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On the solvability of a mixed problem for partial differential equations of parabolic type with involution

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Abstract. Mixed partial differential equation of parabolic type with involution is considered. The sufficient conditions of the existence and uniqueness of the solution for the parabolic type equations with involution is obtained. The Fourier method is used to find a classical solution of the mixed problem for the transformed parabolic type equation with involution.

1. Introduction

Many works [1]-[6] are dedicated to the study of partial differential equations (PDE) of parabolic type. Parabolic PDEs are used to describe a wide variety of time-dependent phenomena, including heat conduction, particle diffusion, and pricing of derivative investment instrument. This paper is devoted to investigating the problems of existence and uniqueness of the solution for the parabolic type equations with involution. We quote here only some latest papers on the problems with involution. The fractional analogue of Helmholtz equation with an involution perturbation in a rectangular domain is considered in [7]. The spectral problem for the second-order differential operators with involution and boundary conditions of Dirichlet type is studied in [8], where the Green's function related to the boundary problem is constructed and uniform estimates of the latter are obtained. The equi-convergence of eigenfunction expansions of two second-order differential operators with involution and boundary conditions of Dirichlet type for any function in $L_2(-1,1)$ is established. The solvability of the main boundary value problems for the nonlocal Poisson equation is studied in [9], where necessary and sufficient conditions for existence and uniqueness for the considered problems are obtained.

In this paper, we study boundary value problems for heat equation with involution and obtained the necessary and sufficient condition for existence and uniqueness of a solution of the considered problems by using the spectral method.

Let $\Pi = \{(x, y) : 0 < x < p, 0 < y < q\}$, $p, q > 0$ be rectangular domain. In this paper we deal with problems on the solvability of the mixed heat problem with involution. Let $u(t, x, y)$ be function



defined for $(x, y) \in \Pi, t > 0$. With the help of Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ we introduce the operator of the following form

$$Lu(t, x) \equiv a_0 \cdot \Delta u(x, y, t) + a_1 \cdot \Delta u(p - x, q - y, t), \quad (t, x, y) \in \Omega = \mathbb{R}_+ \times \Pi,$$

where $\Delta u(t, p - x, q - y)$ we interpret as $\Delta u(t, p - x, q - y) = \Delta u(t, \xi, \eta) \Big|_{(\xi, \eta) = (p-x, q-y)}$.

We deal with the following problem: find a function $u(t, x, y) \in C(\bar{\Omega}) \cap C_{t,x,y}^{1,2}(\Omega)$, satisfying the partial differential equation

$$\frac{\partial u}{\partial t}(T - t, x, y) + Lu(t, x, y) = f(t, x, y), \quad (t, x, y) \in G = (0, T) \times \Pi, \tag{1}$$

and initial condition

$$u(x, y, 0) = 0, \quad 0 \leq x \leq p, 0 \leq y \leq q \tag{2}$$

and boundary conditions

$$u(0, y, t) = u(p, y, t) = 0, \quad 0 \leq y \leq q, 0 \leq t \leq T, \tag{3}$$

$$u(x, 0, t) = u(x, q, t) = 0, \quad 0 \leq x \leq p, 0 \leq t \leq T. \tag{4}$$

A function $u(t, x, y) \in C(\bar{G}) \cap C_{t,x,y}^{1,2}(G)$ is said to be regular solution of the problem (1)-(4), if $u(t, x, y)$ satisfies (1) in classical means, and furthermore, satisfies the initial and boundary conditions (2),(3) and (4).

