5) into equation (10), one gets:

$$Cf_x\sqrt{Re_x} = \frac{f''(0)}{(1-\phi_1)^{2.5}(1-\phi_2)^{2.5}}, \quad \frac{Nu_x}{\sqrt{Re_x}} = -\frac{k_{hnf}}{k_f}\theta'(0), \tag{11}$$

where $Re_x = \frac{xu_w}{v_f}$ is the local Reynolds number.

Results and Discussion

In order to solve equations (7) and (8) and along with the boundary conditions, the method of bvp4c has been implied in Matlab. The applicable initial guesses for $f''(0)$ and $-\theta'(0)$ are made to satisfy the asymptotic boundary conditions. First and second solutions will be carried out to obtain the numerical solutions within a tolerance limit of 10^{-5} level. Then the obtained results will be represented in graphs. A comparison between the present work and previous studies has been made in Table 3 and excellent agreement towards the finding results is found. Therefore, the accuracy of the present work is considered reliable and convincing.

Table 3. Comparison of $C_f\sqrt{Re_x}$ and $Nu_x/\sqrt{Re_x}$ for certain values of nanoparticle Cu volume fraction ϕ_1, when nanoparticle Al_2O_3 volume fraction $\phi_2 = 0.1$

	$C_f\sqrt{Re_x}$			$Nu_x/\sqrt{Re_x}$		
ϕ_1	Devi and Devi [6]	Waini et al. [7]	Present	Devi and Devi [6]	Waini et al. [7]	Present
0.005	-1.32731	-1.327098	-1.327098	1.961686	1.961769	1.961676
0.04	-1.520894	-1.520721	-1.520721	2.026368	2.026442	2.026405
0.06	-1.634279	-1.634119	-1.634119	2.064075	2.064146	2.064119
0.1	-	-1.869764	-1.869764	-	2.141644	2.141603

The graphical results for the influence of viscous dissipation on the boundary layer flow with Cu-Al_2O_3 hybrid nanofluid are obtained. In this work, we set the Prandtl number to 6.2 by considering water (H_2O) as the base fluid and it is compatible with the room temperature. A few of the parameters have been investigated like suction parameter s, Eckert number Ec and nanoparticle volume fraction $(\phi_1$ and $\phi_2)$. The impacts of these parameters over the reduced skin friction coefficient $f''(0)$ and Nusselt number $-\theta'(0)$ are illustrated, respectively.

Figure 1 depicts dual solutions for some values of suction parameter s with the shrinking/stretching parameter λ on $f''(0)$. From this graph, a unique solution obviously exists for the first and second solutions on λ_c, the critical value. Meanwhile, the dual solutions only occur when $\lambda \geq \lambda_c$. It also can be noticed that the greater the value of suction parameter, the smaller the critical value. Besides, the suction parameter $s = 3$ has the larger range of solution compared to the lower values. It also can be seen that when the sheet is stretched $(\lambda > 0)$ and shrunk $(\lambda < 0)$, the magnitude of the reduced skin friction coefficient $|f''(0)|$ increases as the suction parameter increases. Meanwhile, in Figure 2, the various values of Eckert number have the same critical value λ_c on the reduced Nusselt number. The presence of viscous dissipation causes the transformation of kinetic energy to heat and contributes the higher temperature to the fluid. At the same time, the Eckert number can be defined as the ratio of kinetic energy and enthalpy which lead to the decrement in the rate of heat transfer. As a result, increase in Eckert number causes a decrement in the reduced Nusselt number.

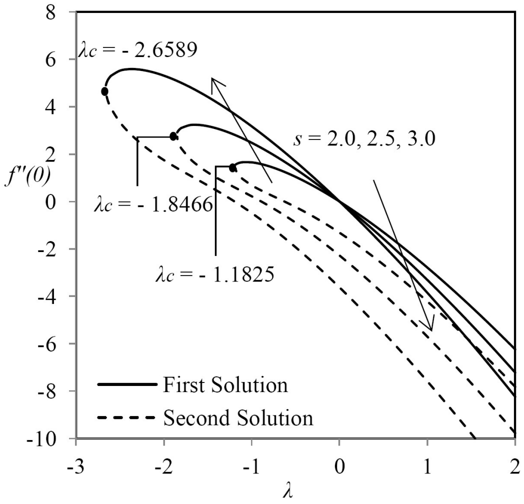

Figure 1. Variations of $f''(0)$ for several values of suction parameter s.

