COMPATIBLE ACTIONS FOR FINITE CYCLIC GROUPS OF *p*-POWER ORDER

(TINDAKAN SERASI BAGI KUMPULAN KITARAN BERPERINGKAT KUASA-*p*)

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IN THE NAME OF ALLAH, THE MOST GRACIOUS, THE MOST MERCIFULL

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ABSTRAK

Konsep hasil darab tensor kumpulan tak abelan berasal daripada teori K-algebra dan juga teori homotopi. Konsep ini ditakrifkan sebagai tindakan yang serasi antara satu sama lain. Pasangan tindakan serasi yang berbeza menghasilkan satu hasil darab tensor tak abelan yang berlainan. Bilangan maksimum hasil darab tensor tak abelan yang berbeza bergantung kepada bilangan pasangan tindakan yang serasi. Oleh itu, kajian ini memberi tumpuan bagi menentukan bilangan pasangan tindakan yang serasi antara dua kumpulan kitaran yang berperingkat kuasa-p di mana p adalah nombor perdana ganjil. Penyelidikan ini bermula dengan menentukan syarat-syarat perlu dan cukup untuk tindakan-tindakan yang berperingkat kuasa-p bertindak serasi. Seterusnya, bilangan automorfisma yang berperingkat kuasa-p dicari bagi kumpulan yang sedemikian, yang mana mewakili tindakan. Dengan menggunakan syarat-syarat perlu dan cukup, bilangan pasangan tindakan serasi telah dikenalpasti berdasarkan peringkat bagi tindakan. Tambahan pula, graf tindakan serasi dan ciri-cirinya adalah diperkenalkan bagi kumpulan yang sedemikian.

ABSTRACT

The concept of the nonabelian tensor product of groups has its origins in the algebraic K-theory and the homotopy theory. This concept is defined on the actions which are compatible to each other. A different compatible pairs of actions can give a different nonabelian tensor products. The maximum different nonabelian tensor product depends on the number of compatible pairs of actions. Thus, this research focuses on determining the number of compatible pairs of actions between two finite cyclic groups of p-power order, where p is an odd prime. This research starts with determining the necessary and sufficient conditions for the actions that have the p-power order to be compatible. Then, the number of the automorphisms that have the p-power order for such type of groups, which present the actions are found. By the necessary and sufficient conditions, the number of the compatible pairs of actions has been determined according to the order of the action. Furthermore, the compatible action graph and its subgraph were introduced for the finite cyclic groups of p-power order, where p is an odd prime. Then, some properties of the compatible action graph are also presented.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

1	identity element
$a \bmod b$	a modulo b
a^{-1}	inverse of a
$\langle a \rangle$	cyclic subgroup generated by a
$[a,b]$	commutator of a and b
$\mathrm{Aut}(G)$	automorphism group of group G
C_n	the cyclic group of order n
$\deg^{+}(v)$	the out-degree of a vertex v
$\deg^{-}(v)$	the in-degree of a vertex v
D_n	the dihedral group of order $2n$
$E(\Gamma_{p\ G\otimes H})$	the set of edges of compatible action graph
(g^k, h^l)	a pair of compatible actions for nonabelian tensor product of $G \otimes H$
G	a finite group G
$\|G\|$	the order of G
$G \otimes H$	the nonabelian tensor product of G and H
$G \otimes G$	the nonabelian tensor square of a group G
$G \wedge G$	the nonabelian tensor exterior product of a group G
$G \boxtimes H$	the box tensor product of groups G and H
$G \cong H$	the groups G and H are isomorphic
$G \setminus Z(G)$	the set of vertices of the non-commuting graph $\nabla(G)$
$G \setminus \{e\}$	the set of vertices of the non-coprime graph $\prod_G$
${}^{g}h$	an action of g on h
$H \leq G$	H is a subgroup of G
$\mathbb{N}$	natural number
Q_n	the quaternion group of order n
S_n	the symmetric group of order n
$t \mid s$	t divides s
$t \nmid s$	t not divides s
$V(\Gamma_{p\ G\otimes H})$	the set of vertices of compatible action graph
$Z(G)$	the center of a group G
$\mathbb{Z}$	set of integers
$\in$	element of

$\Gamma_{p\ G\otimes H}$	compatible action graph of $G \otimes H$
Γ_G^{Ω}	the orbit graph of the group G
Γ_G	the associate graph of the group G
$\Gamma_{1(G)}$	the prime graph of the group G
$\Gamma_{(G)}$	the commuting graph of the group G
$\nabla(G)$	the non-commuting graph of the group G
$\prod_G$	the non-coprime graph of the group G
Ω	the set of all subsets of all commuting elements of size two
Φ	a mapping from G to $\mathrm{Aut}(H)$
T	a mapping from H to $\mathrm{Aut}(G)$

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CHAPTER 1

INTRODUCTION

1.1 An Overview

This chapter is an introduction chapter to the whole thesis, which contains research background, problem statement, research objectives, research scope, research significance, and thesis organization.

1.2 Research Background

The concept of the nonabelian tensor product of groups was introduced by Brown and Loday (1984). This concept is defined for a pair of groups G and H, which acts on each other, providing the actions that satisfies the following compatibility conditions:

$$ {}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \quad \text{and} \quad {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h')) $$

for all $g, g' \in G$ and $h, h' \in H$. The originated of this concept was in connection with a generalized Van Kampen Theorem. Then, the structure of this concept has its origins in the algebraic K-theory and also in the homotopy theory. If G and H are groups that act compatibly on each other, then, the nonabelian tensor product $G \otimes H$ is a group generated by the symbols $g \otimes h$ with relations:

$$ gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h') $$

for all $g, g' \in G$ and $h, h' \in H$.

The research on the group theoretical aspects of the nonabelian tensor product was initiated by Brown and Loday (1987). They focused on the group theoretic properties, specifically on the computation of the nonabelian tensor square. Also, they gave a list of open problems concerning the nonabelian tensor product and the nonabelian tensor square which include a problem in the cyclic group. Thus, the open problems, which have been given by Brown and Loday (1987) were the motivation for many researchers as well as this research to investigate the group theoretical aspects of the nonabelian tensor product of groups.

The nonabelian tensor square $G \otimes G$ has been established by Brown and Loday (1984), which is finite for the finite group G. Then, Ellis (1987) extended the results to the nonabelian tensor product and he showed that the nonabelian tensor product is of p-power order if G and H are of the p-power order. McDermott (1998) computed the nonabelian tensor product $G \otimes H$ when G is a p-group and H is q-group, where p and q are prime numbers. Moreover, Visscher (1998) continued the study on the nonabelian tensor product of the p-power order and he focused on the cyclic groups. Mohamad (2012) studied the compatibility conditions and the nonabelian tensor product of the finite cyclic groups of the p-power order, where p is an odd prime as well as $p = 2$, and provided the necessary and sufficient conditions for the pair of two finite cyclic groups to act compatibly on each other. Then, Sulaiman et al. (2015) continued with Mohamad's work and focused only on the compatible pairs of nontrivial actions that have the 2-power order for the finite cyclic groups of 2-power order. Thus, this research is focusing on the case of the p-power order for such type of groups, where p is an odd prime in order to investigate the compatible pairs of nontrivial actions that have the p-power order. Consequently, the following table illustrates the different cases that have been discovered on the compatibility conditions for such type of groups by the previous works as well as this research.

Table 1.1 Different cases that have been discovered the compatibility conditions for the finite cyclic groups of p-power order by different researchers.

Authors and year	Finding
Visscher (1998)	Provided an action of p-power order that satisfying the compatibility conditions.
Mohamad (2012)	Used the order of the action as a condition for the actions to be compatible on each other.
Sulaiman (2017)	Focused only on the case that the actions have the 2-power order.
Mohammed (2018)	Provided new action which different from the previous results that satisfying the compatibility conditions for such type of groups.

1.3 Problem Statement

The nonabelian tensor product of groups is defined for a pair of groups, which acts on each other, such that the actions satisfying the compatibility conditions. According to the definition of the nonabelian tensor product, the pair of the actions are required to be compatible in order for the nonabelian tensor product to be computed. Thus, different compatible pairs of actions can give different nonabelian tensor product. Many researchers considered the nonabelian tensor product with trivial actions. However, only some of them are considered the nontrivial actions for computing the nonabelian tensor product. Others computed the nonabelian tensor product and the compatibility conditions of nontrivial actions for the finite cyclic groups of the p-power order, where p is an odd prime. In this research, the finite cyclic groups of the p-power order, where p is an odd prime are considered in order to find and prove the exact number of the compatible pairs of actions for the given nonabelian tensor product for such type of groups, which gives the maximum number of different nonabelian tensor product for any two finite cyclic groups of the p-power. Then, the results have been extended to introduce a types of graph, namely the compatible action graph and its subgraph for the nonabelian tensor product of such type of groups by studying the relationship between the group theory and graph theory to present all pairs of compatible actions as edges and the actions as vertices.

1.4 Research Objectives

The objectives of this study are:

(i) to determine the necessary and sufficient conditions for a pair of finite cyclic groups of the p-power order, where p is an odd prime to act compatibly on each other.

(ii) to find the number of the automorphisms of the finite cyclic groups of the p-power order, where p is an odd prime with a specific order.

(iii) to find the exact number of the compatible pairs of actions for the finite cyclic groups of the p-power order, where p is an odd prime.

(iv) to find the properties of the compatible action graph and the intersection of its subgraph for the finite cyclic groups of the p-power order, where p is an odd prime.

1.5 Research Scope

This research focused on the compatible actions, and the groups considered are limited to the finite cyclic groups of the p-power order, where p is an odd prime.

1.6 Research Significance

The contribution of this thesis is to provide a necessary and sufficient conditions on the pair of finite cyclic groups of the p-power order, which act compatibly on each other.

In addition, new results in determining the number of the compatible pairs of actions for the finite cyclic groups of the p-power order, where p is an odd prime are presented. Furthermore, the number of the automorphisms with a specific order for such type of groups have been determined by using some properties in number theory.

The results have been extended to introduce a new types of graph, which is called the compatible action graph and its subgraph by using the theoretical relationship between group theory and graph theory. Some properties of the compatible action graph and its subgraph are also provided.

1.7 Thesis Organisation

Chapter one presented as an introduction chapter to the whole thesis. This chapter contains research background, problem statement, research objectives, research scope, and research significance.

Chapter 2 focuses on the details of literature reviews on the compatibility conditions and the concept of the nonabelian tensor product of groups with some recent works had done on the relation between group theory and graph theory.

Some definitions and preliminary results of the automorphism groups, compatibility conditions, and graph theory are given in Chapter 3. By using Groups, Algorithms and Programming (GAP) software, the compatible actions and the number of the compatible pairs of actions are found. All results in this chapter are used in subsequent chapters to prove the new results.

Chapter 4 focuses on the automorphism and the compatible actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime. This chapter then included some properties of the automorphism of such type of groups and the necessary and sufficient conditions for a pair of actions that have *p*-power order to act compatibly on each other. Furthermore, some examples are presented when $G = H$ and $G \neq H$, to illustrate the compatible actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime.

The main results of this thesis are given in Chapter 5, which is divided into two parts. The first part, concerning the number of the automorphisms of the finite cyclic groups of the *p*-power order with the specific order, while the second part is concerning the number of the compatible pairs of actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime. The results illustrated that the number of the compatible pairs of nontrivial actions for a given nonabelian tensor product of such type of groups are equal.

Chapter 6 shows the connection between the group theory and the graph theory. In this chapter, a new graph, namely the compatible action graph and its subgraph have been presented and some theoretical properties of the compatible action graph for the finite cyclic groups of *p*-power order are given.

Lastly, Chapter 7 contains the summary of this research and some suggestions for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This chapter presents the details of the literature review on the compatible actions, the nonabelian tensor product of groups and some related results in graph theory.

2.2 Compatible Actions and The Nonabelian Tensor Product of Groups

The concept of the nonabelian tensor product of groups has been discussed since 1984. Brown and Loday in 1984 and 1987 were the researchers who introduced the concept of the nonabelian tensor product to extend the ideas of Whitehead (1950). This concept is defined by a pair of groups G and H, providing that the groups act on each other in such a way where the actions are satisfying the compatibility conditions. The paper by Brown et al. (1987) has motivated many researchers to investigate the group theoretical aspects of the nonabelian tensor product.

A study by Brown et al. (1987) focused on the group theoretic properties, precisely to compute the nonabelian tensor square. They also provided a list of open problems concerning the nonabelian tensor product and the nonabelian tensor square and one of the open problems concerning to the cyclic groups, which stated that whether the nonabelian tensor product of two cyclic groups is cyclic. Thus, our focus in this research is on the compatible actions without computing the nonabelian tensor product. Again, Brown et al. (1987) established that the nonabelian tensor square $G \otimes G$ is a finite for a finite G. In addition, they showed that the nonabelian tensor square of a nilpotent group is nilpotent.

In addition, some result on the nonabelian tensor product for both G and H with p-power order was proven. Furthermore, the computation of the nonabelian tensor square for groups of order up to 30 was given by using GAP programming. Meanwhile, Ellis (1987) extended the results for the nonabelian tensor product without any analytical proof. Furthermore, he shows that the nonabelian tensor product is of the p-power order if G and H are of the p-power order.

Gilbert and Higgins (1989) studied the concept of the nonabelian tensor product of groups and they found that there is an isomorphism from the subgroup $\left[G, H^{\varphi}\right]$ of $\eta(G,H)$ onto the nonabelian tensor product such that $\left[g, h^{\varphi}\right] \to g \otimes h$ for $g \in G$ and $h \in H$. This isomorphism is useful to study the nonabelian tensor product inside of $\eta(G,H)$. Two years later, Rocco (1991) gives a bound for the order of the nonabelian tensor square $G \otimes G$ if G has order p^n. Bacon and Kappe (1993) studied the nonabelian tensor square $G \otimes G$ and determined the nonabelian tensor square of 2-generator p-groups of nilpotency class 2, where p is an odd prime. In addition, they also showed that if G is a nilpotent group of class 2, then the nonabelian tensor square $G \otimes G$ is abelian.

Ellis and Leonard (1995) modified a method which can be used to compute the nonabelian tensor product $G \otimes H$ for all pairs of normal subgroups G and H of order up to 14. They also computed the nonabelian tensor square and Schur multiplier of Burnside groups, which are $B(2,4)$ and $B(3,3)$ of order 2^{12} and 3^{7}, respectively. However, they provided an alternative description for the nonabelian tensor product which stated that, there is an isomorphism $((G \otimes H) \rtimes H) \rtimes G \cong G * H/J$, where $\rtimes$ is the semi-direct product and J is the subgroup of $G * H$. Then, their results concluded that there is an isomorphism $G \otimes H \cong \overline{G} \cap \overline{H}$, where $\overline{G}$ and $\overline{H}$ are the normal closure in $G * H/J$ of G and H.

An overview on some of the developments on the nonabelian tensor product of groups since the appearance of the paper of Brown et al. (1987) with literature results up to 1997 was illustrated by Kappe (1997). After that, McDermott (1998) developed an algorithm to compute the nonabelian tensor product $G \otimes H$ and implemented the algorithm with the help of GAP software. Meanwhile, he also determined the order of the nonabelian tensor product $G \otimes H$ by using GAP software for all normal subgroup G and H of the quaternion group of order 32. In addition, he gave both the nonabelian tensor product of quaternion group and dihedral group of order eight and split them into two cases, such that the actions act compatibly on each other and the actions do not act compatibly on each other. Besides that, Ellis and McDermott (1998) improved the Rocco's bound in 1991 and extended it to the case of the nonabelian tensor product $G \otimes H$ of prime power groups G and H.

Extended from Ellis and McDermott's work, Visscher (1998) continued the study on the nonabelian tensor product of the p-power order and he focused on the cyclic groups. He clarified more descriptions of the action for the cyclic group of prime power order in the first part of his thesis before using the results to compute the nonabelian tensor product. Moreover, he computes some of the nonabelian tensor product of cyclic groups of the p-power order and presents a complete classification of all nonabelian tensor product of cyclic groups of 2-power order with mutual nontrivial actions of order two. In addition, Visscher (1998) gave the bounds on the nilpotency class and solvability length of $G \otimes H$, provided such information is given in context with G and H. The bounds are given in terms of $D_H(G)$, the derived subgroup of G afforded by the action of H on G, and $D_G(H)$, the analogue's subgroup of H. Furthermore, Visscher (1998) determined the characterisation of the compatibility condition and provided some necessary and sufficient number in theoretical conditions for a pair of cyclic groups of the p-power order, where p is an odd prime, as well as $p = 2$ to act compatibly with each other.

Nakaoka (2000) studied the nonabelian tensor product of solvable groups, and gave the description of the derived and the lower central series of $G \otimes H$. Besides that, Nakaoka (2000) obtained the bound for the order of $G \otimes G$ for a finite solvable group G. As a result, she obtained that there is an isomorphism from the subgroup $[G, H^{\varphi}]$ of $\eta(G, H)$ to $G \otimes H$ such that $[g, h^{\varphi}] \rightarrow g \otimes h$ for $g \in G$ and $h \in H$.