2. On the eigenfunctions and eigenvalues of a spectral problem with involution

In this section we solve a spectral problem (1)-(4) by using separation (Fourier's) method. If the solution of the problem (1)-(4) is represented as $u(t, x, y) = v(t) \cdot w(x, y)$, then the functions $v(t)$ and $w(x, y)$ are solutions to the following spectral problems

$$v'(p - t) = \mu v(t), \tag{5}$$

$$v(0) = 0, \tag{6}$$

and

$$Lw(x, y) = -\gamma \cdot w(x, y), \quad (x, y) \in \Pi \tag{7}$$

$$w(x, y) = 0, \quad (x, y) \in \partial \Pi \tag{8}$$

where we used the notation $\gamma = \mu - \lambda$. The following result on the eigenvalues and eigenfunctions of the (5),(6) :

Lemma 2.1. The spectral problem (5), (6) has an infinite set of eigenvalues

$$\mu_n = (-1)^n \left(n + \frac{1}{2} \right) \frac{\pi}{T}, \quad n = 0, 1, 2, \dots, \tag{9}$$

and their corresponding eigenfunctions

$$v_n(t) = \sqrt{\frac{2}{T}} \sin \left(n + \frac{1}{2} \right) \frac{\pi t}{T}, \quad n = 0, 1, \dots, \tag{10}$$

which form an orthonormal basis of the space $L_2(0, T)$.

For the proof we refer to [10]. We proceed to the study of the problem (7),(8).

Theorem 1. Let H_j - eigenvector of the matrix A , and ε_j is corresponding eigenvalue. If coefficients a_0, a_1 in (7),(8) the coefficients are that for $\varepsilon_j \neq 0$ and $w(x, y)$ is eigenfunction, and γ corresponding eigenvalue, then a function $z_j(x, y) = h_{1,j}w(x, y) + h_{2,j}w(p - x, q - y)$ is eigenfunction corresponding to eigenvalue $\nu_j = \frac{\gamma}{\varepsilon_j}$ for the Dirichlet problem

$$-\Delta z_j(x, y) = \nu_j \cdot z_j(x, y), \quad (x, y) \in \Pi \tag{11}$$

$$z_j(x, y) = 0, \quad (x, y) \in \partial \Pi. \tag{12}$$

Proof. Let $w(x, y)$ and γ is an eigenfunction and an eigenvalue of the problem (7), (8). By putting in (7) $(p - x, q - y)$ instead (x, y) , we obtain the system of equations

$$\begin{cases} a_0 \cdot \Delta w(x, y) + a_1 \cdot \Delta w(p - x, q - y) = -\gamma \cdot w(x, y), \\ a_1 \cdot \Delta w(x, y) + a_0 \cdot \Delta w(p - x, q - y) = -\gamma \cdot w(p - x, q - y), \end{cases} \tag{13}$$

which can be written in the matrix form as follows

$$ALW = -\gamma W$$

where $W = (w(x), w(p - x, q - y))^T$, and the matrix A has a form $A = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix}$

It is easy to show that $H_1 = (1, 1)^T; H_2 = (1, -1)^T$ are eigenvectors of a matrix A and $\varepsilon_1 = a_0 + a_1, \varepsilon_2 = a_0 - a_1$ are corresponding eigenvalues:

$$AH_j = \varepsilon_j H_j \Leftrightarrow \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} h_{1j} \\ h_{2j} \end{pmatrix} \equiv \begin{pmatrix} a_0 h_{1,j} + a_1 h_{2,j} \\ a_0 h_{2,j} + a_1 h_{1,j} \end{pmatrix} = \varepsilon_j \begin{pmatrix} h_{1j} \\ h_{2j} \end{pmatrix},$$

here $h_{1j} = 1, j = 1, 2$ and $h_{2j} = (-1)^{j-1}, j = 1, 2$. Then from the equation (13)

$$\varepsilon_j \Delta [h_{1,j}w(x, y) + h_{2,j}w(p - x, q - y)] = -\gamma [h_{1,j}w(x, y) + h_{2,j}w(p - x, q - y)]$$

Using the notation

$$z_j(x, y) = h_{1,j}w(x, y) + h_{2,j}w(p - x, q - y)$$

We conclude that in the case when $\varepsilon_j \neq 0$ a function defined in above is a solution of the spectral problem (11), (12) with $\nu_j = \frac{\gamma}{\varepsilon_j}, j = \overline{1, 2}$.

