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On the spectral properties of the differential operators with involution

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Abstract. In this paper we deal with the problems of the eigenfunction expansions related to the differential operators with involution. The mean value formula for the eigenfunction is obtained with application of the transformation methods of the operators in the symmetric regions. The obtained formula is applied to estimate the eigenfunctions of the given differential operator in the ball. For domains with smooth boundary, the solution to these differential operator problems involves eigenfunction expansions associated with biharmonic-type operator with Navier boundary conditions.

1. Introduction

In the present paper, we deal with the biharmonic-type operator, which inherit some main properties of the biharmonic operators and represent the self-adjoint Fourth order differential operators. The biharmonic-type equations are used to describe various vibrating systems in physics, which play one of the essential roles in development of the spectral theory of the differential operators. The latter is mostly used for mathematical description of the physical processes taking place in real space. The partial differential equations (PDEs), particularly biharmonic-type equations are widely used to model the most difficult engineering problems concerning heat and mass transfer processes. The properties of fourth order partial differential equations on general domains remained largely beyond reach. The present research establishes estimations of the solutions to the biharmonic-type equation and their derivatives in arbitrary bounded region up to boundary, without any restrictions on the geometry of the underlying domain. Let Ω be a bounded domain in R^2 with smooth boundary $\Gamma = \partial\Omega \in C^{2,\alpha}$. Let $C^\infty(\Omega)$ denotes a set of infinite times differentiable functions on $\Omega \subset R^2$. The class of functions from $C^\infty(\Omega)$ with compact support is denoted by $C_0^\infty(\Omega)$. Let recall the definition of the Sobolev space

$$W_p^l(\Omega) = \{ f : \partial^\alpha f \in L_p(\Omega), \forall \alpha : |\alpha| \le l \},$$

for more properties of the Sobolev space see [1]. To capture the influence of the smoothness of the being expanding function to convergence of the corresponding spectral expansions we will investigate properties of the spectral decompositions of the biharmonic operator in Sobolev's, Nikolskii's classes. Many useful facts can be obtained by proving the analogous isomorphism theorems of Alimov [2]. The investigation of convergence problems of arbitrary spectral decompositions started as a continuation of the theory of multiple trigonometric series with spherical sums. The first results concerning the problems of the convergence and summability of the spectral decompositions of the Laplace operator on closed domain are obtained in [3], where it was shown that the eigenfunction expansions corresponding to the functions from the Sobolev classes $W_p^{\frac{N}{2}+1}$ uniformly converges on closed domain. The uniformly convergence of the spectral expansions of the functions from $W_p^{\frac{N-1}{2}+\varepsilon}$, $\varepsilon > 0$ on closed domain related to the elliptic differential operator of order $2m$ are proved in [4]. In [5] it is proved that the eigenfunction expansions of the second order elliptic operator corresponding to the first boundary condition uniformly converges if

$$\sum_{k=1}^{\infty} \lambda_k^{\frac{N-1}{2}} (\ln \lambda_k)^{2+\varepsilon} f_k^2 < \infty,$$

for all functions $f \in W_p^{\frac{N-1}{2}}$, $p > \dfrac{2N}{N-1}$, where f_k are Fourier coefficients of the function f. In this paper we estimate the eigenfunctions of the biharmonic operator, which guarantee the uniform convergence of the eigenfunction expansions. The mean value formula for the eigenfunctions of the polyharmonic operator is obtained in [7].

2. Preliminaries

In this paper, we deal with biharmonic-type operator. Namely, we consider the operator $\Lambda = \Delta^2 - a \cdot \Delta$,
where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ −Laplace operator and a is nonnegative constants. Let $\mathcal{F}_\Omega$ denotes a set of all functions u from $W_2^4(\Omega)$, satisfying the boundary conditions $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$. Let consider the operator Λ with the domain of definition $\mathcal{F}_\Omega$. Such defined operator Λ is symmetric:

$$\iint_\Omega \Lambda u(\xi,\eta) \cdot \overline{v(\xi,\eta)} d(\xi,\eta) = \iint_\Omega u(\xi,\eta) \cdot \overline{\Lambda v(\xi,\eta)} d(\xi,\eta),$$