Nakaoka and Rocco (2001) studied the nonabelian tensor product for two groups, which are the nilpotent groups, where the actions act on each other in the nilpotently way. In addition, they also present that the nonabelian tensor square for finite group G is cyclic.

Besides that, Morse (2005) gives an overview and literature study on some of the developments and computation on the nonabelian tensor square of groups.

The nonabelian tensor product of polycyclic groups has been studied by Moravec (2007) and he showed that the nonabelian tensor product $G \otimes H$ is polycyclic where G and H are two polycyclic groups that act compatibly with each other. Besides that, Moravec (2008) proved that the exponent of the nonabelian tensor product of two locally finite groups can be bounded in terms of exponents in the given groups. He presented that the exponent of the nonabelian tensor square divides the exponent of *G*, when *G* is a group of nilpotent of the class ≤ 3 and of the finite exponent.

Blyth and Morse (2009) developed a theory for computing the nonabelian tensor square $G \otimes G$ and related computations for finitely presented groups, specialising on the polycyclic groups. The results gave the computations and the basis of an algorithm for computing the nonabelian tensor square for any polycyclic group. Meanwhile, Moravec (2009) studied the nonabelian tensor square for powerful *p*-groups. He provided some fundamental properties of nonabelian tensor square focuses on powerful *p*-groups such like, if *G* is powerful, then the exponent of $G \otimes G$ divides the exponent of *G*.

Thomas (2010) introduced a homology free proof that the nonabelian tensor product of two finite groups is finite, which gives an algebraic proof for the study by Ellis (1987). Besides that, he provided an explicit proof that the nonabelian tensor product of two finite *p*-groups is a finite *p*-group. Later on, Blyth et al. (2010) studied the nonabelian tensor square $G \otimes G$ for the class of group *G* and they characterised the exterior square $G \wedge G$ in terms of a presentation of *G*. They also applied the results to some classes of groups, such as the classes of free solvable and free nilpotent groups of finite rank, as well as some classes of the finite *p*-groups. Furthermore, Russo (2010) showed that the nonabelian tensor product of two Chernikov groups is Chernikov group. Then, Moravec (2010) introduced the notion of powerful action of a *p*-group upon another *p*-group. In addition, he derived some properties of powerful actions and studied faithful powerful actions. Then, he showed that the nonabelian tensor product of powerful *p*-groups acting powerfully and compatibly upon each other is again a powerful *p*-group.

Russo (2011) proved that if $G, H \in \chi$, then $G \otimes H \in \chi$, where χ represent a given classes of groups, such as the class of all finite groups, nilpotent groups, polycyclic groups, locally finite groups, and Chernikov groups. In addition, Thomas (2012) introduced a generalisation to the concept of the nonabelian tensor product, which is called the box-tensor product and denoted by $G \boxtimes H$. However, he extended various results for the concept of the nonabelian tensor product to the box-tensor product, such as the finiteness of the product when each factor is finite. Besides that, he showed that the finiteness of the box-tensor product $G \boxtimes H$ when both G and H are finite. Then, he also proved that $G \otimes H$ is finite if the mutual actions are half compatible.

Mohamad (2012) studied the concept of the nonabelian tensor product and focused on the finite cyclic groups. He proved that the nonabelian tensor product of finite cyclic groups of the p-power order are cyclic when p is an odd prime. Mohamad (2012) also showed that the nonabelian tensor product of cyclic groups of 2-power order with two-sided actions is also cyclic, when both actions have order greater than two. In addition, Mohamad et al. (2012) studied the computation of the nonabelian tensor product for cyclic groups of order p^2 where p is an odd prime. They provided the necessary and sufficient conditions for the finite cyclic groups of the p-power order that act on each other in the compatible ways where the order of the actions is included as one of the conditions. Moreover, they showed that the nonabelian tensor product of the finite cyclic groups of order p^2 is also cyclic when the actions have order p.

Next, Otera et al. (2013) investigated some algebraic and topological properties for the nonabelian tensor product in viewpoint of the classes of a group. Besides that, Rashid et al. (2013) studied the nonabelian tensor square and its capability, focusing on the groups of order $8q$ where q is an odd prime. They also computed the capability of the group using the Schur multiplier of the groups of order $8q$. Fauzi et al. in (2014) computed the nonabelian tensor square of Biebierbach group of dimension five with dihedral point group of order eight denoted by $B_l(5)$. They also proved that the nonabelian tensor square of the first Biebierbach group of dimension five with a dihedral point group of order eight can be generated by ten elements and they verified the results by using the GAP software.

Sulaiman et al. (2015) computed the exact number of compatible pairs of actions between the two cyclic groups of 2-power order. In addition, he used some necessary and sufficient number theoretical conditions for a pair of cyclic groups of 2-power order with nontrivial actions that act compatibly on each other to investigate some properties in finding the exact number of compatible pairs of actions. He also provided some results on the compatible pairs of nontrivial actions of order two and four. In the same year, Donadze et al. (2015) investigated the closure and the finiteness properties for the nonabelian tensor product of groups. They showed that some classes are closed under the formation of the nonabelian tensor product, such as solvable by finite, nilpotent by finite, polycyclic by finite, nilpotent of nilpotency class *n*, and super solvable groups.

Next, Shahoodh et al. (2016) computed the compatible pairs of nontrivial actions for two finite cyclic groups of 3-power order. Meanwhile, Jafari (2016) categorised the nonabelian tensor square for the finite *p*-groups by the order. In addition, he also computed the Schur multiplier and showed that $G \otimes G$ for the finite generalised extra special *p*-groups are not capable. In 2016, Russo studied the topology of the nonabelian tensor product of profinite groups. Then, he proved the nonabelian tensor products of projective limits of finite of such type of groups. In the same year, Sulaiman et al. determined the exact number of the compatible pairs of actions for the finite cyclic groups of 2-power order and he only focused for a case when one of the actions has an order greater than two. In 2016 Sulaiman et al. have studied the compatible pairs of the nontrivial actions for the finite cyclic groups of 2-power order. Ghorbanzadeh et al. (2017) investigated the nonabelian tensor square of *p*-groups of order p^4, then they obtained the Schur multiplier, exterior center and the tensor center of such type of groups. However, Mohamad et al. (2017) provided the exact number of the compatible pairs of nontrivial actions for the same cyclic groups of 2-power order with the actions that have the same order.

Previously, there were only three researchers, which focused on the finite cyclic groups, namely Visscher (1998), Mohamad (2012) and Sulaiman (2016). The first two researchers, which are Visscher (1998) and Mohamad (2012) investigated the nonabelian tensor product of the finite cyclic groups of the *p*-power order with the compatibility conditions and they provided some necessary and sufficient number of theoretical conditions for the two finite cyclic groups of the *p*-power order that the actions act on each other in a compatible way. Sulaiman et al. (2016) focused on the compatible pairs of actions for the finite cyclic groups of 2-power order with nontrivial actions. Then, they only covered for such type of groups with the actions that have 2-power order. In this research, we are interested in finding the number of the compatible pairs of actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime with mutual nontrivial actions that have the *p*-power order.

In the next section, the previous works in the relationship between group theory and graph theory are presented.

2.3 Some Relations Between Group Theory and Graph Theory

The study of an algebraic structure motivated many researchers to investigate the properties of the graphs such as Moghaddamfar et al. (2005) defined the non-commuting graph, which is denoted by $\nabla(G)$ and is defined as follows: the set of vertices of $\nabla(G)$ is $G \setminus Z(G)$ with two vertices x and y joined by an edge whenever the commutator of x and y is not the identity. They proved for some finite group G and H if $\nabla(G) \cong \nabla(H)$ then $|G| = |H|$. However, Abdollahi et al. (2006) studied the associate graph, which is called the non-commuting graph of G denoted by Γ_G where G is the nonabelian group and $Z(G)$ is the centre of G. As the results, some of the properties of the non-commuting graph are determined, such as Γ_G are Hamiltonian and planarity when G is an isomorphic to one of the groups S_3, D_8 or Q_8.

Iranmanesh and Jafarzadeh (2007) constructed some graphs, which are called the commuting graph, the non-commuting graph, and the prime graph of the group G, which are denoted respectively by $\Gamma_{(G)}, \nabla(G)$ and $\Gamma_{1(G)}$. In addition, they studied the relation between the commuting graph and the prime graph for the finite groups and they showed that if G is any finite group, such that $\Gamma_{(M)} \cong \Gamma_{(G)}$ then $M \cong G$, where M be a finite simple group. Zhang and Shi (2009) proved the conjecture *AAM*'s, which stated that, "If M is a finite nonabelian simple group and G is a group such that $\nabla(G) \cong \nabla(M)$, then $G \cong M$," is also true for some simple groups with the connected prime graph. This conjecture was provided by Abdollahi et al. (2006). In the same year, Darafsheh (2009) extended the work on the non-commuting graph of G by investigating the groups with the same non-commuting graph. Then, he provided that if $|G| = |H|$ then $G \cong H$. Besides that, he illustrated that the graph isomorphism $\Gamma_G \cong \Gamma_H$ implies $G \cong H$. Jahandideh et al. (2015) studied the conditions on the edges and vertices of the non-commuting graph.

Furthermore, they provided some properties of the non-commuting graph such as the number of the edges, which is denoted by $|E(\Gamma_G)|$, the degree of the vertex of the non-commuting graph and the number of the conjugacy class of the finite group.

In the connection between the group theory and graph theory, a paper by Mansoori et al. (2016) defined the non-coprime graph associated to the group G, which is denoted by $\prod_G$ where the vertex set is $G \setminus \{e\}$ and two distinct vertices are adjacent connected by the edge with the orders, relatively the non-prime. Besides that, they investigated some properties of the non-coprime graph for the nilpotent and abelian groups, and the relation between the non-coprime graph and known prime are presented. They determined the general properties of the non-coprime graph, such as diameter, girth, connectivity, Hamiltonian, independence number, domination number, and planarity when it is isomorphic to one of the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6$ or S_3. In the same year, Sarmin et al. (2016) computed the probability that an element of G denoted by the dihedral group of order $2n$ fixes the set Ω under the regular action where Ω is the set of all subsets, which of all commuting elements of size two in the form of (a,b) where a and b commute and $|a| = |b| = 2$. The results was obtained by applying the probability into the generalised conjugacy and orbit graph Γ_G^{Ω}. In addition, the properties of the graph, such as the chromatic number and the clique are determined. Also in the same year, Zamri et al. (2016) computed the probability that a group element fixes a set focused on the metacyclic 3-groups of negative type of nilpotency class at least three. By applying the orbit graph, the result was obtained. Then, the metacyclic 3-groups of negative type have been found by using the conjugate action.

From the literature some researchers defined the specific graph on the groups and studied the graph properties for the group, such as Jahandideh et al. (2015) and Mansoori et al. (2016). Thus, one of the main parts of this research is to investigate the theoretical relationship between group theory and graph theory. Therefore, we determine some properties of the compatible action graph and its subgraph for the finite cyclic groups of the p-power order, where p is an odd prime, and an extension from the compatible actions for such type of groups and the number of the compatible pairs of actions by representing the vertex as an automorphism and the edge as a compatible pairs of actions.

2.4 Conclusion

In this chapter, the literature on compatible actions, nonabelian tensor product of groups and graph theory are presented. Some researchers studied the compatibility conditions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime with nontrivial actions, but none of them stated the exact number of compatible pairs of actions for a given nonabelian tensor product for such type groups. Furthermore, some researchers have investigated the theoretical relationship between the group theory and the graph theory but none of them had studied the compatible action graph as an extension from the compatible actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime. Therefore, this research will focus on the number of the compatible pairs of actions. Therefore, some preliminary results on the automorphism groups, number theory, compatibility conditions, graph theory, and GAP software are stated in the following chapter.

CHAPTER 3

PRELIMINARY RESULTS

3.1 Introduction

This chapter presents the preliminary results of related past works, given by several researchers. This chapter contains some definitions and related works on the automorphism groups, number theory, compatibility conditions, graph theory and GAP software. By using GAP software, the number of the compatible pairs of actions are then determined. The results in this chapter will be used in proving the main results in the next chapters.

3.2 Some Properties of Automorphism Groups

It is well known that the actions are required to be compatible with each other before determining the nonabelian tensor product. Since the finite cyclic groups of the *p*-power order are considered in this research, then according to the definition of the compatible actions for the cyclic groups, the actions are automorphisms. Hence, an automorphism for such type of groups is introduced first.

Let G and H be the finite cyclic groups generated by a single element $g \in G$ and $h \in H$ respectively. Then, the automorphism group of the group G is denoted by $\mathrm{Aut}(G)$, which is defined as a mapping $\rho : G \to G$ such that $\rho(g) = g^t$ where t is an integer and $\gcd(t, |g|) = 1$. The automorphism group of the finite cyclic group of the *p*-power order is a direct product of two finite cyclic groups as given in the following theorem.

Theorem 3.1 (Dummit and Foote, 2004)

Let p be an odd prime and $\alpha \in \mathbb{N}$. If G is a cyclic group of order p^{α}, then

$\mathrm{Aut}(G) \cong C_{p-1} \times C_{p^{\alpha-1}} \cong C_{(p-1)p^{\alpha-1}}$ and $\left|\mathrm{Aut}(G)\right| = \varphi(p^{\alpha}) = (p-1)p^{\alpha-1}$.

The next theorem describes the isomorphism property for the cyclic groups.

Theorem 3.2 (Fraleigh, 2003)

Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $(\mathbb{Z},+)$. If G has finite order n, then G is isomorphic to $(\mathbb{Z}_n,+_n)$.

Now, the Euler Phi-function for a given positive integer is stated in the following definition.

Definition 3.1 Euler's $\boldsymbol{\varphi}$-function (Burton, 2005)

For $m \geq 1$, the Euler's Phi-function, denoted by $\varphi(m)$, is the number of the positive integers not exceeding m that are relatively prime with m.

The following theorem described the order of any power of any integer a, as stated bellow.

Theorem 3.3 (Burton, 2005)

If the integer a has order k modulo n and $h > 0$, then a^h has order $\left(\dfrac{k}{\gcd(h,k)}\right)$ modulo n.

3.3 The Compatibility Conditions

In this section, some definitions and previous results on the compatible conditions that are stated. We start with the definition of the action of the group G on the group H, which is given as follows.

Definition 3.2 Action (Visscher, 1998)

Let G and H be groups. An action of the group G on the group H is a mapping $\Phi : G \to \mathrm{End}(H)$ such that $\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$ for all $g, g' \in G$ and $h \in H$.

In the case of the groups G and H are finite cyclic groups, the action Φ of the group G on the group H is required to have the property $\Phi(1_G) = id_H$, such that it is the identity mapping on the group H. Therefore, from this point, the action will be a homomorphism Φ from the group G to the $\mathrm{Aut}(H)$.

Next, the definition of the compatible pairs of actions between the two groups is given.

Definition 3.3 Compatible Action (Brown and Loday, 1987)

Let G and H be the groups, which act on each other and each of which acts on itself by conjugation. Then these mutual actions are said to be compatible if

$$ {}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \tag{3.1} $$

and

$$ {}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h')) \tag{3.2} $$

for all $g, g' \in G$ and $h, h' \in H$.

In the case of the abelian groups, the compatibility conditions can be simplified and given in the following proposition.

Proposition 3.1 (Visscher, 1998)

Let G and H be groups, which act on each other. If G and H are abelian, then the mutual actions are compatible if and only if

$$ {}^{({}^{g}h)}g' = {}^{h}g' \text{ and } {}^{({}^{h}g)}h' = {}^{g}h' $$

for all $g, g' \in G$ and $h, h' \in H$.

Next, in order to prove the mutual actions of the groups G and H on each other are compatible, it is enough to show that the compatibility conditions is satisfied for the generators of the groups G and H. This case is presented in the following proposition.

Proposition 3.2 (Visscher, 1998)

Let $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$ be groups with generating sets X and Y and relations R and S, respectively. Furthermore, suppose G and H act on each other. If the compatibility conditions (3.1) and (3.2) hold for elements X and Y, then the mutual actions are compatible.

The following corollary showed that when G is abelian, then the trivial action is always compatible with any other action.

Corollary 3.1 (Visscher, 1998)

Let G and H be groups. Furthermore, let G act trivially on H. If G is abelian, then for any action of H on G, the mutual actions are compatible.

The next proposition gives the necessary and sufficient conditions for the actions of two finite cyclic groups to be compatible on each other.

Proposition 3.3 (Visscher, 1998)

Let $G = \langle x \rangle \cong C_{p^\alpha}$ and $H = \langle y \rangle \cong C_{p^\beta}$ be finite cyclic groups. Then there exist mutual actions of G and H on each other such that ${}^{y}x = x^k$ and ${}^{x}y = y^l$ for $k, l \in \mathbb{Z}$ if and only if the conditions (i) and (ii) below are satisfied. These actions are compatible if and only if condition (iii) is satisfied as well.