The following result generates the method of construction of the eigenfunctions and eigenvalues of the problem (7), (8).

Theorem 2. Let $H = (h_1, h_2)^T$ - eigenvector of the matrix $A = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix}$, and ε is corresponding

eigenvalue. If $z(x, y)$ and ν are eigenfunction and eigenvalue, respectively, of the Dirichlet problem

$$-\Delta z(x, y) = \nu z(x, y), (x, y) \in \Pi, \tag{14}$$

$$z(x, y) = 0, (x, y) \in \partial\Pi \tag{15}$$

Then

$$w(x, y) = h_1 z(x, y) + h_2 z(p - x, q - y) \tag{16}$$

is eigenfunction of the problem (7), (8) corresponding to eigenvalue $\gamma = \varepsilon \cdot \nu$.

Proof. Let a function $z(x, y)$ be eigenfunction of the Dirichlet problem (14),(15). Then

$$-\Delta z(p - x, q - y) = \nu \cdot z(p - x, q - y), (x, y) \in \Pi,$$

$$z(p - x, q - y)|_{\partial\Pi} = 0.$$

For a function $w(x, y)$ we obtain a system of the equation

$$\Delta w(x, y) = -\nu [h_1 z(x, y) + h_2 z(p - x, q - y)]$$

$$\Delta w(p - x, q - y) = -\nu [h_2 z(x, y) + h_1 z(p - x, q - y)]$$

Multiplying first equation by a_0 and second equation by a_1 in the system above we obtain

$$\begin{aligned} & a_0 \Delta w(x, y) + a_1 \Delta w(p - x, q - y) = \\ & -\nu (a_0 h_1 + a_1 h_2) z(x, y) - \nu (a_1 h_1 + a_0 h_2) z(p - x, q - y) \\ & = -\nu \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \begin{pmatrix} z(x, y) \\ z(p - x, q - y) \end{pmatrix} = -\nu AHW \end{aligned} \tag{17}$$

Further, considering the equation $AH = \varepsilon H$, the expression on the right-hand side of **Error!**

Reference source not found. we have

$$\begin{aligned} \varepsilon \nu HW &= -\nu (a_0 h_1 + a_1 h_2) z(x, y) - \nu (a_1 h_1 + a_0 h_2) z(p - x, q - y) \\ &= -\nu \varepsilon h_1 z(x, y) - \nu \varepsilon h_2 z(p - x, q - y) \\ &= -\gamma w(x, y) \end{aligned}$$

Hence, the function from the condition that a function

$$w(x, y) = h_1 z(x, y) + h_2 z(p - x, q - y)$$

satisfies equation (7). It is not difficult to show that this function also satisfies the boundary condition (8), which completes the proof of the theorem.

Here we give one useful property of the eigenfunctions of the Laplace operator, which can be obtained by the using the ideas of the paper [11].

Corollary 2.1. All eigenfunctions of the Dirichlet problem (14), (15) can be chosen so that they have one of the properties of symmetry:

$$z(x, y) + z(p - x, q - y) = 0, \forall (x, y) \in \Pi,$$

or

$$z(x, y) - z(p - x, q - y) = 0, \forall (x, y) \in \Pi.$$

Using the property of the eigenfunctions obtained in the Corollary 2.1 we have

Theorem 3. Let $z(x, y)$ be eigenfunction of the Dirichlet problem (14), (15), and ν is corresponding eigenvalue. If $\varepsilon_1 > 0, \varepsilon_2 > 0$, then a function $w(x, y) \equiv z(x, y)$ is eigenfunction of the problem (7), (8), and corresponding eigenvalues are:

$$\gamma = \begin{cases} \nu\varepsilon_1, & \text{if } z(x, y) = z(p - x, q - y) \\ \nu\varepsilon_2, & \text{if } z(x, y) = -z(p - x, q - y) \end{cases}$$