and semibounded:

$$\iint_\Omega \Lambda u(\xi,\eta) \cdot \overline{u(\xi,\eta)} d(\xi,\eta) \geq 0$$

for all $u, v \in \mathcal{F}_\Omega$. By Friedrich's Theorem [1], the operator Λ has at least one non-negative self-adjoint extension. We denote Λ as the self adjoint extension of the Λ with discrete spectrum. Let $\{\lambda_n\}_{n=1}^\infty$ and $\{v_n(\xi,\eta)\}_{n=1}^\infty$ denotes a set of eigenvalues and eigenfunctions, respectively, of the operator Λ. From the general theory of the elliptic operators follows that the eigenfunctions of Λ form a complete orthonormal system in $L_2(\Omega)$. The self-adjoint extension Λ of the biharmonic type operator can be represented by spectral decomposition of unity $\{E_\lambda\}, \lambda > 0$. The spectral expansion $E_\lambda f$ of the function $f \in L_2(\Omega)$ coincides with the Fourier expansion of the given function in terms of eigenfunctions of the biharmonic type operator. For any $f \in L_2(\Omega)$ we define

$$E_\lambda f(\xi,\eta) = \sum_{\lambda_n < \lambda} f_n v_n(\xi,\eta),$$

where $\{f_n\}_{n=1}^\infty$ is a set of Fourier coefficients of the function f:

$$f_n = \iint_\Omega f(\xi,\eta) v_n(\xi,\eta) d(\xi,\eta), n = 1,2,3, \ldots.$$

In order to investigate the convergence of the partial sums $E_\lambda f$, it is useful to estimate the solution of the biharmonic type equation of the following form

$$\Delta^2 v(\xi,\eta) - a\Delta v(\xi,\eta) + \mu^4 v(\xi,\eta) = f(\xi,\eta), \tag{1}$$

satisfying the boundary conditions

$$v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = 0, \tag{2}$$

where $\mu = \mu_0 + i\tau_0, \mu_0 > 0, \tau_0 \neq 0.$

We can transform the problem (1)–(2) to two Dirichlet problems for second order equation. To do this, let α and β be the roots of the quadratic equation $\xi^2 - a\xi + \mu^4 = 0.$ Assume $\alpha = \alpha_0 + i\tau_1, \alpha_0 > 0, \tau_1 \neq 0,\ \beta = \beta_0 + i\tau_2,\ \beta_0 > 0$ and $\tau_2 \neq 0.$ Put $L_1 = \Delta - \alpha$ and $L_2 = \Delta - \beta$. Consequently, we have $L_1 \cdot L_2 = \Delta^2 - a\cdot\Delta + \mu^4 \cdot u$ and the problem (1)–(2) is decomposed to the following problems:

$$L_1 w(\xi,\eta) = \Delta w(\xi,\eta) - \alpha\cdot w(\xi,\eta) = f(\xi,\eta),\ (\xi,\eta)\in\Omega,$$

$$w|_{\partial\Omega} = 0,$$

and

$$L_2 v(\xi,\eta) = \Delta v(\xi,\eta) - \beta v(\xi,\eta) = w(\xi,\eta) \text{ in } \Omega,$$

$$v|_{\partial\Omega} = 0.$$

From the results of the paper [5], we obtain

$$\|v\|_{C(\bar\Omega)} \le K_1 \sqrt{\frac{\ln^2\beta_0}{\beta_0}}\, \|w\|_{L_2(\Omega)},$$

$$\|w\|_{C(\bar\Omega)} \le K_2 \sqrt{\frac{\ln^2\alpha_0}{\alpha_0}}\, \|f\|_{L_2(\Omega)}.$$

Hence,

$$\|v\|_{C(\bar\Omega)} \le K_1 K_2 \sqrt{\frac{\ln^2\beta_0}{\beta_0}} \sqrt{\frac{\ln^2\alpha_0}{\alpha_0}}\, \|f\|_{L_2(\Omega)}.$$

Considering $\alpha_0\beta_0 \sim \mu_0^2$ and $2\ln\alpha_0 \ln\beta_0 \le \ln^2(\alpha_0\beta_0)$, we obtain

$$\|v\|_{C(\bar\Omega)} \le K\cdot\frac{\ln\mu_0}{\mu_0}\cdot\|f\|_{L_2(\Omega)}. \tag{3}$$