(i) $\gcd(k, p^\alpha) = \gcd(l, p^\beta) = 1$

(ii) $k^{p^\beta} \equiv 1 \pmod{p^\alpha}$ and $l^{p^\alpha} \equiv 1 \pmod{p^\beta}$

(iii) $k^{l-1} \equiv 1 \pmod{p^\alpha}$ and $l^{k-1} \equiv 1 \pmod{p^\beta}$.

For the case of two finite cyclic groups of the p-power order, where p is an odd prime, the following theorem stated the compatibility for the pair of the actions that have the p-power order.

Theorem 3.4 (Mohamad, 2012)

Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be groups where $\alpha, \beta \geq 3$. Furthermore, let $\sigma \in \mathrm{Aut}(G)$ with $|\sigma| = p^k$, where $k = 1, 2, \ldots, \alpha - 1$ and $\sigma' \in \mathrm{Aut}(H)$ with $|\sigma'| = p^{k'}$, where $k' = 1, 2, \ldots, \beta - 1$. Then (σ, σ') is a compatible pair of actions if and only if $k + k' \leq \min\{\alpha, \beta\}$.

The following corollary shows that for the finite cyclic groups of even order and with the actions that have order two, then the actions are always compatible.

Corollary 3.2 (Mohamad, 2012)

Let $G = \langle x \rangle \cong C_m$ and $H = \langle y \rangle \cong C_n$ where *m* and *n* are even integers with both actions of *y* on *x* and *x* on *y* having order two. Then, the actions are compatible.

The next lemma shows that if *G* and *H* are the finite cyclic groups of the *p*-power order, where *p* is an odd prime and each of which act on the other so that ${}^{x}y = y^{p+1}$ and ${}^{y}x = x^{p+1}$, then the actions are compatible with some conditions are fulfilled.

Lemma 3.1 (Mohamad, 2012)

Let $G = H \cong C_{p^2}$ be the finite cyclic groups with $G = \langle x \rangle$ and $H = \langle y \rangle$, $p > 2$, where ${}^{x}y = y^{p+1}$ and ${}^{y}x = x^{p+1}$, then the actions are compatible and the following conditions are hold.

$$ {}^{x^p}y = y \quad \text{and} \quad {}^{y^p}x = x \tag{3.3} $$

$$ {}^{x}y^p = y^p \quad \text{and} \quad {}^{y}x^p = x^p \tag{3.4} $$

In the next section, some fundamental concepts in graph theory are used to investigate the theoretical properties between the group theory and the graph theory.

3.4 Basic Properties on Graph Theory

Let *G* and *H* be finite cyclic groups. In order to investigate the compatible action graph for the nonabelian tensor product of *G* and *H*, we need to use some basic concepts in graph theory in order to define each action as a vertex and each pair of actions as an edge in the compatible action graph. Hence, in this section, some of these fundamental concepts that are needed in this research are given. These basic concepts can be found in (Rosen, 2012) and (Bollobas, 2013).

A graph G is a mathematical structure containing two sets, which are denoted by $V(G)$ and $E(G)$ which are called the set of the vertices and the set of the edges respectively. Then, the order of the graph G, is the number of the vertices in the graph G which is denoted by $|V(G)|$. Furthermore, a graph G is connected if there is a path between every pair of distinct vertices, and is disconnected otherwise. On the other hand, the graph G, is said to be complete if each ordered pair of the vertices are adjacent to each other and denoted by K_n, where n is the number of adjacent vertices.

Additionally, a simple graph G is called Bipartite graph, if its vertex set can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph G connects vertex in V_1 and vertex in V_2, and no edge in the graph G connects either two vertices in V_1 or two vertices in V_2. The directed graph, is the graph consist of the set of vertices and the set of directed edges, such that the directed edges are associated with the ordered pair (u,v) is said to start at u and end at v, where $u,v \in V$. Moreover, the degree of the vertex v in the directed graph has two types, the out-degree and the in-degree. The out-degree is denoted by $\deg^+(v)$ which is the number of the edges with v as their initial vertex, while the in-degree is the number of the edges with v as their terminal vertex which is denoted by $\deg^-(v)$. For the directed graph G, the path of the length n from u to v, where n is positive integer, is defined as a sequence of edges $e_1, e_2, ..., e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2) and so on, with e_n is associated with (x_{n-1}, x_n), where $x_0 = u$ and $x_1 = v$.

Next, the definition of the compatible action graph for the nonabelian tensor product of two finite cyclic groups of 2-power order is given as follows.

Definition 3.11 Compatible Action Graph (Sulaiman, 2017)

Let $G = \langle g \rangle$ and $H = \langle h \rangle$ be the two finite cyclic groups of 2-power order and (σ, σ') be the pair of the compatible actions for the nonabelian tensor product of $G \otimes H$ where $\sigma \in \mathrm{Aut}(G)$ and $\sigma' \in \mathrm{Aut}(H)$. Then, $\Gamma_{G\otimes H} = (V(\Gamma_{G\otimes H}), (E(\Gamma_{G\otimes H}))$ is a compatible action graph with the set of vertices $V(\Gamma_{G\otimes H})$, which is the nonempty set of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$, and the set of edges $E(\Gamma_{G\otimes H})$, which is the nonempty set of all compatible pair of actions.

The next section, some of the results on the finite cyclic groups of the *p*-power order, where *p* is an odd prime, by using the GAP software for computing the compatible pairs of actions for the finite cyclic groups of the *p*-power order.

3.5 The Groups, Algorithms and Programming (GAP) Software

The Groups, Algorithms and Programming (GAP) is a free software package for computation in discrete abstract algebra with particular emphasis on computational group theory (GAP, Version 4.8.8, 2017). The GAP software provides a programming language with many functions implementing algebraic algorithm written in the GAP language. The GAP programming is used in the research and teaching for studying groups and their representations, rings, vector spaces, algebras, and combinatorial structures. This system includes the source, which is free, can be easily modified and extended for a special use.

Next, let *G* and *H* be the finite cyclic groups of the *p*-power order, then, the GAP code in Figure 3.1, is used to create a conjecture for some of the main results of this research such as the number of the automorphisms with respective order and the number of the compatible pairs of actions when one of the actions is trivial which have been given in Chapter 5 and proved in Proposition 5.1 and Proposition 5.4. Furthermore, The input of GAP code in Figure 3.1, is the finite cyclic groups of the *p*-power order. Hence, the automorphisms for the finite cyclic groups of *p*-power order with their specific order which satisfies the compatibility conditions and the total number of the compatible pairs of nontrivial actions are found and presented in appendix A.

```
NumberCompatibleAction:=
function(m,n)
local k,l,ghg,hgh,a,b,x,y,p,q,z ;
z:=0;
for k in [2..m-1] do
    for l in [2..n-1] do
            a:=k;
        b:=l;
            if Gcd(m, k)=1 and Gcd(n, l)=1 then
                for x in [1..m] do
                        if a<>1 then
                                a:=k^x mod m;
                        fi;
                        if a=1 then
                                p:=x;
                                break;
                        fi;
                od;
                for y in [1..n] do
                        if b<>1 then
                                b:=l^y mod n;
                        fi;
                        if b=1 then
                                q:=y;
                                break;
                    fi;
                    od;
        fi;
ghg:=k^l mod m;
hgh:=l^k mod n;
if ghg=k and hgh=l then
z:=z+1;
Print("k=",k," (order action=",p,")",",l=",l," (order action=",q,")");
Print(" Compatible","\n");
fi;
od;
od;
Print("No of Compatible",z);
end;
```

Figure 3.1 GAP Coding for The Number of Compatible Actions

Next, by using Theorem 5.1 in Chapter 5, the exact number of the compatible pairs of actions that have the p-power order for the finite cyclic groups of the p-power order, where p is an odd prime are given in Table 3.1. Hence, Table 3.1, shows the exact number of the compatible pairs of actions for $C_{p^{\alpha}} \otimes C_{p^{\beta}}$ with the same prime p and $\alpha, \beta \in \mathbb{N}$.

Table 0.1 The Number of Compatible Pairs of Actions for $C_{p^{\alpha}} \otimes C_{p^{\beta}}$.

p	α	β	No. of Pairs	p	α	β	No. of Pairs	p	α	β	No. of Pairs
	3	3	54		4	3	54		5	3	54
	3	4	90		4	4	226		5	4	226
	3	5	198		4	5	334		5	5	810
3	3	6	522	3	4	6	658	3	5	6	1134
	3	7	1494		4	7	1630		5	7	2106
	3	8	4410		4	8	4546		5	8	5022
	3	9	13158		4	9	13294		5	9	13770
	3	3	300		4	3	300		5	3	300
	3	4	700		4	4	2000		5	4	2000
	3	5	2700		4	5	4000		5	5	12500
5	3	6	12700	5	4	6	14000	5	5	6	22500
	3	7	62700		4	7	64000		5	7	72500
	3	8	312700		4	8	314000		5	8	322500
	3	9	1562700		4	9	1564000		5	9	1572500
	3	3	882		4	3	882		5	3	882
	3	4	2646		4	4	8232		5	4	8232
	3	5	14994		4	5	20580		5	5	72030
7	3	6	101430	7	4	6	107016	7	5	6	158466
	3	7	706482		4	7	712068		5	7	763518
	3	8	4941846		4	8	4947432		5	8	4998882
	3	9	34589394		4	9	34594980		5	9	34646430

Example 3.1

Let $G = C_{3^3}$ and $H = C_{3^4}$ be the two finite cyclic groups of 3-power order. Table 3.1, illustrates that there are 90 compatible pairs of actions for $G \otimes H$ whereas there are 54 compatible pairs of actions for $H \otimes G$. Since $G \otimes H \neq H \otimes G$, therefore, the number of the compatible pairs of actions for the finite cyclic groups of the p-power for the given nonabelian tensor product $G \otimes H$ and $H \otimes G$ is not necessary equal when $G \neq H$. Only the case that all the actions are nontrivial it will be the same number for $G \otimes H$ and

$H \otimes G$ for the actions that have the p-power order for the finite cyclic groups of the p-power order, where p is an odd prime.

3.6 Conclusion

In this chapter, all related results by the previous researchers were given. The GAP software have been used to find the compatible actions. The GAP outputs can provide the compatible actions with their orders for the finite cyclic groups of the p-power order.

CHAPTER 4

AUTOMORPHISM AND THE COMPATIBLITY CONDITIONS

4.1 Introduction

In this chapter, the necessary and sufficient conditions of the compatible mutual actions for a pair of finite cyclic groups of the *p*-power order, where *p* is an odd prime to act compatibly with each other are provided. Some results on the automorphism of the finite cyclic groups of the *p*-power order are found and presented before characterising the compatible mutual actions.

4.2 Characterisation of an Automorphism for Cyclic Groups of *p*-Power Order

The compatible actions are important before computing the nonabelian tensor product of groups. According to Definition 3.3, the actions are automorphisms for the finite cyclic groups. In this section, some properties of the automorphism that have the *p*-power order for the finite cyclic groups of the *p*-power order, where *p* is an odd prime are given. We start with the following number theory result.

Lemma 4.1

Let *p* be an odd prime number. Then

$$(2p+1)^{p^{n-2}} = a_n 2p^n + 2p^{n-1} + 1$$

for some integer a_n where $n \geq 2$.

Proof:

The proof is by the induction on n. For $n=2$, the statement is true by letting $a_2=0$. Next, assume that the statement is true for some $n>2$. By Corollary of Fermat's theorem (Burton,PP,88, 2005), observe that

$$\begin{aligned}(2p+1)^{p^{(n-2)+1}} &= \left((2p+1)^{p^{n-2}}\right)^{p}\\ &= \left(a_n 2p^n + 2p^{n-1}+1\right)^p\\ &= \left((a_n p+1)2p^{n-1}+1\right)^p.\end{aligned}$$

Now, by using the Binomial theorem, we have

$$\begin{aligned}\left((a_n p+1)2p^{n-1}\right)^p &= \left((a_n p+1)2p^{n-1}\right)^p + \binom{p}{1}\left((a_n p+1)2p^{n-1}\right)^{p-1}+\cdots\\ &\quad + \binom{p}{p-1}(a_n p+1)2p^{n-1}+1.\\ &= \left((a_{n+1}p+1)2p^{n-1}\right)^p + \binom{p}{1}\left((a_{n+1}p+1)2p^{n-1}\right)^{p-1}+\cdots\\ &\quad + \left(a_{n+1}2p^{n+1}+2p^n\right)+1.\end{aligned}$$

Without loss of generality, let

$$\left((a_{n+1}p+1)2p^{n-1}\right)^p + \binom{p}{1}\left((a_{n+1}p+1)2p^{n-1}\right)^{p-1}+\cdots+\left(a_{n+1}2p^{n+1}+2p^n\right)+1 = K2p^{n+1}$$

except the term $\left(a_{n+1}2p^{n+1}+2p^n\right)+1$ for some integer K. Then, we have

$$\begin{aligned}&K2p^{n+1}+a_{n+1}2p^{n+1}+2p^n+1\\ &= a_{n+1}2p^{n+1}+2p^n+1,\end{aligned}$$

where $a_{n+1}=K+a_n$ and K is some integer. Thus, the claim is true for $n+1>2$. By the principle of the mathematical induction, it follows that the claim is true for all $n\geq 2$. □

The automorphism group of the finite cyclic groups of the p-power order is the direct product of the two finite cyclic groups as given in Theorem 3.1. Theorem 3.1 also proved that, the automorphism of the p-power order is isomorphic to the group $C_{p^{\alpha-1}}$ where $\alpha\in\mathbb{N}$. Thus, the generator of the finite cyclic groups of the p-power order that give the order of the automorphisms that have the *p*-power order need to be found and is given in the following theorem.

Theorem 4.1

Let $G=\langle g\rangle \cong C_{p^{\alpha}}$ be a group with p is an odd prime and $\alpha \geq 2.$ Then, $\sigma : \mathrm{g} \rightarrow \mathrm{g}^{2p+1}$ is an automorphism of order $p^{\alpha-1}.$

Proof:

Let $G=\langle g\rangle \cong C_{p^{\alpha}}$ with ,1.3By Theorem $\alpha \geq 2.$ an odd prime and be p , $|G| = p^{\alpha}$

$\mathrm{Aut}(\mathrm{C}_{p^{\alpha}}) \cong C_{p-1} \times C_{p^{\alpha-1}} \cong C_{(p-1)p^{\alpha-1}}$ and $\left|\mathrm{Aut}(\mathrm{C}_{p^{\alpha}})\right| = (p-1)p^{\alpha-1}.$ To prove our claim that σ is of order $p^{\alpha-1},$ it can be shown as follows:

(i) $(2p+1)^{p^{\alpha-1}} \equiv 1 \pmod{p^{\alpha}},$

(ii) $(2p+1)^{p^{\alpha-2}} \not\equiv 1 \pmod{p^{\alpha}}.$

By Lemma 4.1, if the equation is raised to the power of *p*, then

$$(2p+1)^{p^{\alpha-1}} = a_{\alpha+1}2p^{\alpha+1} + 2p^{\alpha} + 1 \equiv 1 \pmod{p^{\alpha}}$$

which hold for all $\alpha \geq 2.$ Hence, (ii) follows by Lemma 4.1. This implies that $\sigma^{p^{\alpha-1}}(g) = g$ and $\sigma^{p^{\alpha-2}}(g) \neq g$ which is a proof that σ is an automorphism of order $p^{\alpha-1}.$

□

The generators of the automorphism groups of the *p*-power order are important because they explain the structure of the automorphisms of the finite cyclic groups of the *p*-power order, which is defined as an action for the nonabelian tensor product. If *G* is a finite cyclic group of the *p*-power order generated by the single element, $g \in G,$ then any automorphism of the group *G* is given by the mapping $\rho : g \rightarrow g^{t},$ where *t* is an integer with $\gcd(t, p^{\alpha}) = 1.$

Therefore, the general descriptions of the integer *t* for every automorphisms of the finite cyclic groups of *p*-power order cannot be determined in this research because we have seen from Theorem 3.1 that $\mathrm{Aut}(\mathrm{C}_{p^{\alpha}}) \cong C_{p-1} \times C_{p^{\alpha-1}}.$ We have tried to provide the general presentation of an automorphism of the finite cyclic groups of the *p*-power order but there is no general pattern. Thus, our focuses in this research is the second part of the direct product of an automorphism group of the p-power order. In addition, we can determine the $\gcd(t-1, p^{\alpha})$, which is the order of the automorphisms of the finite cyclic groups of the p-power order that have the order of the p-power. The immediate result on the $\gcd(t-1, p^{\alpha})$ for the automorphisms of the finite cyclic groups of the p-power order that have order $p^{\alpha-k}$ with $k=1,2,\ldots,\alpha-1$ is given in the following proposition.