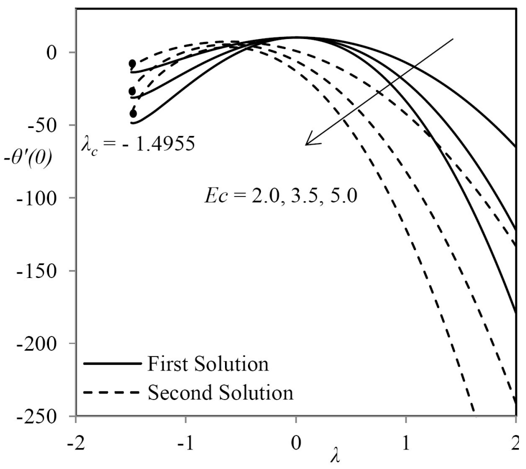

Figure 2. Variations of $-\theta'(0)$ for several values of Eckert number *Ec*.

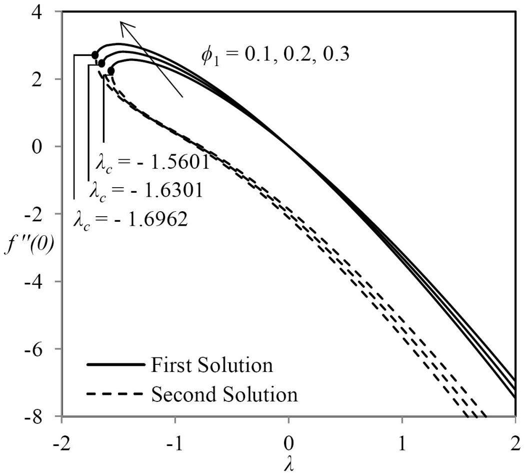

Figure 3. Variations of $f''(0)$ for several values of Cu nanoparticle volume fraction.

Figure 3 illustrates the influence of copper (Cu) nanoparticle volume fraction ϕ_1 on the skin friction coefficient. The magnitude of the reduced skin friction coefficient $|f''(0)|$ increases as the copper nanoparticles concentration increases. On the other hand, different trend is observed in Figure 4 when aluminium oxide (Al_2O_3) nanoparticle volume fraction ϕ_2

increases. Physically, when the concentration of Al_2O_3 is constant and the concentration of Cu nanoparticles is augmented, the reaction towards the skin friction is higher. Otherwise, a contradictory reaction occurs if the volume fraction of Cu remains the same and the volume fraction of Al_2O_3 varies. This is due to the fact that the copper has higher density than the aluminium oxide. Therefore, the increment in the copper nanoparticles concentration will decelerate the fluid motion and hence, accelerate the drag force. Besides, the inclusion of viscous dissipation in this model is also believed to be the contributing factor of the obtained results since the viscous dissipation can enhance the surface shear stress [8].

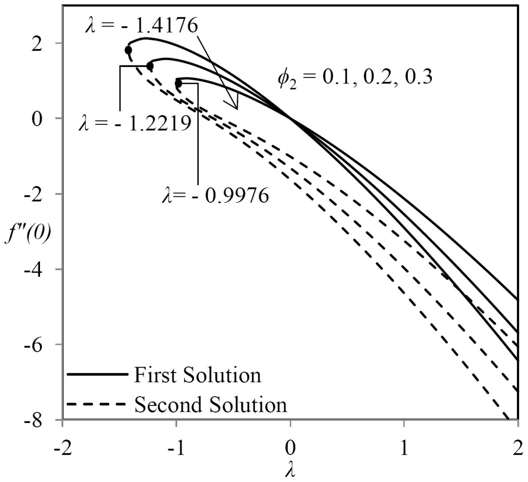

Figure 4. Variations of $f''(0)$ for several values of Al_2O_3 nanoparticle volume fraction.

Conclusion

The problem of two dimensional, steady and incompressible viscous flow of a $Cu\text{-}Al_2O_3$ hybrid nanofluid flow on a stretching/shrinking surface has been solved numerically through bvp4c function in Matlab. The dual solutions were obtained for some values of the governing parameters. The magnitude of the reduced skin friction coefficient increases as the suction parameter increases and as λ approaching the critical point λ_c. Since the viscous dissipation induces the conversion of kinetic energy to heat, thus, the rise in Eckert number increases the hybrid nanofluid's temperature and consequently reduces the rate of heat transfer. Also, the magnitude of the reduced skin friction coefficient enhances as the copper nanoparticles concentration increases but declines as the aluminium oxide nanoparticles concentration increases.

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