Theorem 1. Let $v \in C(\overline{\Omega}) \cap W_2^4(\Omega)$ be a solution of the biharmonic type equation:

$$\Delta^2 v(\xi,\eta) - a\Delta v(\xi,\eta) + \mu^4 v(\xi,\eta) = f(\xi,\eta)$$

with boundary conditions

$$v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = 0,$$

where $\mu = \mu_0 + ia,\ a \neq 0.$ Then,

$$\|v\|_{C(\bar\Omega)} \le K\cdot\frac{\ln\mu_0}{\mu_0}\cdot\|f\|_{L_2(\Omega)}.$$

We use the statement of the following theorem to prove uniform convergence of the eigenfunction expansions of the biharmonic type operator in closed domain.

Theorem 2 For the eigenfunctions $\{v_n(\xi,\eta)\}$ of the biharmonic type operator, corresponding to the boundary conditions $v_n|_{\partial\Omega} = \Delta v_n|_{\partial\Omega} = 0$, we have

$$\sum_{|\sqrt[4]{\lambda_n}-\lambda|<1} v_n^2(\xi,\eta) = O(1)\lambda^5 \ln\lambda, \qquad \lambda\to\infty,$$

for all $(\xi,\eta)\in\overline{\Omega} = \Omega\cup\partial\Omega$.

Proof. Let $\Re_\mu(\xi,\eta,z,t)$ be a resolvent of the problem (1), (2). Then the solution of the equation

$$\left(\Lambda+\mu^4\right)v(\xi,\eta)=f(\xi,\eta),$$

can be represented as follows

$$v(\xi,\eta)=\int_\Omega \Re_\mu(\xi,\eta,z,t)f(z,t)dzdt,$$

where we assumend that the solution is subject to the following boundary conditions $v|_{\partial\Omega}=\Delta v|_{\partial\Omega}=0$
This representation for the solution of biharmonic type equation and estimation (3) give

$$\sqrt{\iint_\Omega \left|\Re_\mu(\xi,\eta,z,t)\right|^2 dzdt}\le K\cdot\frac{\ln\mu_0}{\mu_0}.$$

Let $v_n(\xi,\eta)$ denote the eigenfunction of the biharmonic-type operator Λ with the domain of definition $\mathcal{F}_\Omega$ corresponding to the eigenvalue λ_n:

$$\Lambda v_n(\xi,\eta)=-\lambda_n v_n(\xi,\eta)$$

Which can be written as follows

$$\left(\Lambda+\mu^4\right)v_n(\xi,\eta)=\left(\mu^4-\lambda_n\right)v_n(\xi,\eta),$$

Application of the resolvent leads us to the representation

$$v_n(\xi,\eta)=\left(\mu^4-\lambda_n\right)\int_\Omega \Re_\mu(\xi,\eta,z,t)v_n(z,t)dzdt.$$

Finally, using the Bessel's inequality we obtain the following

$$\sum_{|\sqrt[4]{\lambda_n}-\mu_0|<1} v_n^2(\xi,\eta)\le K\cdot\mu_0^6\cdot\sqrt{\iint_\Omega \left|\Re_\mu(\xi,\eta,z,t)\right|^2 dzdt}\le K\cdot\mu_0^6\cdot\frac{\ln\mu_0}{\mu_0}\le K\mu_0^5\ln\mu_0,\qquad \mu_0\to\infty.$$

The proof of the Theorem 2 is completed.

3. Conclusion

This research focuses on the estimation of the eigenfunctions of the biharmonic-type operators, which will be applied to solve the problems on the conditions for the convergence and summability of eigenfunctions expansions to biharmonic-type operator in closed domain. The biharmonic-type operator is the elliptic operator of order 2 with domain consists of classes of infinitely differentiable functions with compact support, which is a symmetric and nonnegative linear operator and has a self-adjoint extension. For domains with smooth boundary, the solution to these differential operator problems involves eigenfunction expansions associated with biharmonic-type operator with Navier boundary conditions.

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