Proposition 4.1

Let σ be an automorphism of a group $C_{p^{\alpha}} = \langle g \rangle$ of order $p^{\alpha-k}$ with $k=1,2,\ldots,\alpha-1$. Then $\sigma = \rho^{l}$ with $\rho(g) = g^{2p+1}$ where $\gcd(l, p^{\alpha}) = p^{k-1}$ with l is positive integer. Furthermore, $\gcd(t-1, p^{\alpha}) = p^{k}$, where $t = (2p+1)^{l}$.

Proof:

Let σ be an automorphism of a group $C_{p^{\alpha}}$ of order $p^{\alpha-k}$ with $k=1,2,\ldots,\alpha-1$. By Theorem 4.1, $\rho(g) = g^{2p+1}$ is an automorphism of order $p^{\alpha-1}$. Thus, $\sigma = \rho^{l}$ generates all automorphisms of p-power order. If $l \nmid p^{\alpha}$, then clearly $\gcd(l, p^{\alpha}) = 1$. Consider $l \mid p^{\alpha}$, then $l \mid p^{\alpha-k}$. Thus $\gcd(l, p^{\alpha}) = p^{k-1}$ by Theorem 3.3. Let $t = (2p+1)^{l}$, then $\gcd(t-1, p^{\alpha}) = \gcd((2p+1)^{l} - 1, p^{\alpha}) = p^{k}$. □

Next, the following corollary contains a pair of number theoretic results, where the proof can be found in Bacon (1992). The purpose of this corollary is to rewrite the action of p-power order in other form, in order to satisfying the compatibility conditions for the finite cyclic groups of p-power order where p is an odd prime.

Corollary 4.1

Let $p, r, \gamma, M \in \mathbb{N}$ where p is an odd prime and $\gcd(p, r) = 1$. For $n \in \mathbb{N}$, denote with $[n]_p$, the highest p-power dividing n. Then $\left[(2p+1)^{rp^{\gamma}} - 1\right]_p = Mp^{\gamma+1}$.

The next lemma, give a description of the form of the action of one finite cyclic group of the p-power order upon the other.

Lemma 4.2

Let $G=\langle g\rangle \cong C_{p^{\alpha}}$ and $H=\langle h\rangle \cong C_{p^{\beta}}$ be finite cyclic groups of the *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. If G acts on H, then there exist $l \in \mathbb{Z}$ so that ${}^{g}h=h^{l}$ and $l=(2p+1)^{kp^{\gamma}}$ where $k, \gamma \in \mathbb{N}$ such that $\gcd(k, p)=1$ and $\max(1, \beta-\alpha-1) \leq \gamma \leq \beta-1$.

Proof:

Since G acts on *H*, then there exist an action $\Phi: G \rightarrow \operatorname{Aut}(H)$. By Theorem 3.1, $\operatorname{Aut}(C_{p^{\beta}}) \cong C_{p-1} \times C_{p^{\beta-1}} \cong C_{(p-1) p^{\beta-1}}$ and by Theorem 4.1, the direct factor $C_{p^{\beta-1}}$ is generated by the automorphism $\varphi: H \rightarrow H$ which is defined by $\varphi(h)=h^{(2p+1)}$.

Now, since the action $\Phi: G \rightarrow \operatorname{Aut}(H)$ is homomorphism, then $\Phi(G)$ is a cyclic subgroup of $\operatorname{Aut}(H)$ of the *p*-power order. Thus, $\Phi(g)=\varphi^{kp^{\gamma}}$ for some $k, \gamma \in \mathbb{N}$ with $\gcd(k, p)=1$. Since $|\varphi|=p^{\beta-1}$ it follows that $\gamma \leq \beta-1$. Again, since $|\varphi|=p^{\beta-1}$ we obtain $\alpha+\gamma \geq \beta-1$ or equivalently $\gamma \geq \beta-\alpha-1$. Since γ is positive integer, so we have the bound $\max(1, \beta-\alpha-1) \leq \gamma \leq \beta-1$. Finally, with $l=(2p+1)^{kp^{\gamma}}$, we have

$${}^{g}h=\Phi(g)(h)=\varphi^{kp^{\gamma}}(h)=h^{(2p+1)^{kp^{\gamma}}}=h^{l}. \qquad \square$$

The next section is the necessary and sufficient number theoretical conditions for the pair of finite cyclic groups of the *p*-power order, where *p* is an odd prime to act compatibly on each other are presented.

4.3 The Necessary and Sufficient Conditions for The Compatible Actions

In this section, the necessary and sufficient conditions for the compatible mutual actions for the pair of the finite cyclic groups of the *p*-power order to act compatibly on each other are given. The characterisation has been developed according to the necessary and sufficient conditions. This characterisation includes the new generator for the finite cyclic groups of the *p*-power order, which makes a difference with the previous results obtained by Mohamad (2012). The characterisation is presented in the following theorem.

Theorem 4.2

Let $G = \langle g \rangle \cong C_{p^\alpha}$ and $H = \langle h \rangle \cong C_{p^\beta}$ be finite cyclic groups of the *p*-power order, where *p* is an odd prime and $\alpha, \beta \geq 3.$ Furthermore, let G and H act on each other so that

$${}^{h}g = g^{(2p+1)^{kp^\gamma}} \qquad \text{and} \qquad {}^{g}h = h^{(2p+1)^{lp^\delta}}$$

for $k, l, \gamma, \delta \in \mathbb{N}$ with $\gcd(k, p) = \gcd(l, p) = 1.$ Then, G and H act compatibly on each other if and only if $\gamma + \delta \geq \max\{\alpha - 2, \beta - 2\}.$

Proof:

Let G and H be cyclic groups of *p*-power order and each of which act on the other such that

$${}^{h}g = g^{(2p+1)^{kp^\gamma}} \qquad \text{and} \qquad {}^{g}h = h^{(2p+1)^{lp^\delta}} \tag{4.1}$$

for each $k, l, \gamma, \delta \in \mathbb{N}$ with $\gcd(k, p) = \gcd(l, p) = 1.$ By Lemma 4.2, the bounds of γ and δ are $\max(\alpha - \beta - 1) \leq \gamma \leq \alpha - 1$ and $\max(\beta - \alpha - 1) \leq \delta \leq \beta - 1$ respectively.

Next, let $\Phi : G \to \mathrm{Aut}(H)$ and $\mathrm{T} : H \to \mathrm{Aut}(G)$ be the actions of G and H on each other. The actions can be written as follows:

$$\mathrm{T}(h)(g) = g^{(2p+1)^{kp^\gamma}} \tag{4.2}$$

$$\Phi(g)(h) = h^{(2p+1)^{lp^\delta}} \tag{4.3}$$

By Proposition 3.2, the mutual actions are compatible if and only if the compatibility conditions are satisfied on the generators of the groups G and $H.$ Since the groups G and H are abelian, by Proposition 3.1, then the mutual actions are compatible if and only if ${}^{{}^{g}h}g = {}^{h}g$ and ${}^{{}^{h}g}h = {}^{g}h.$ Therefore, by the notation of Φ and T for the actions of G and H act compatibility on each other if and only if

$$\mathrm{T}\big(\Phi(g)(h)\big)(g) \ = \ \mathrm{T}(h)(g). \tag{4.4}$$

$$\Phi\big(\Gamma(h)(g)\big)(h) \ = \ \Phi(g)(h) \tag{4.5}$$

By using Eq (4.3), we obtain for the right-hand side of Eq (4.4),

$$\mathrm{T}\big(\Phi(g)(h)\big)(g) \ = \mathrm{T}(h^{(2p+1)^{lp^\delta}})(g).$$

Combining Eq (4.2) and the fact that T is a homomorphism yields the following congruence between the exponents of g in Eq (4.4)

$$\left((2p+1)^{kp^{\gamma}}\right)^{(2p+1)^{lp^{\delta}}} \equiv (2p+1)^{kp^{\gamma}} \pmod{p^{\alpha}}.$$

By Corollary 4.1, we have $(2p+1)^{lp^{\delta}} = Mp^{\delta+1}+1,$ where $M \in \mathbb{Z}$ such that $\gcd(M,p)=1.$ Thus,

$$\begin{aligned}\left((2p+1)^{kp^{\gamma}}\right)^{(2p+1)^{lp^{\delta}}} &= \left((2p+1)^{kp^{\gamma}}\right)^{Mp^{\delta+1}+1} \\ &= (2p+1)^{kMp^{\gamma+\delta+1}}(2p+1)^{kp^{\gamma}}.\end{aligned}$$

Again, by using Corollary 4.1, $(2p+1)^{kMp^{\gamma+\delta+1}} = Np^{\gamma+\delta+2}+1,$ where $N \in \mathbb{Z}$ such that $\gcd(N,p)=1.$ Thus,

$$\begin{aligned}\left((2p+1)^{kp^{\gamma}}\right)^{(2p+1)^{lp^{\delta}}} &= (Np^{\gamma+\delta+2}+1)(2p+1)^{kp^{\gamma}} \\ &= Np^{\gamma+\delta+2}(2p+1)^{kp^{\gamma}}+(2p+1)^{kp^{\gamma}} \\ &= (2p+1)^{kp^{\gamma}} \pmod{p^{\alpha}}.\end{aligned}$$

Now, notice that $\gcd(p,(2p+1)^{kp^{\gamma}})=1$ and recall that $\mathrm{T}(h)(g)=g^{(2p+1)^{kp^{\gamma}}}$, thus we have

$$\mathrm{T}\big(\Phi(g)(h)\big)(g) = g^{Kp^{\gamma+\delta+2}}\,\mathrm{T}(h)(g),$$

where $K \in \mathbb{Z}$ such that $\gcd(K,p)=1.$ Thus, Eq (4.4) is holds if and only if $g^{Kp^{\gamma+\delta+2}} = 1_{\langle g\rangle}.$ Since $|g| = p^{\alpha}$, this holds if and only if $\gamma+\delta+2 \geq \alpha,$ or equivalently $\gamma+\delta \geq \alpha-2.$ Similarly Eq (4.5) holds if and only if $\gamma+\delta \geq \beta-2.$ We conclude that G and H act compatibility on each other if and only if $\gamma+\delta \geq \max\{\alpha-2,\beta-2\}.$ □

The following corollary focuses on the special case in which the groups G and H act compatibly on each other when the actions on each other are $2p+1.$

Corollary 4.2

Let $G = \langle g \rangle$ and $H = \langle h \rangle$ be the finite cyclic groups of the *p*-power order with *p* is an odd prime and $G = H \cong C_{p^2}$. Furthermore, let the actions of *G* and *H* and on each other such that ${}^{g}h = h^{2p+1}$ and ${}^{h}g = g^{2p+1}$. Then, the actions are compatible.

Proof:

Let *G* and *H* be the finite cyclic groups of the *p*-power order. Let the actions of *G* and *H* on each other as in the hypothesis. By Lemma 3.1, it follows that the actions are compatible. □

In the next section, some examples on the compatibility conditions

4.4 Some Examples on The Compatible Actions

In this section, some examples of the same and different groups are presented to clarify the characterisation of the compatible mutual actions for the finite cyclic groups of the *p*-power order. By using Theorem 4.1, the compatible pair of actions for such type of groups have been determined when $G = H$ and $G \neq H$, then summarized in Table 4.1 and Table 4.2 respectively. Recall that, if $G = H \cong C_{3^3}$ be the finite cyclic groups of 3-power order, then the compatible pairs of actions have been determined as follows.

By Theorem 4.1, the actions on each other such that ${}^{g}h = h^{(2p+1)^{kp^{\gamma}}}$ and ${}^{h}g = g^{(2p+1)^{lp^{\delta}}}$. Furthermore, let $k = l = 1$, then $\gcd(1,3) = \gcd(1,3) = 1$. Now, since $\alpha = \beta = 3$, then by Lemma 4.2, we have the bounds for γ and δ are as follows.

$\max(1,3\text{-}3\text{-}1) \leq \gamma \leq 3-1$ and $\max(1,3\text{-}3\text{-}1) \leq \delta \leq 3-1$ or equivalently $1 \leq \gamma \leq 2$ and $1 \leq \delta \leq 2$. Now, since $\gamma = \delta = 1$, then the actions on each other can be written as ${}^{g}h = h^{(7)^3} = h^{19}$ (mod 3^3) and ${}^{h}g = g^{(7)^3} = g^{19}$ (mod 3^3). By Proposition 3.1, the actions are compatible if ${}^{{}^{g}h}g = {}^{h}g$ and ${}^{{}^{h}g}h = {}^{g}h$. Now, consider the first compatibility conditions.

$$
\begin{aligned}
{}^{{}^{g}h}g &= {}^{h^{19}}g \\
&= g^{19^{19}} \\
&= g^{19} \qquad \text{since } 19^{19} \equiv 19 \pmod{3^3} \\
&= {}^{g}h.
\end{aligned}
$$

Similarly with the second compatibility conditions. Thus, we conclude that the actions ${}^{g}h = h^{19}$ and ${}^{h}g = g^{19}$ are compatible in $C_{3^3} \otimes C_{3^3}$. Next, if $G \neq H$, then by the similar way, the compatible pair of actions for the finite cyclic groups of *p*-power order have been determined and presented in Table 4.2.

Table 0.1 Compatible Pairs of Actions for Cyclic Groups of *p*-Power Order when $G = H$.

Groups	p	k	γ	g	p	l	δ	h	(g,h)
$G = H = C_{3^3}$	3	1	1	19	3	1	1	19	(19,19)
	3	4	1	10	3	1	2	1	(10,1)
	3	5	2	1	3	7	1	19	(1,19)
	3	8	2	1	3	7	2	1	(1,1)
$G = H = C_{5^3}$	5	1	1	51	5	1	1	51	(51,51)
	5	2	1	101	5	2	2	1	(101,1)
	5	6	2	1	5	8	1	26	(1,26)
	5	12	2	1	5	16	2	1	(1,1)
$G = H = C_{7^3}$	7	1	1	99	7	1	1	99	(99,99)
	7	2	2	1	7	4	1	50	(1,50)
	7	6	1	246	7	8	2	1	(246,1)
	7	10	2	1	7	12	2	1	(1,1)
$G = H = C_{11^3}$	11	1	2	1	11	1	1	243	(1,243)
	11	2	1	485	11	4	1	969	(485,969)
	11	3	2	1	11	2	1	485	(1,485)
	11	6	2	1	11	10	2	1	(1,1)
$G = H = C_{13^3}$	13	1	1	339	13	1	1	339	(339,339)
	13	4	1	1353	13	2	2	1	(1353,1)
	13	6	2	1	13	4	1	1353	(1,1353)
	13	10	2	1	13	8	2	1	(1,1)
$G = H = C_{17^3}$	17	1	1	579	17	1	1	579	(579,579)
	17	2	1	1157	17	4	2	1	(1157,1)
	17	6	2	1	17	2	1157	1	(1,1157)
	17	8	2	1	17	10	2	1	(1,1)
$G = H = C_{19^3}$	19	1	1	723	19	1	1	723	(723,723)
	19	2	2	1	19	4	2	1	(1,1)
	19	4	1	2889	19	6	2	1	(2889,1)
	19	8	2	1	19	10	1	362	(1,362)

Table 0.2 Compatible Pairs of Actions for Cyclic Groups of p-Power Order when $G \neq H$.

Groups	p	k	γ	g	p	l	δ	h	(g,h)
$G = C_{3^3}$	3	1	1	19	3	1	1	19	(19,19)
and	3	2	2	1	3	4	1	73	(1,73)
$H = C_{3^4}$	3	4	1	19	3	2	3	1	(19,1)
	3	8	2	1	3	6	2	28	(1,1)
$G = C_{5^3}$	5	1	1	51	5	1	1	426	(51,426)
and	5	2	2	1	5	4	1	451	(101,451)
$H = C_{5^4}$	5	4	1	76	5	2	2	501	(76,501)
	5	8	2	1	5	6	3	1	(1, 1)
$G = C_{7^3}$	7	1	2	1	7	1	2	687	(1,687)
and	7	2	1	197	7	4	1	50	(197,50)
$H = C_{7^4}$	7	10	1	295	7	8	1	99	(295,99)
	7	4	2	1	7	5	3	1	(1,1)
$G = C_{11^3}$	11	1	2	1	11	1	1	12222	(1,12222)
and	11	2	1	485	11	4	2	10649	(485,10649)
$H = C_{11^4}$	11	6	1	122	11	8	2	6656	(122,6656)
	11	3	2	1	11	2	3	1	(1,1)
$G = C_{13^3}$	13	1	2	1	13	1	2	4395	(1,4395)
and	13	8	1	508	13	2	1	20450	(508,20450)
$H = C_{13^4}$	13	2	1	677	13	4	3	1	(677,1)
	13	6	2	1	13	3	1	8789	(1,16394)
$G = C_{17^3}$	17	1	1	579	17	1	2	9827	(579,9827)
and	17	6	2	1	17	2	3	1	(1,1)
$H = C_{17^4}$	17	8	1	4625	17	3	3	1	(4625,1)
	17	10	1	868	17	6	1	28034	(868,28034)
$G = C_{19^3}$	19	1	1	723	19	1	2	13719	(723,13719)
and	19	2	2	1	19	3	3	1	(1,1)
$H = C_{19^4}$	19	8	1	5777	19	2	3	1	(5777,1)
	19	4	2	1	19	10	1	362	(1,362)

In the next section, some related results on the compatible actions are discussed.

4.5 Compatible Actions That Have Even Order of Actions

In this section, more results on the compatible actions for the finite cyclic groups are presented. Thus, the following proposition shows that if the groups of even order that act on each other with both actions have order two, then, the actions are compatible.

Proposition 4.2

Let $G = \langle g \rangle \cong C_{p-1}$ and $H = \langle h \rangle \cong C_{q-1}$, where p and q are different prime numbers that are greater than three with actions of g on h and h on g are having order two. Then, the actions are compatible.

Proof:

Let $G = \langle g \rangle \cong C_{p-1}$ and $H = \langle h \rangle \cong C_{q-1}$, where p and q are different prime numbers that are greater than three with actions of g on h and h on g having order two and given by

$$ {}^{g}h = h^{l} \text{ and } {}^{h}g = g^{k} $$

where l and k are positive integers. We need to prove that the actions satisfy the compatible conditions as stated in Proposition 3.1. Since G and H are groups of even order then the values of l and k must be odd since $\gcd(q-1,l) = \gcd(p-1,k) = 1$ for the automorphisms. Hence, $l = 2s+1$ and $k = 2t+1$ for the positive integers s and t. Thus, $l \equiv 1 (\text{mod})\ p-1$ and $k \equiv 1$ (mod) $q-1$. Since the actions have order two, it follows Corollary 3.2 that ${}^{{}^{g}h}g = {}^{h}g$ and ${}^{{}^{h}g}h = {}^{g}h$. Thus, the actions always act compatibly if they have order two. □

The following corollary is the specific case from Proposition 4.2, where p and q are equal.

Corollary 4.3

Let $G = H \cong C_{p-1}$ be the cyclic groups with p as an odd prime greater than three. If both actions of G on H and H on G have order two, then, the actions are compatible.

Proof:

Suppose that $G = H \cong C_{p-1}$ with p as an odd prime greater than three and both actions of G on H and H on G have order two. From Proposition 4.2, it follows that if the actions having order two, then, the actions are always compatible. □

The following corollary shows the actions are compatible where one of the actions is trivial.

Corollary 4.4

Let $G = \langle g \rangle \cong C_{p-1}$ and $H = \langle h \rangle \cong C_{q-1}$ be the finite cyclic groups with p and q are different prime numbers and each of which acts on the other. If one of the actions is trivial, then any pair of actions of C_{p-1} and C_{q-1} are compatible.

Proof:

Let $G = \langle g \rangle \cong C_{p-1}$ and $H = \langle h \rangle \cong C_{q-1}$ be the finite cyclic groups with p and q are different prime numbers and each of which acts on the other. Without loss of generality, let the action of g on h is trivial, that is ${}^{g}h = h$ and the action of h on g be ${}^{h}g = g^{k}$ with k is any positive integer. We need to show that the actions satisfy the compatibility conditions in Proposition 3.1. Since ${}^{g}h = h$ observe that ${}^{{}^{g}h}g = {}^{h}g$, then, the first condition is hold. Since the action of g on h is trivial, observe that

$$ {}^{{}^{h}g}h = {}^{g^{k}}h = {}^{g\overbrace{(g.g.\cdots.g)}^{(k-1)-\text{times}}}h = {}^{g}h. $$

then the second condition is hold. Thus, the actions are compatible. □

4.6 Conclusion

In this chapter, the characterisation of the compatible mutual actions for a pair of the finite cyclic groups of the p-power order to act compatibility on each other has been characterised. Some examples also have been presented to explain the characterisation of the compatible mutual actions with same and different groups that act compatibly with each other.

CHAPTER 5

THE NUMBER OF AUTOMORPHISMS AND COMPATIBLE ACTIONS

5.1 Introduction

In this chapter, the automorphisms of the finite cyclic groups of the p-power order, where p is an odd prime are investigated. Some number theory results are used in order to find the number of the automorphisms. This chapter contains the number of the compatible pairs of actions for the finite cyclic groups of the p-power order by using the necessary and sufficient conditions for a pair of such type of groups to act compatibly with each other.

5.2 The Number of Automorphisms with Specific Order

In this section, the number of the automorphisms of such type of groups with their specific order are found. Since our consideration groups are G and H be the finite cyclic groups of the p-power order, then, the action of the group G on the group H is a homomorphism from G to Aut(H) and the action of the group H on the group G is a homomorphism from H to Aut(G). Therefore, the number of the automorphisms of such type of groups need to be found before the number of the compatible pairs of actions can be determined. Hence, the number of the automorphisms for the finite cyclic groups of the p-power order with the respective order is given in the following proposition.

Proposition 5.1

Let $G \cong C_{p^\alpha}$ be a finite cyclic group of the *p*-power order with *p* is an odd prime and $\alpha \geq 2.$ Then, there exist $(p-1)p^{k-1}$ automorphisms of order p^k where $k = 1, 2, ..., \alpha - 1.$

Proof:

Let $G \cong C_{p^\alpha}$ be a finite cyclic group of the *p*-power order with *p* is an odd prime and $\alpha \geq 2.$ Without a loss of generality, suppose that *H* be a finite cyclic *p*-subgroup of *G* such that $|H| = p^k$ where $k = 1, 2, ..., \alpha - 1.$ Thus, each element that relatively prime with p^k has an order p^k. Since *H* is a cyclic subgroup, then by Definition 3.1, $\varphi(p^k) = (p-1)p^{k-1}$, which gives the number of the automorphisms that have order p^k.

□

Next, the following proposition gives the total number of the automorphisms that have the *p*-power order for any finite cyclic group of the *p*-power order where *p* is an odd prime.

Proposition 5.2

Let $G \cong C_{p^\alpha}$ be a finite cyclic group of the *p*-power order with *p* is an odd prime and $\alpha \geq 2$. Then, there exists $p^{\alpha-1} - 1$ automorphisms that have the *p*-power order.

Proof:

Let $G \cong C_{p^\alpha}$ be a finite cyclic group of the *p*-power order with *p* is an odd prime and $\alpha \geq 2$. From Proposition 5.1, there are $(p-1)p^{k-1}$ automorphisms of order p^k where $k = 1, 2, ..., \alpha - 1.$ By using the generating function, the total number of the automorphisms that have the *p*-power is

$$p + p^2 + \cdots + p^k = \sum_{k=1}^{\alpha-1}(p-1)p^{k-1} = \frac{(p-1)}{p}\sum_{k=1}^{\alpha-1}p^k = \frac{(p-1)}{p}\left(\frac{1-p^{\alpha+1-1}}{1-p} - 1\right) = p^{\alpha-1} - 1. \quad \square$$

The next proposition shows that there is one element that has order two in each automorphism group of the finite cyclic group of the *p*-power order.

Proposition 5.3

For any automorphism group of the finite cyclic group of the *p*-power order, there is only one element of order two.

Proof:

Let $G = \langle g \rangle \cong C_{p^{\alpha}}$ be a finite cyclic group of the *p*-power order with *p* is an odd prime and $\alpha \geq 2$. By Theorem 3.1, $\text{Aut}(C_{p^{\alpha}}) \cong C_{p-1} \times C_{p^{\alpha-1}} \cong C_{(p-1)p^{\alpha-1}}$ and by Theorem 3.2, any finite cyclic group of even order is an isomorphic to $\mathbb{Z}_{2n}$. Thus, $C_{(p-1)p^{\alpha-1}} \cong \mathbb{Z}_{2n}$. Therefore, the element that have order two in $\mathbb{Z}_{2n}$ is the solution of the congruence $g \equiv 0$ (mod *n*). Hence, the only element that has order two in $\mathbb{Z}_{2n}$ is $[n]_{2n}$. □

By Theorem 3.1, the automorphism of the finite cyclic groups of the *p*-power order is the direct product of two finite cyclic groups. Thus, in this research only the second part of the direct product which $C_{p^{\alpha-1}}$ of an automorphism group of such type groups are considered because we have focus on the compatible actions that have the *p*-power order and the trivial action. However, we include some result on the first part of the direct product which C_{p-1} in the following corollary.

Corollary 5.1

Let C_{p-1} be a finite cyclic group of even order. Then, there are $\varphi(p-1)$ elements that have order $p-1$.

Proof:

Let C_{p-1} be a finite cyclic group of even order. Furthermore, let $g \in C_{p-1}$. Without loss of generality, if $\gcd(g, p-1) \neq 1$, then $|g|$ is one of the factors of $p-1$. Otherwise, $|g| = p-1$. Thus, by Definition 3.1, there are $\varphi(p-1)$ elements that have order $p-1$. □

Next, Table 5.1 illustrates the number of the elements with their specific orders.

Table 0.1 Number of Elements with Specific Orders.

Order of element	Number of elements
1	1
2	1
p^k	$(p-1)p^{k-1}$
$(p-1)p^k$	$\varphi(p-1)(p-1)p^{k-1}$
$(p-1)$	$\varphi(p-1)$

The following example provides an explanation for the number of the automorphisms with their orders.

Example 5.1

Let $G=\langle g\rangle \cong C_{3^3}$. Then

(i) Aut(G) has only one automorphism of order one.

(ii) Aut(G) has only one automorphism of order two.

(iii) Aut(G) has $3^{3-1}-1=8$ automorphisms that have 3-power order.

The next section explain about the number of the compatible pairs of actions for the finite cyclic groups of the p-power order.

5.3 The Number of Compatible Pairs of Actions

In this section, the number of the compatible pairs of actions can be determined by using the necessary and sufficient number conditions for the two finite cyclic groups of the p-power order, where p is an odd prime to act compatibly with each other. According to the order of the actions, the number of the compatible pairs of actions has been computed.

The following proposition gives the number of the compatible pairs of actions for such type of groups where one of the actions has an order one.

Proposition 5.4

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order where *p* is an odd prime such that $\alpha, \beta \geq 1$. If one of the actions is ρ where $\rho \in \text{Aut}(G)$ and $|\rho| = 1$, then the number of the compatible pairs of actions is $(p-1)p^{\beta-1}$.

Proof:

Let $\rho \in \text{Aut}(G)$ where $|\rho| = 1$ and $\alpha, \beta \geq 1$. By Corollary 3.1, if *G* acts trivially on *H*, then any action of *H* on *G*, the mutual actions are compatible. By Theorem 3.1, $|\text{Aut}(H)| = (p-1)p^{\beta-1}$, which is the number of the compatible pairs of actions. □

Next, the number of the compatible pairs of actions for the two finite cyclic groups of the *p*-power order, where *p* is an odd prime has been determined when one of the actions has an order p^k where $k = 1, 2, ..., \alpha - 1$. By using the necessary and sufficient number theoretical conditions for such type groups, the number of the compatible pairs of actions for the specific value of *k* is given in the following proposition.

Proposition 5.5

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be finite cyclic groups of *p*-power order where *p* is an odd prime. Furthermore, let $\rho \in \text{Aut}(G)$ with $|\rho| = p^k$ where $k = 1, 2, ..., \alpha - 1$ and $\rho' \in \text{Aut}(H)$ with $\alpha, \beta \geq 3$. Then, the number of the compatible pairs of actions is $(p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ where $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, ..., \alpha - 1$.

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be finite cyclic groups of *p*-power order where *p* is an odd prime. Furthermore, let $\rho \in \text{Aut}(G)$ with $|\rho| = p^k$ where $k = 1, 2, ..., \alpha - 1$ and $\rho' \in \text{Aut}(H)$ with $\alpha, \beta \geq 3$. By Proposition 5.1, there are $(p-1)p^{k-1}$ automorphisms of order p^k

where $k=1,2,\ldots,\alpha-1$ and by Theorem 3.4, the actions are compatible with $|\rho'|=1$ and $|\rho'|=p^{k'}$. Thus, we shall consider the two cases as follow:

Case I: Suppose that $|\rho'|=1$, then by Corollary 3.1, when one of the actions is trivial, then the actions are compatible. Thus, there are $(p-1)p^{k-1}$ compatible pairs of actions under this case.

Case II: Suppose that $|\rho'|=p^{k'}$ where $k'=1,2,\ldots,\beta-1$ and by Theorem 3.4, the actions are compatible when $k+k'\leq \min\{\alpha,\beta\}$. Hence, the number of the compatible pairs of actions for every k is the summation of the possibilities of the actions to be compatible, which are

$$\left[\left((p-1)p^{1-1}+(p-1)p^{2-1}+(p-1)p^{3-1}\right)+\left((p-1)p^{1-1}+(p-1)p^{2-1}\right)+\left((p-1)p^{1-1}\right)\right]+\cdots+$$

$$\left[(p-1)p^{\min\{\alpha,\beta\}-k-1}\right]=\sum_{i=1}^{r}(p-1)p^{i-1}$$

where $r=\min\{\alpha,\beta\}-k$. By Proposition 5.1, there are $(p-1)p^{k-1}$ automorphisms of order p^k. Thus, there are $(p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ compatible pairs of actions under this case. Therefore, in total there are $(p-1)p^{k-1}+(p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ compatible pairs of actions with $r=\min\{\alpha,\beta\}-k$.

□

The following proposition gives the total number of the compatible pairs of actions for two finite cyclic groups of the p-power order when one of the actions has the p-power order.

Proposition 5.6

Let $G\cong C_{p^\alpha}$ and $H\cong C_{p^\beta}$ be finite cyclic groups of p-power order where p is an odd prime such that $\alpha,\beta\geq 3$. Then, the total number of the compatible pairs of actions is $\sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$, where $r=\min\{\alpha,\beta\}-k$ and $k=1,2,\ldots,\alpha-1$.

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order where *p* as an odd prime such that $\alpha, \beta \geq 3$. Furthermore, let $\rho \in \text{Aut}(G)$ and $\rho' \in \text{Aut}(H)$ with $|\rho| = p^k$ where $k = 1, 2, ..., \alpha - 1$. From Proposition 5.5, there are $(p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ compatible pairs of actions where $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, ..., \alpha - 1.$ Thus, in total if all *k*'s are considered, then the number of the compatible pairs of actions are given in the following.

$$\begin{aligned}
(p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1} &= \left((p-1)p^{1-1} + (p-1)p^{1-1}\sum_{i=1}^{r}(p-1)p^{i-1} \right) + \\
&\quad \left((p-1)p^{2-1} + (p-1)p^{2-1}\sum_{i=1}^{r}(p-1)p^{i-1} \right) + \cdots + \\
&\quad \left((p-1)p^{(\alpha-1)-1} + (p-1)p^{(\alpha-1)-1}\sum_{i=1}^{r}(p-1)p^{i-1} \right) \\
&= \sum_{k=1}^{\alpha-1}\left((p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1} \right) \\
&= \sum_{k=1}^{\alpha-1}(p-1)p^{k-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1} \\
&= \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1 + \sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1} \right].
\end{aligned}$$

Therefore, there are $\sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1 + \sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1} \right]$ compatible pairs of actions where $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, ..., \alpha - 1.$

□

In general, the number of the compatible pairs of actions for two finite cyclic groups of the *p*-power order, where *p* is an odd prime for a given nonabelian tensor product $C_{p^\alpha} \otimes C_{p^\beta}$ can be found. The result is given in the following theorem.

Theorem 5.1

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order where *p* as an odd prime such that $\alpha, \beta \geq 3.$ Then, there exist $(p-1)p^{\beta-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$ compatible pairs of actions that have order p^k where $k = 1,2,\ldots,\alpha-1$ and $r = \min\{\alpha,\beta\} - k.$

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order where *p* is an odd prime. Furthermore, let $\rho \in \mathrm{Aut}(G)$ where $\alpha, \beta \geq 3.$ The number of the compatible pairs of actions with specific order can be determined by separating them into two cases as follows.

Case I: Suppose that $|\rho| = 1.$ By Proposition 5.4, when one of the actions is trivial, then, the number of the compatible pairs of actions is $(p-1)p^{\beta-1}.$

Case II: Suppose that $|\rho| = p^k$ where $k = 1,2,\ldots,\alpha-1.$ By Proposition 5.6, the total number of the compatible pairs of actions is $\sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$, where $r = \min\{\alpha,\beta\} - k$ and $k = 1,2,\ldots,\alpha-1.$

Hence, in total, the number of the compatible pairs of actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime are

$$(p-1)p^{\beta-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right],$$

where $r = \min\{\alpha,\beta\} - k$ and $k = 1,2,\ldots,\alpha-1.$ □

By using Theorem 5.1, the number of the compatible pairs of actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime has been determined.

Next, with the help of GAP code in Figure 3.1, by input the different finite cyclic groups of 3-power order which are C_{3^3} and C_{3^4}, the following table illustrated the output of GAP which represented the compatible pairs of actions with their orders for the nonabelian tensor product of C_{3^3} and C_{3^4}.

Table 0.2. Compatible Pairs of Actions for $C_{3^3} \otimes C_{3^4}$.

$\lvert\rho\rvert$	k	l	$\lvert\rho'\rvert$	$\lvert\rho\rvert$	k	l	$\lvert\rho'\rvert$
1	1	1	1	1	1	68	54
1	1	2	54	1	1	70	27
1	1	4	27	1	1	71	18
1	1	5	54	1	1	73	9
1	1	7	27	1	1	74	54
1	1	8	18	1	1	76	27
1	1	10	9	1	1	77	54
1	1	11	54	1	1	79	27
1	1	13	27	1	1	80	2
1	1	14	54	9	4	1	1
1	1	16	27	9	4	28	3
1	1	17	18	9	4	55	3
1	1	19	9	9	7	1	1
1	1	20	54	9	7	28	3
1	1	22	27	9	7	55	3
1	1	23	54	3	10	1	1
1	1	25	27	3	10	10	9
1	1	26	6	3	10	19	9
1	1	28	3	3	10	28	3
1	1	29	54	3	10	37	9
1	1	31	27	3	10	46	9
1	1	32	54	3	10	55	3
1	1	34	27	3	10	64	9
1	1	35	18	3	10	73	9
1	1	37	9	9	13	1	1
1	1	38	54	9	13	28	3
1	1	40	27	9	13	55	3
1	1	41	54	9	16	1	1
1	1	43	27	9	16	28	3
1	1	44	18	9	16	55	3
1	1	46	9	3	19	1	1
1	1	47	54	3	19	10	9
1	1	49	27	3	19	19	9
1	1	50	54	3	19	28	3
1	1	52	27	3	19	37	9
1	1	53	6	3	19	46	9
1	1	55	3	3	19	55	3
1	1	56	54	3	19	64	9
1	1	58	27	3	19	73	9
1	1	59	54	9	22	1	1
1	1	61	27	9	22	28	3
1	1	62	18	9	22	55	3
1	1	64	9	9	25	1	1
1	1	65	54	9	25	28	3
1	1	67	27	9	25	55	3

From Table 5.2 there are 90 compatible pairs of actions for $C_{3^3} \otimes C_{3^4}$. Hence, the result from Theorem 5.1 is equivalent with the number of the compatible pairs of actions given in Table 5.2.

Next, an example is given by illustrating the number of the compatible pairs of actions for the given two finite cyclic groups of the *p*-power order.

Example 5.2

Let $G \cong C_{3^3}$ and $H \cong C_{3^4}$ be the finite cyclic groups of 3-power order. Now, consider the actions of *G* and *H* act on each other such that ${}^{g}h = h^{l}$ and ${}^{h}g = g^{k}$ for $g \in G$ and $h \in H$ with $k, l \in \mathbb{N}$. From Theorem 5.1, the number of the compatible pairs of actions, when the actions that have order one and 3^k, where $k = 1, 2$ is given as follows:

(i) when the action has order one, then the number of the compatible pairs of actions is $(p-1)p^{\beta-1} = (3-1)3^{4-1} = 54.$

(ii) when the action has order $3^k, k = 1, 2,$ then the number of the compatible pairs of actions is

$$\sum_{k=1}^{\alpha-1}(p-1)p^{k-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\sum_{k=1}^{\alpha-1}\sum_{i=1}^{\min\{\alpha,\beta\}-k}(p-1)p^{i-1}$$

$$= \sum_{k=1}^{2}(3-1)3^{k-1} + \sum_{k=1}^{2}(3-1)3^{k-1}\sum_{k=1}^{2}\sum_{i=1}^{3-k}(3-1)3^{i-1}$$

$$= 36$$

Hence, in total there are 54 + 36 = 90 compatible pairs of actions.

In the next section, the number of the compatible pairs of actions when the groups are the same are presented.

5.4 The Number of Compatible Pairs of Actions When *G* = *H*

It is well known from the definition of the nonabelian tensor product that when two groups are the same, the nonabelian tensor product is the nonabelian tensor square. Thus, from Theorem 5.1, the number of the compatible pairs of actions is given in the following corollary.

Corollary 5.2

Let $G = H \cong C_{p^{\alpha}}$ be the finite cyclic groups of the *p*-power order where *p* is an odd prime and $\alpha \geq 3$. Then there are

$$(p-1)p^{\alpha-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{\alpha-k}(p-1)p^{i-1}\right]$$

compatible pairs of actions.

Proof:

Let $G = H \cong C_{p^{\alpha}}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha \geq 3$. Since $G = H$, then $\alpha = \min\{\alpha, \alpha\}$. Therefore, from Theorem 5.1, there are $(p-1)p^{\alpha-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{\alpha-k}(p-1)p^{i-1}\right]$ compatible pairs of actions. □

In particular, the number of the compatible pairs of nontrivial actions for a given nonabelian tensor product $C_{p^{\alpha}} \otimes C_{p^{\beta}}$ and $C_{p^{\beta}} \otimes C_{p^{\alpha}}$ for the finite cyclic groups of the *p*-power order is equal. This result is given in the following corollary.

Corollary 5.3

Let $G = \langle x \rangle \cong C_{p^{\alpha}}$ and $H = \langle y \rangle \cong C_{p^{\beta}}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha, \beta \geq 3$. Then, the number of the compatible pairs of nontrivial actions that have the *p*-power order for the nonabelian tensor products, $C_{p^{\alpha}} \otimes C_{p^{\beta}}$ and $C_{p^{\beta}} \otimes C_{p^{\alpha}}$ are equal.

Proof:

Let $G = \langle x \rangle \cong C_{p^\alpha}$ and $H \cong C_{p^\beta} \langle y \rangle$ be the finite cyclic groups of the *p*-power order where *p* is an odd prime and $\alpha, \beta \geq 3.$ From proposition 5.6, there are $\sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ compatible pairs of nontrivial actions where $k = 1, 2, \ldots, \alpha - 1$ and $r = \min\{\alpha, \beta\} - k.$ Since $r = \min\{\alpha, \beta\} - k = \min\{\beta, \alpha\} - k,$ then for any given nonabelian tensor product $C_{p^\alpha} \otimes C_{p^\beta}$ and $C_{p^\beta} \otimes C_{p^\alpha},$ the number of compatible pairs of nontrivial actions are the same. □

5.5 Conclusion

In this chapter, the number of the automorphisms of the finite cyclic groups of the *p*-power order with the respective order were determined. Furthermore, the number of the compatible pair of actions that have the *p*-power order between the two finite cyclic groups of the *p*-power order, where *p* is an odd prime were determined. By using the necessary and sufficient conditions, the number of the compatible pairs of actions has been computed according to the order of the action.

CHAPTER 6

THE COMPATIBLE ACTION GRAPH AND ITS SUBGRAPH

6.1 Introduction

This chapter investigate the connection between the group theory and the graph theory. By extending the results on the compatible actions, a new graph and its subgraph specifically on the cyclic groups of *p*-power order with actions that have the *p*-power order is defined. Thus, some properties of the compatible action graph for such type of groups are given.

6.2 Motivation of Compatible Action Graph

The idea which makes us investigated the compatible actions for the subgroup of the finite cyclic groups of the *p*-power order is that, usually we think that the subgroup *H* from the group *G* should be all exists in the group *G*, but it is not necessary. Thus, with the compatible actions, there are some of the actions are existed in the automorphism of the subgroup *H* but not in the automorphism of the group *G*. Thus, the following example is given to show that there are compatible actions which are existed in subgroup $C_{3^4} \otimes C_{3^4}$ but not in the group $C_{3^5} \otimes C_{3^5}$.

Example 6.1

Let $G = \langle g \rangle \cong C_{3^5}$ and $H = \langle h \rangle \cong C_{3^5}$ be finite cyclic groups of 3-power order. Furthermore, let $\rho \in \operatorname{Aut}(G)$ and $\rho' \in \operatorname{Aut}(H)$ be two actions such that $\rho(g) = g^{10}$ and

$\rho'(h) = h^{10}$ or equivalently ${}^{g}h = h^{10}$ and ${}^{h}g = g^{10}$. Then, for the first compatibility condition,

$$\begin{aligned} {}^{{}^{h}g}h &= {}^{g^{10}}h \\ &= h^{10^{10}} \\ &= h^{91} \qquad \text{since } 10^{10} \equiv 91 \left(\bmod 3^{5}\right) \\ &\neq {}^{g}h. \end{aligned}$$

Hence, the actions ρ and ρ' are not compatible in $C_{3^5} \otimes C_{3^5}$.

Now, let $G = \langle g \rangle \cong C_{3^4}$ and $H = \langle h \rangle \cong C_{3^4}$ be finite cyclic groups of 3-power order where G and H are subgroup of C_{3^5} . Furthermore, let $\rho \in \mathrm{Aut}(G)$ and $\rho' \in \mathrm{Aut}(H)$ be two actions such that $\rho(g) = g^{10}$ and $\rho'(h) = h^{10}$, then by Proposition 3.1, the actions are compatible if ${}^{{}^{g}h}g = {}^{h}g$ and ${}^{{}^{h}g}h = {}^{g}h.$ or equivalently ${}^{g}h = h^{10}$ and ${}^{h}g = g^{10}$. Now, consider the first compatibility conditions.

$$\begin{aligned} {}^{{}^{h}g}h &= {}^{g^{10}}h \\ &= h^{10^{10}} \\ &= h^{10} \qquad \text{since } 10^{10} \equiv 10 \left(\bmod 3^{4}\right) \\ &= {}^{g}h. \end{aligned}$$

Same goes for the second compatibility conditions. Thus the actions ρ and ρ' are compatible in $C_{3^4} \otimes C_{3^4}$.

More generally, the example below is given to illustrate the idea and the intersection between the group and the subgroup.

Example 6.2

Let $G \cong C_{3^5}$ be a finite cyclic group of 3-power order, and let $H \cong C_{3^4}$ be a subgroup of G. For the nonabelian tensor product of the subgroup $C_{3^4} \otimes C_{3^4}$, the pairs (10,10), $(19,19)$, $(37,37)$, $(46,46)$, $(64,64)$ and $(73,73)$ are compatible in $C_{3^4} \otimes C_{3^4}$ but not in $C_{3^5} \otimes C_{3^5}$. However, the pairs $(10,28),(10,55),(19,28),(19,55),(28,10),(28,19),(28,28),$ $(28,37),(28,46),(28,55),(28,64),(28,73),(37,28),(37,55),(46,28),(46,55),(55,10),$

$(55,19),(55,28), (55,37),(55,46),(55,55),(55,64),(55,73),(64,28),(64,28),(73,28)$ and $(73,55)$ are compatible and all represent the intersection between $C_{3^5} \otimes C_{3^5}$ and $C_{3^4} \otimes C_{3^4}$.

Therefore, to find the intersection between the group and the subgroup, the compatible action graph has been defined for this case.

In the next section, the properties of the compatible action graph are presented.

6.3 The Properties of the Compatible Action Graph

In this section, the theoretical relationship and the connection between the group theory and the graph theory were studied. The graph Γ can be described as a discrete structure consisting of two sets, which are the set of vertices, which is denoted by $V(\Gamma)$ and the set of edges connect these vertices, which is denoted by $E(\Gamma)$.

This research is focusing on the compatible actions for the finite cyclic groups of the *p*-power order with the actions that have the *p*-power order. Thus, a new notation namely $\Gamma_{p\ G\otimes H}$ is introduced to present the compatible action graph with actions that only have the *p*-power order. Thus, the following definition is extended from Sulaiman (2017) to the finite cyclic groups of the *p*-power order with all the actions that have the *p*-power order, where *p* is an odd prime and is given as follows.

Definition 6.1 Compatible Action Graph of *p*-Power Order

Let G and H be two finite cyclic groups of the *p*-power order with p is an odd prime. Furthermore, let (ρ, ρ') be a pair of the compatible actions for the nonabelian tensor product of $G \otimes H$, where $\rho \in \mathrm{Aut}(G)$ and $\rho' \in \mathrm{Aut}(H)$. Then, $\Gamma_{p\ G\otimes H} = (V(\Gamma_{p\ G\otimes H}),$ $(E(\Gamma_{p\ G\otimes H}))$ is a compatible action graph with the set of vertices $V(\Gamma_{p\ G\otimes H})$, which is the set of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$, and the set of edges, $E(\Gamma_{p\ G\otimes H})$ which is the set of all compatible pairs of actions (ρ, ρ').

The order of the compatible actions graph for the finite cyclic groups of p-power order are studied. From Definition 3.6, the order of the graph G is defined as the number of the vertices in the graph G, which is denoted by $|G|$. Hence, the order of the compatible action graph has been found and is denoted by $\left|\Gamma_{p\ G\otimes H}\right|$. Therefore, the order of the compatible action graph is considered into two cases, which are $G \neq H$ and $G = H$. Thus, the order of the compatible action graph is given in the following proposition.

Proposition 6.1
Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. Then, the order of the compatible action graph is;

(i) $\left|\Gamma_{p\ G\otimes H}\right| = (p-1)(p^{\alpha-1} + p^{\beta-1})$ if $G \neq H$.

(ii) $\left|\Gamma_{p\ G\otimes H}\right| = (p-1)p^{\alpha-1}$ if $G = H$.

Proof:
Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. From Definition 3.6, $\left|\Gamma_{p\ G\otimes H}\right| = \left|V(\Gamma_{p\ G\otimes H})\right|$. Furthermore, from Definition 3.11, $V(\Gamma_{p\ G\otimes H})$ is the nonempty set of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$. Thus, there are two cases needed to be considered, which are $G \neq H$ and $G = H$.

Case I: Suppose that $G \neq H$. Then,
$\left|V(\Gamma_{p\ G\otimes H})\right| = |\mathrm{Aut}(G)| + |\mathrm{Aut}(H)| = (p-1)p^{\alpha-1} + (p-1)p^{\beta-1} = (p-1)(p^{\alpha-1} + p^{\beta-1})$.

Case II: Suppose that $G = H$. Without loss of generality, let α be the order where $\alpha = \beta$, then $\left|V(\Gamma_{p\ G\otimes H})\right| = |\mathrm{Aut}(G)| = (p-1)p^{\alpha-1}$.

Therefore, $\left|\Gamma_{p\ G\otimes H}\right| = (p-1)(p^{\alpha-1} + p^{\beta-1})$ when $G \neq H$ and $\left|\Gamma_{p\ G\otimes H}\right| = (p-1)p^{\alpha-1}$ when $G = H$. □

Since the action of G on H is the mapping $\Phi : G \to \mathrm{Aut}(H)$, then the compatible action graph of the finite cyclic groups of the *p*-power order is directed multigraph.

Thus, there exist vertices that are connected by multiple edges. However, the loop is only present when $G = H$. The cardinality of edge of the compatible action graph for the finite cyclic groups of the *p*-power order is given in the following proposition.

Proposition 6.2

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. Then,

$$\left|E(\Gamma_{p\ G\otimes H})\right| = (p-1)p^{\beta-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right],$$

where $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, ..., \alpha - 1$.

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha, \beta \geq 3$. From Definition 6.1, $E(\Gamma_{p\ G\otimes H})$ is the set of all compatible pairs (ρ, ρ'), where $\rho \in \mathrm{Aut}(G)$ and $\rho' \in \mathrm{Aut}(H)$. Then, by Theorem 5.1,

$$\left|E(\Gamma_{p\ G\otimes H})\right| = (p-1)p^{\beta-1} + \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right].$$

where $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, ..., \alpha - 1$. □

In the terminology of the graphs with directed edges, the edges have directions in the directed graph. Thus, the initial vertex of the direction is called as the initial vertex and the ending vertex is the terminal vertex.

Hence, the compatible pairs of actions (ρ, ρ') be defined as the directed edge of the compatible action graph. Therefore, from Definition 3.4, the vertex ρ is considered as an initial vertex of (ρ, ρ') and ρ' is the terminal vertex of (ρ, ρ').

In addition, the out-degree of the vertex *v* in the directed graph is denoted by $\deg^+(v)$, where it needs to be found in order to investigate the number of the edges with *v* as their initial vertex. Thus, the number of the directed edges, the out-degree of the vertex *v* is presented in the following proposition.

Proposition 6.3

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v \in V(\Gamma_{p\ G \otimes H})$ where $v \in \text{Aut}(G)$ and $|v| = p^k$. Then $\deg^{+}(v)$ is one of the following;

(i) $(p-1)p^{\beta-1}$ if $k = 0$.

(ii) $(p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$, with $r = \min\{\alpha, \beta\} - k$ if $k = 1, 2, \ldots, \alpha - 1$.

Proof:

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v \in V(\Gamma_{p\ G \otimes H})$ where $v \in \text{Aut}(G)$ and $|v| = p^k$. Then, there are two cases are considered as follows.

Case I: Let $k = 0$, then by Proposition 5.4, the actions are compatible when the action of G on H is trivial. Thus, $\deg^{+}(v) = (p-1)p^{\beta-1}$.

Case II: Let $k = 1, 2, \ldots, \alpha - 1$, then by Proposition 5.5, there are $(p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ compatible pairs of actions where $r = \min\{\alpha, \beta\} - k$ and $k = 1, 2, \ldots, \alpha - 1$. Thus

$$\deg^{+}(v) = (p-1)p^{k-1} + (p-1)p^{k-1}\sum_{i=1}^{r}(p-1)p^{i-1}. \qquad \square$$

From Definition 3.5, the in-degree of the vertex *v* is denoted by $\deg^{-}(v)$, which is the number of the edges with *v* as their terminal vertex. The following proposition shows the number of the directed edges, where the in-degree of the vertex *v* are given.

Proposition 6.4

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p-power order with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v \in V(\Gamma_{p\ G\otimes H})$ where $v \in \text{Aut}(H)$ and $|v| = p^{k'}$. Then $\deg^-(v)$ is one of the following;

(i) $(p-1)p^{\alpha-1}$ if $k' = 0$.

(ii) $(p-1)p^{k'-1} + (p-1)p^{k'-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ with $r = \min\{\alpha, \beta\} - k'$ if $k' = 1, 2, ..., \beta - 1$

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p-power order with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v \in V(\Gamma_{p\ G\otimes H})$ where $v \in \text{Aut}(H)$ and $|v| = p^{k'}$. Then there are two cases are considered as follows.

Case I: From Proposition 5.4, the actions are compatible when the action of H on G is trivial. Thus, $\deg^-(v) = (p-1)p^{\alpha-1}$.

Case II: From Proposition 5.5, there are $(p-1)p^{k'-1} + (p-1)p^{k'-1}\sum_{i=1}^{r}(p-1)p^{i-1}$ compatible pairs of actions, where $k' = 1, 2, ..., \beta - 1$ and $r = \min\{\alpha, \beta\} - k'$. Thus, $\deg^-(v) = (p-1)p^{k'-1} + (p-1)p^{k'-1}\sum_{i=1}^{r}(p-1)p^{i-1}$. □

In the compatible action graph, when $G = H$, the number of the directed edges, the out-degree of the vertex v, and the number of the directed edges from the in-degree of the vertex v are equivalent. This result is presented in the following corollary.

Corollary 6.1

Let G and H be the finite cyclic groups of the p-power order with p is an odd prime and $v \in V(\Gamma_{p\ G\otimes H})$. If $G = H$, then $\deg^{-}(v) = \deg^{+}(v)$ for $\Gamma_{p\ G\otimes G}$.

Proof:

Let G and H be the finite cyclic groups of the p-power order with p as an odd prime and $G = H$. From Propositions 6.3 and 6.4, $\deg^{-}(v) = \deg^{+}(v)$ for any $v \in V(\Gamma_{p\ G\otimes G})$. □

Next, the connectivity of the compatible action graph is studied. The compatible action graph is connected when there is a path between the pair of the vertices. Thus, the connectivity of the compatible action graph for the finite cyclic groups of the p-power order, where p is an odd prime are presented in the following proposition.

Proposition 6.5

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the p-power order with p as an odd prime and $\alpha, \beta \geq 3$. Then, $\Gamma_{p\ G\otimes H}$ is the connected graph.

Proof:

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the p-power order with p as an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $v_1 \in V(\Gamma_{p\ G\otimes H})$ with $v_1 \in \mathrm{Aut}(G)$ and v_1 is trivial action. By Proposition 6.3, $\deg^{+}(v_1) = (p-1)p^{\beta-1}$. Since $|\mathrm{Aut}(H)| = (p-1)p^{\beta-1}$, then v_1 is compatible with every $v \in \mathrm{Aut}(H)$. Similarly, let $v_2 \in V(\Gamma_{p\ G\otimes H})$ with $v_2 \in \mathrm{Aut}(H)$ and v_2 is trivial action, then from Proposition 6.4, the $\deg^{-}(v_2) = (p-1)p^{\alpha-1}$, since $|\mathrm{Aut}(G)| = (p-1)p^{\alpha-1}$, then v_2 is compatible with every $v \in \mathrm{Aut}(G)$. Thus, $\Gamma_{p\ G\otimes H}$ is the connected graph. □

Supposed that $G \neq H$, then, the compatible action graph has the property that the vertex can be partitioned into two sets, namely V_1 and V_2. Therefore, every edge in the compatible action graph connects a vertex in V_1 and V_2, then, the compatible action graph became a bipartite graph. Thus, this result is presented in the following proposition.

Proposition 6.6

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha, \beta \geq 3$. Then, $\Gamma_{p\ G\otimes H}$ is the bipartite graph if and only if $G \neq H$.

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. First need to show that if the compatible action graph $\Gamma_{p\ G\otimes H}$ is a bipartite graph then $G \neq H$. By contradiction method, assume that $G = H$ and, let v_1 be the trivial action of Aut(*G*). By Proposition 6.3, $\deg^+(v_1) = (p-1)p^{\beta-1}$, which gives the actions compatible with any other action. Now, let $v_2 \in V(\Gamma_{p\ G\otimes H})$, such that v_1 and v_2 are compatible. Then, by Corollary 3.1, v_1 and v_2 are also compatible. Thus, v_1 and v_2 could not be partitioned into two disjoint sets, which contradicts on the assumption. Thus, $G \neq H$.

Next, by contradiction method, suppose that $G \neq H$ and assume that the compatible action graph $\Gamma_{p\ G\otimes H}$ is not a bipartite graph. Since $\Gamma_{p\ G\otimes H}$ is not a bipartite graph, then, there exists $v_1, v_2 \in V(\Gamma_{p\ G\otimes H})$, such that v_1 and v_2 could not be partitioned into two disjoint sets with v_1 and v_2 are both compatible on each other. Thus, from Definition 6.1, this happens only when $G = H$, which contradicts to the assumption. Therefore, $\Gamma_{p\ G\otimes H}$ is a bipartite graph. □

The complete graph K_n contains exactly one edge between each pair of the vertices. As a result, the compatible action graph is not a complete graph. This result is given as follows.

Proposition 6.7

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha, \beta \geq 3$. Then, $\Gamma_{p\ G \otimes H}$ is not a complete graph.

Proof:

By Theorem 3.1, $\left|\mathrm{Aut}(C_{p^{\alpha}})\right| = (p-1)p^{\alpha-1}$ and by Proposition 5.1, there are $(p-1)p^{k-1}$ automorphisms of order p^{k} where $k = 1, 2, \ldots, \alpha - 1.$ That is mean $(p-1)p^{k-1} < (p-1)p^{\alpha-1}$ which mean that there exist some nontrivial actions in $\mathrm{Aut}(C_{p^{\alpha}})$ are not of *p*-power order. Then, by Theorem 3.4, these actions are not compatible which mean there is no edges connect these vertices. Thus, $\Gamma_{p\ G \otimes H}$ is not a complete graph. □

6.4 Subgraph of Compatible Action Graph

Let $C_{p^{\alpha}}$ and $C_{p^{\beta}}$ be two finite cyclic groups of the *p*-power order where *p* is an odd prime and $\alpha, \beta \geq 3$. Furthermore, suppose that $C_{p^{\alpha-i}}$ and $C_{p^{\beta-i}}$ are two subgroups of $C_{p^{\alpha-i}}$ and $C_{p^{\alpha-i}}$ respectively with $i = 1, 2, \ldots, \min\{\alpha, \beta\} - 2$. This section concern on the intersect between the two compatible action graphs which are $\Gamma_{C_{p^{\alpha}} \otimes C_{p^{\beta}}}$ and $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$ to investigate the number of the edges and vertices. Therefore, this section presented the necessary and sufficient conditions for the cyclic subgroups of the *p*-power order acting on each other in a compatible way when the order of the subgroups are reduced by the same power order from the order of the groups. Then the order of the subgraph and the number of the edges of the subgraph of compatible action graph are investigated. Thus, the following proposition shows the necessary and sufficient conditions for $C_{p^{\alpha-i}}$ and $C_{p^{\beta-i}}$ to act compatibly on each other.

Proposition 6.8

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p-power order with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let (ρ, ρ') is a compatible pair of actions for $C_{p^\alpha} \otimes C_{p^\beta}$ where $\rho(g) = g^k$ and $\rho'(h) = h^l$ with $k, l \in \mathbb{N}$. Then, (ρ, ρ') is a compatible pair of actions for $C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}$ where $\rho(g) = g^{k \bmod p^{\alpha-i}}$ and $\rho'(h) = h^{l \bmod p^{\beta-i}}$ with $i = 1, 2, \ldots, \min\{\alpha, \beta\} - 2$.

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the p-power order with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let (ρ, ρ') is a compatible pair of actions for $C_{p^\alpha} \otimes C_{p^\beta}$ where $\rho(g) = g^k$ and $\rho'(h) = h^l$ with $k, l \in \mathbb{N}$. Without loss of generality, assume that $C_{p^{\alpha-i}} \leq C_{p^\alpha}$ and $C_{p^{\beta-i}} \leq C_{p^\beta}$, then $C_{p^{\alpha-i}} = \langle g' \rangle$ and $C_{p^{\beta-i}} = \langle h' \rangle$ for some $g' \in G$ and $h' \in H$. Since (ρ, ρ') is a compatible pair of actions for $C_{p^\alpha} \otimes C_{p^\beta}$, then there exist a mutual actions of G and H on each other such that ${}^h g = g^k$ and ${}^g h = h^l$ for $k, l \in \mathbb{N}$. In order to prove that $\rho(g) = g^{k \bmod p^{\alpha-i}}$ and $\rho'(h) = h^{l \bmod p^{\beta-i}}$ is a compatible pair of actions for $C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}$, by Proposition 3.3, there are three conditions need to be satisfied as follows.

(i) $\gcd\left(k, p^{\alpha-i}\right) = \gcd\left(l, p^{\beta-i}\right) = 1.$

Define that $\rho: G \to G$ with $\rho(g) = g^k$ is an automorphism if and only if $\gcd(k, p^\alpha) = 1$. Since p^α is an odd number because p is odd, then k must be even. Therefore, $\gcd(k, p^{\alpha-i}) = 1$. Similarly, there exist a mutual actions of G and H such that ${}^g h = h^l$. Since $\gcd(l, p^\beta) = 1$, then $\gcd(l, p^{\beta-i}) = 1$. Hence, $\gcd(k, p^{\alpha-i}) = \gcd(l, p^{\beta-i}) = 1$, and the first condition is satisfied.

(ii) $k^{p^{\beta-i}} \equiv 1 \pmod{p^{\alpha-i}}$ and $l^{p^{\alpha-i}} \equiv 1 \pmod{p^{\beta-i}}$.

Let H acts on G, then there exist a mutual action of H on G such that ${}^{h}g = g^{k}$, then $g = {}^{1H}g = {}^{h^{p^{\beta-i}}}g = g^{k^{p^{\beta-i}}}$. Thus $k^{p^{\beta-i}} \equiv 1 \bmod p^{\alpha-i}$. Similarly, if G acts on H, there exist a mutual action of G on H such that ${}^{g}h = h^{l}$, then $l^{p^{\alpha-i}} \equiv 1 \bmod p^{\beta-i}$. Hence, the second condition is satisfied.

(iii) $k^{l-1} \equiv k\left(\bmod p^{\alpha-i}\right)$ and $l^{k-1} \equiv 1 \pmod{p^{\beta-i}}$.

By Proposition 3.1, G and H act compatibly on each other if and only if ${}^{({}^{g}h)}g = {}^{h}g$ and ${}^{({}^{h}g)}h = {}^{g}h$. From the first condition, ${}^{{}^{g}h}g = {}^{h^{l}}g = g^{k^{l}}$ and ${}^{h}g = g^{k}$. Thus $k^{l} \equiv k\left(\bmod p^{\alpha-i}\right)$ or equivalently $k^{l-1} \equiv k\left(\bmod p^{\alpha-i}\right)$ since $\gcd\left(k, p^{\alpha-i}\right) = 1$. Similarly for the second condition is $l^{k} \equiv 1 \pmod{p^{\beta-i}}$ or equivalently $l^{k-1} \equiv 1 \pmod{p^{\beta-i}}$ since $\gcd(l, p^{\beta-i}) = 1$. Hence, the third condition is hold. Therefore, $\rho(g) = g^{k \bmod p^{\alpha-i}}$ and $\rho'(h) = h^{l \bmod p^{\beta-i}}$ is a compatible pair of actions for $C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}$. □

Next, the order of $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$ is investigated. From Proposition 6.1, the order of $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$ is considered into two cases which are $G \neq H$ and $G = H$. Thus, the following proposition gives the order for $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$.

Proposition 6.9

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the p-power order with p as an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $\Gamma_{C_{p^{\alpha}} \otimes C_{p^{\beta}}}$ and $\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$ be two compatible action graphs with $i = 1, 2, \ldots, \alpha - 2$. Then, the order of the subgraph of compatible action graph is

$$\left|\Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}\right| = \frac{1}{p^{i}}\left|\Gamma_{C_{p^{\alpha}} \otimes C_{p^{\beta}}}\right|.$$

Proof:

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of *p*-power order with *p* is an odd prime and $\alpha, \beta \geq 3$. From Proposition 6.1, the order of the compatible action graph considered into two cases which are $G \neq H$ and $G = H$. Thus, two cases are considered as follows.

Case I: Suppose that $G \neq H$. By Proposition 6.1(i), $\left|\Gamma_{p\ G \otimes H}\right| = (p-1)(p^{\alpha-1} + p^{\beta-1})$. Thus,

$$\left|\Gamma_{p\ C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}\right| = (p-1)(p^{\alpha-i-1} + p^{\beta-i-1}) = \frac{(p-1)}{p^i}(p^{\alpha-1} + p^{\beta-1}) = \frac{1}{p^i}\left|\Gamma_{p\ C_{p^\alpha} \otimes C_{p^\beta}}\right|.$$

Case II: Suppose that $G = H$. From Proposition 6.1(ii), $\left|\Gamma_{p\ G \otimes H}\right| = (p-1)p^{\alpha-1}$. Thus,

$$\left|\Gamma_{p\ C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}\right| = (p-1)p^{\alpha-i-1} = \frac{(p-1)}{p^i}p^{\alpha-1} = \frac{1}{p^i}\left|\Gamma_{p\ C_{p^\alpha} \otimes C_{p^\beta}}\right|. \qquad \square$$

The next proposition shows the number of the edges of the subgraph of compatible action graph. From Theorem 5.1, there are two cases are considered as follows.

Proposition 6.10

Let $G \cong C_{p^\alpha}$ and $H \cong C_{p^\beta}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $\Gamma_{p\ C_{p^\alpha} \otimes C_{p^\beta}}$ and $\Gamma_{p\ C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$ be two compatible action graphs when $i = 1, 2, \ldots, \alpha-2$ and $v \in V(\Gamma_{p\ C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}})$. Then

(i) If $k = 0$, then $\left|E(\Gamma_{p\ C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}})\right| = \dfrac{(p-1)}{p^i}p^{\beta-1}$.

(ii) if $k = 1, 2, \ldots, \alpha-1$, then $\left|E(\Gamma_{p\ C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}})\right| = \sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1 + \sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$

when $r = \min\{\alpha-i, \beta-i\} - k$,

Proof:

Let $G \cong C_{p^{\alpha}}$ and $H \cong C_{p^{\beta}}$ be the finite cyclic groups of the *p*-power order with *p* as an odd prime and $\alpha, \beta \geq 3$. Furthermore, let $\Gamma_{p\, C_{p^{\alpha}} \otimes C_{p^{\beta}}}$ and $\Gamma_{p\, C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$ be two compatible action graphs where $i = 1, 2, \ldots, \alpha - 2$ and $v \in V(\Gamma_{p\, C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}})$, then two cases are considered as follows.

Case I: Let $k = 0$, then by Proposition 6.3(i), $\deg^{+}(v) = (p-1)p^{\beta-1}$ and *v* represent the trivial automorphism. By Corollary 3.1, *v* is compatible with any vertex and by Proposition 5.4, there exist $(p-1)p^{\beta-1}$ compatible pairs of actions. Thus,

$$\left| E(\Gamma_{p\, C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}) \right| = (p-1)p^{\beta-i-1} = \frac{(p-1)}{p^{i}} p^{\beta-1}.$$

Case II: Let $k = 1, 2, \ldots, \alpha - 1$, then by Theorem 3.4, the actions are compatible when $k + k' \leq \min\{\alpha, \beta\}$. By Proposition 6.3(ii),

$$\deg^{+}(v) = (p-1)p^{k-1} + (p-1)p^{k-1} \sum_{i=1}^{r} (p-1)p^{i-1},$$

where $r = \min\{\alpha, \beta\} - k$. From the assumption we have $v \in \Gamma_{C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}$. Thus,

$$\left| E(\Gamma_{p\, C_{p^{\alpha-i}} \otimes C_{p^{\beta-i}}}) \right| = \sum_{k=1}^{\alpha-1} (p-1)p^{k-1} \left[1 + \sum_{k=1}^{\alpha-1} \sum_{i=1}^{r} (p-1)p^{i-1} \right], \text{ with } r = \min\{\alpha - i, \beta - i\} - k. \ \square$$

Next, the number of the compatible pairs of actions in the intersection between the compatible action graph and its subgraph has been determined. Thus, for this case only when $i = 1$ is considered as a reduce for the power of the subgroups of the finite cyclic groups of *p*-power order. Therefore, when one of the actions is trivial, then the number of the compatible pairs of actions in the intersection between the compatible action graph and its subgraph is presented in the following lemma.

Lemma 6.1

Let $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}}$ and $\Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$ be two compatible action graphs with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let ρ be the trivial action in $\Gamma_{C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$. Then there are $\dfrac{(p-1)p^{\beta-1}}{p}$ compatible pairs of actions in $\Gamma_{C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$.

Proof:

Let $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}}$ and $\Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$ be two compatible action graphs with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let ρ be the trivial action in $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$. By Corollary 3.1, the action ρ is compatible with any other action. Since $C_{p^{\alpha-1}}$ and $C_{p^{\beta-1}}$ are subgroups from $C_{p^{\alpha}}$ and $C_{p^{\beta}}$, then $(\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}) = \Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$. Thus, the action ρ is compatible with any other action in $\Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$. By Proposition 6.10(i), when one of the actions is trivial, there are $\dfrac{(p-1)}{p^{i}} p^{\beta-1}$ compatible pairs of actions. Since $i=1$, then there are $\dfrac{(p-1)}{p} p^{\beta-1}$ compatible pairs of actions in $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$.

□

The next lemma gives the number of the compatible pairs of actions in the intersection between the compatible action graph and its subgraph when one of the actions that has the *p*-power order.

Lemma 6.2

Let $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}}$ and $\Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$ be two compatible action graphs with p is an odd prime and $\alpha, \beta \geq 3$. Furthermore, let ρ be the nontrivial action in $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$. Then, there are $\displaystyle\sum_{k=1}^{\alpha-2}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-2}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$ compatible pairs of actions in $\Gamma_{p\,C_{p^{\alpha}} \otimes C_{p^{\beta}}} \cap \Gamma_{p\,C_{p^{\alpha-1}} \otimes C_{p^{\beta-1}}}$, where $r = \min\{\alpha, \beta\} - k - 1$ and $k = 1, 2, \ldots, \alpha-2$.

Proof:

From Proposition 6.10(ii), the number of the compatible pairs of actions for the subgraph $\Gamma_{p\,C_{p^{\alpha-i}}\otimes C_{p^{\beta-i}}}$ is $\sum_{k=1}^{\alpha-1}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-1}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$, where $k=1,2,\ldots,\alpha-1$ and $r=\min\{\alpha-i,\beta-i\}-k$. Since k is hold for each values of $1,2,\ldots,\alpha-1$, then k is also hold for $1,2,\ldots,\alpha-2$. Since the order of the actions is reduced, then the bound represent the power of the order of the subgroups. β and α where $r=\min\{\alpha,\beta\}-k-i$, Therefore, there are $\sum_{k=1}^{\alpha-2}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-2}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$ compatible pairs of actions in $\Gamma_{p\,C_{p^{\alpha}}\otimes C_{p^{\beta}}}\cap\Gamma_{p\,C_{p^{\alpha-1}}\otimes C_{p^{\beta-1}}}$ where $r=\min\{\alpha,\beta\}-k-1$ and $k=1,2,\ldots,\alpha-2$. □

In general, the number of the compatible pairs of actions in the intersection between the compatible action graph and its subgraph for the finite cyclic groups of the p-power order where p is an odd prime is given in the following theorem.

Theorem 6.1

Let $\Gamma_{p\,C_{p^{\alpha}}\otimes C_{p^{\beta}}}$ and $\Gamma_{p\,C_{p^{\alpha-1}}\otimes C_{p^{\beta-1}}}$ be two compatible action graphs with p is an odd prime and $\alpha,\beta\geq 3$. Then, there are $\frac{(p-1)p^{\beta-1}}{p}+\sum_{k=1}^{\alpha-2}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-2}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$where where $r=\min\{\alpha,\beta\}-k-1$ and $k=0,1,2,\ldots,\alpha-2$.

Proof:

Let $\Gamma_{p\,C_{p^{\alpha}}\otimes C_{p^{\beta}}}$ and $\Gamma_{p\,C_{p^{\alpha-1}}\otimes C_{p^{\beta-1}}}$ be two compatible action graphs with p is an odd prime and $\alpha,\beta\geq 3$. the number of the compatible pairs of actions in the intersection between the compatible action graph and its subgraph can be determined by separating into two cases as follows.

Case I: Suppose that the action is trivial. By Lemma 6.1, when the action is trivial, there are $\frac{(p-1)}{p}p^{\beta-1}$ compatible pairs of actions in $\Gamma_{p\,C_{p^{\alpha}}\otimes C_{p^{\beta}}}\cap\Gamma_{p\,C_{p^{\alpha-1}}\otimes C_{p^{\beta-1}}}$.

Case II: Suppose that the action is nontrivial of *p*-power order. By Lemma 6.2, when the action that have the *p*-power order, there are $\sum_{k=1}^{\alpha-2}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-2}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$ compatible pairs of actions in $\Gamma_{p\, C_{p^\alpha}\otimes C_{p^\beta}} \cap \Gamma_{p\, C_{p^{\alpha-1}}\otimes C_{p^{\beta-1}}}$, where $r = \min\{\alpha,\beta\}-k-1$ and $k=1,2,\ldots,\alpha-2$.

Thus, in total, there are $\frac{(p-1)p^{\beta-1}}{p}+\sum_{k=1}^{\alpha-2}(p-1)p^{k-1}\left[1+\sum_{k=1}^{\alpha-2}\sum_{i=1}^{r}(p-1)p^{i-1}\right]$ compatible pairs of actions in $\Gamma_{p\, C_{p^\alpha}\otimes C_{p^\beta}} \cap \Gamma_{p\, C_{p^{\alpha-1}}\otimes C_{p^{\beta-1}}}$, where $r = \min\{\alpha,\beta\}-k-1$ and $k=1,2,\ldots,\alpha-2$. □

6.5 Conclusion

In this chapter, the compatible action graph and it is subgraph for the finite cyclic groups of the *p*-power order, where *p* is an odd prime are introduced. Some properties of the compatible action graph are presented, such that the cardinality of the edge, the order of the compatible action graph, the number of the directed edges from the in-degree and the out-degree of the vertex *v*, the bipartite graph, the connectivity of the compatible action graph, and the compatible action graph is not complete. Furthermore, new necessary and sufficient conditions for the subgraph of compatible action graph to act compatibly on each other are provided. Then, the number of the edges and the order of the subgraph of compatible action graph are presented. However, the number of the compatible pairs of actions in the intersection between the compatible action graph and its subgraph are also given.

CHAPTER 7

SUMMARY AND CONCLUSION

7.1 Summary of The Research

This thesis started with the first chapter, which is an introduction chapter. This chapter contains research background, problem statement, objectives of the research, research scope, research significance, and thesis organisation.

Chapter 2 focuses on the literature review of this research, which focuses on the compatible actions and the nonabelian tensor products of the groups. Various works related to the compatible actions, nonabelian tensor product of groups, and graph theory by different researchers were discussed in this chapter.

Some definitions and preliminary results on the automorphisms of the finite cyclic groups of the p-power order, compatible conditions, number theory, graph theory and GAP coding are given in Chapter 3. By using the GAP software, the number of the compatible pairs of actions for the finite cyclic groups of the p-power order has been computed and it is then verified with the theorem. All results in this chapter are used in the next chapters in order to prove the new results.

Meanwhile, some properties of the automorphisms for the finite cyclic groups of the p-power order, where p is an odd prime, are presented in Chapter 4. However, the necessary and sufficient conditions for a pair of actions that have the p-power order to act compatibly on each other have been determined. This chapter contains the compatibility for the actions that have order two.

Chapter 5 focuses in determining the number of compatible pairs of actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime. The number of the automorphisms for the finite cyclic groups of the *p*-power order, where *p* is an odd prime with the specific order were given first. According to the order of the actions, there are two cases in determining the number of the compatible pairs of actions for such type of groups, which are the trivial action and the actions that have the *p*-power order. From the results, the number of the compatible pairs of nontrivial actions that have the *p*-power order for the given nonabelian tensor product for such type of groups are the same.

Lastly, the compatible action graph and its subgraph have been introduced for the finite cyclic groups of the *p*-power order, where *p* is an odd prime. The compatible action graph was denoted by $\Gamma_{p\ G\otimes H}$ and consists of two nonempty sets; the set of the vertices $V(\Gamma_{p\ G\otimes H})$, which is a nonempty set of Aut(G) and Aut(H), and the set of the edges $E(\Gamma_{p\ G\otimes H})$, which is a nonempty set of all the compatible pairs of actions (ρ, ρ'). Consequently, some necessary and sufficient conditions for the subgroups of such type of groups to act compatibly on each other are provided. Then, the number of compatible pairs of actions which represents the intersection between the compatible action graph and its subgraph has been given .

7.2 Recommendation for Future Research

This research focuses only on the finite cyclic groups of the *p*-power order, where *p* is an odd prime. The main concern of this research is to find the maximum different nonabelian tensor product by determining the exact number of the compatible pair of actions for $C_{p^\alpha} \otimes C_{p^\beta}$ without finding the nonabelian tensor product. Thus, some suggestions for further research are presented as follows:

(i) Determine the general presentation for the automorphism group of the finite cyclic groups of the *p*-power order, where *p* is an odd prime.

(ii) Determine the compatible actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime by representation as a matrix.

(iii) Find the compatible actions for the finite cyclic groups of the *p*-power order, where *p* is an odd prime with the actions that have an even order.

(iv) Find the nonabelian tensor product for the finite cyclic groups of the p-power order, where p is an odd prime when the actions that have even order.

(v) Find the intersection between the compatible action graph and its subgraph for the finite cyclic groups of the p-power order, where p is an odd prime when the value of i greater than one.

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5.6

5.7

APPENDIX A

THE OUTPUT OF GAP SOFTWARE

The outputs for the GAP coding given in Figure 3.1 are stated as below. This output presents the list of the automorphisms with their specific order that satisfying the compatible conditions and the total of the number of the compatible actions.

---

gap> CompatibleAction(9,9);
k=4 (order action=3),l=4 (order action=3) Compatible
k=4 (order action=3),l=7 (order action=3) Compatible
k=7 (order action=3),l=4 (order action=3) Compatible
k=7 (order action=3),l=7 (order action=3) Compatible
No of Compatible4
gap> CompatibleAction(27,27);
k=4 (order action=9),l=10 (order action=3) Compatible
k=4 (order action=9),l=19 (order action=3) Compatible
k=7 (order action=9),l=10 (order action=3) Compatible
k=7 (order action=9),l=19 (order action=3) Compatible
k=10 (order action=3),l=4 (order action=9) Compatible
k=10 (order action=3),l=7 (order action=9) Compatible
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k=10 (order action=3),l=13 (order action=9) Compatible
k=10 (order action=3),l=16 (order action=9) Compatible
k=10 (order action=3),l=19 (order action=3) Compatible
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k=10 (order action=3),l=25 (order action=9) Compatible
k=13 (order action=9),l=10 (order action=3) Compatible
k=13 (order action=9),l=19 (order action=3) Compatible
k=16 (order action=9),l=10 (order action=3) Compatible
k=16 (order action=9),l=19 (order action=3) Compatible

k=19 (order action=3),l=4 (order action=9) Compatible
k=19 (order action=3),l=7 (order action=9) Compatible
k=19 (order action=3),l=10 (order action=3) Compatible
k=19 (order action=3),l=13 (order action=9) Compatible
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k=19 (order action=3),l=25 (order action=9) Compatible
k=22 (order action=9),l=10 (order action=3) Compatible
k=22 (order action=9),l=19 (order action=3) Compatible
k=25 (order action=9),l=10 (order action=3) Compatible
k=25 (order action=9),l=19 (order action=3) Compatible
No of Compatible28
gap> CompatibleAction(25,25);
k=6 (order action=5),l=6 (order action=5) Compatible
k=6 (order action=5),l=11 (order action=5) Compatible
k=6 (order action=5),l=16 (order action=5) Compatible
k=6 (order action=5),l=21 (order action=5) Compatible
k=11 (order action=5),l=6 (order action=5) Compatible
k=11 (order action=5),l=11 (order action=5) Compatible
k=11 (order action=5),l=16 (order action=5) Compatible
k=11 (order action=5),l=21 (order action=5) Compatible
k=16 (order action=5),l=6 (order action=5) Compatible
k=16 (order action=5),l=11 (order action=5) Compatible
k=16 (order action=5),l=16 (order action=5) Compatible
k=16 (order action=5),l=21 (order action=5) Compatible
k=21 (order action=5),l=6 (order action=5) Compatible
k=21 (order action=5),l=11 (order action=5) Compatible
k=21 (order action=5),l=16 (order action=5) Compatible
k=21 (order action=5),l=21 (order action=5) Compatible
No of Compatible16
gap> CompatibleAction(49,49);
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k=8 (order action=7),l=15 (order action=7) Compatible

k=8 (order action=7),l=22 (order action=7) Compatible

k=8 (order action=7),l=29 (order action=7) Compatible

k=8 (order action=7),l=36 (order action=7) Compatible

k=8 (order action=7),l=43 (order action=7) Compatible

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k=15 (order action=7),l=15 (order action=7) Compatible

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k=15 (order action=7),l=29 (order action=7) Compatible

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k=15 (order action=7),l=43 (order action=7) Compatible

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k=29 (order action=7),l=15 (order action=7) Compatible

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k=29 (order action=7),l=43 (order action=7) Compatible

k=31 (order action=6),l=19 (order action=6) Compatible

k=31 (order action=6),l=31 (order action=6) Compatible

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k=36 (order action=7),l=15 (order action=7) Compatible

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k=43 (order action=7),l=29 (order action=7) Compatible

k=43 (order action=7),l=36 (order action=7) Compatible

k=43 (order action=7),l=43 (order action=7) Compatible

No of Compatible40

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APPENDIX B

LIST OF PUBLICATIONS

Some results of this research conducted have been published/ presented in seminar/ conferences or submitted as listed in the following.

P1. Shahoodh, **K**, **M**., Mohamad, M. S., Yusof, Y. & Sulaiman, S. A., (2017). Number of Compatible Pair of Actions For Finite Cyclic Groups of *p*-Power Order. Journal Teknologi. (Submitted).

P2. Shahoodh, **K**, **M**., Mohamad, M. S., Yusof, Y. & Sulaiman, S. A., (2017). Number of Compatible Pair of Actions for Finite Cyclic Groups of 3-Power Order. International Journal of Simulation, System, Sciences and Technology.(accepted).-SCOPUS Index.

P3. Sulaiman, S. A., Mohamad, M. S., Yusof, Y. **Shahoodh**, **K**. **M**., (2017). The Number of Compatible Pair of Actions For Cyclic Groups of 2-Power Order. International Journal of Simulation, System, Sciences and Technology.(accepted).-SCOPUS Index.

P4. Mohd Sham Mohamad, Sahimel Azwal Sulaiman, Yuhani Yusof & **Mohammed Khalid Shahoodh**., (2017). Compatible Pair of Nontrivial Action for Finite Cyclic 2-Groups. 1st International Conference on Applied & Industrial Mathematics and Statistics 2017 (ICoAIMS 2017) Universiti Malaysia PAHANG. –SCOPUS Index.

P5. Shahoodh, **K**, **M**., Mohamad, M. S., Yusof, Y. & Sulaiman, S. A., (2016). Compatible Pair of Nontrivial Actions for Finite Cyclic Groups of 3-Power Order. Proceeding of 3rd National Conference for Postgraduate Research (NCON-PGR 2016): Knowledge Discovery For Wealth Creations. Universiti Malaysia Pahang, Pekan, Pahang, Malaysia:24-25 September 2016. Proceeding Conference.

P6. Sulaiman, S. A., Mohamad, M. S., Yusof, Y. **Shahoodh**, **K**. **M**., (2016). Number of Compatible Pair for Nontrivial Actions of Finite Cyclic 2-Group. Proceedings of 4th International Science Postgraduate Conference 2016 (ISPC2016) © Faculty of Science, Universiti Teknologi Malaysia.

P7. Sulaiman, S. A., Mohamad, M. S., Yusof, Y. & **Shahoodh**, **K**. **M**., (2016) Compatible Pair of Nontrivial Action For Finite Cyclic 2-Groups. Proceeding of 3rd National Conference for Postgraduate Research (NCON-PGR 2016): Knowledge Discovery For Wealth Creations. Universiti Malaysia Pahang, Pekan, Pahang, Malaysia:24-25 September 2016. Proceeding Conference.