Proof. To prove the theorem, we use the statement of Theorem 2. First, we note that the system (17) can be rewritten in the following matrix form $HZ = W$, where

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, Z = \begin{pmatrix} z(x, y) \\ z(p - x, q - y) \end{pmatrix}, W = \begin{pmatrix} w_1(x, y) \\ w_2(x, y) \end{pmatrix}.$$

It is easy to show that the inverse of the matrix to the matrix H has a form $H^{-1} = \frac{1}{2}H$. Further, using the matrix H^{-1} we construct the following system:

$$w_j(x, y) = \frac{1}{2} [h_{j,1}z(x, y) + h_{j,2}z(p - x, q - y)], j = \overline{1, 2},$$

where $h_{j,i}, j, i = 1, 2$ elements of the matrix H^{-1} . From the Theorem 2 we conclude that functions $w_j(x, y)$ are eigenfunctions of the problem (7),(8) and corresponding eigenvalues have a form $\gamma_j = \varepsilon_j \cdot \nu$.

Let for the function $z(x, y)$ we have $z(x, y) = z(p - x, q - y), (x, y) \in \Pi$. Then, we obtain

$$h_{j,1}z(x, y) + h_{j,2}z(p - x, q - y) = (h_{j,1} + h_{j,2})z(x, y).$$

co we have

$$w_j(x, y) = \frac{1}{2} [(h_{j,1} + h_{j,2})z(x, y)], j = \overline{1, 2}.$$

Next considering the definition of $h_{j,1}$ and $h_{j,2}$, we have

$$w_1(x, y) = \frac{1}{2} [z(x, y) + z(p - x, q - y)]$$

and

$$w_2(x) = \frac{1}{2} [z(x, y) - z(p - x, q - y)].$$

Corollary 2.2. The eigenfunctions of the (14), (15)

$$z_{k,m}(x, y) = \sqrt{\frac{2}{p}} \sin \frac{k\pi}{p} x \cdot \sqrt{\frac{2}{q}} \sin \frac{m\pi}{q} y, \quad k, m = 1, 2, \dots,$$

Corresponding to the eigenvalues

$$v_{k,m} = \left(\frac{k\pi}{p}\right)^2 + \left(\frac{m\pi}{q}\right)^2, \quad k, m = 1, 2, \dots$$

form complete orthonormal system in the space $L_2(\Pi)$.

Using this statement and Theorem 3, we can construct the eigenfunctions and eigenvalues of the problem (7),(8). According to Corollary 2.2, it easy to show that the eigenfunctions of the Dirichlet problem have the following form:

$$z_{k,m}(x, y) = \sqrt{\frac{2}{p}} \sin \frac{k\pi}{p} x \cdot \sqrt{\frac{2}{q}} \sin \frac{m\pi}{q} y, \quad k, m = 1, 2, \dots,$$

From here we find

$$\begin{aligned} z_{k,m}(p-x, q-y) &= \sqrt{\frac{2}{p}} \sin \frac{k\pi}{p} (p-x) \cdot \sqrt{\frac{2}{q}} \sin \frac{m\pi}{q} (q-y) \\ &= (-1)^{k+m} \sqrt{\frac{2}{p}} \sin \frac{k\pi}{p} x \cdot \sqrt{\frac{2}{q}} \sin \frac{m\pi}{q} y = (-1)^{k+m} z_{k,m}(x, y) \end{aligned}$$

Then the eigenfunctions and the corresponding eigenvalues of the problem (7),(8) have the form if $k = 2i - 1, m = 2j - 1, i, j = 1, 2, \dots$, then $z_{2i-1,2j-1}(p-x, q-y) = z_{2i-1,2j-1}(x, y)$,

and hence,

$$\begin{aligned} w_{2i-1,2j-1}(x, y) &= \sqrt{\frac{2}{p}} \sin \frac{(2i-1)\pi}{p} x \cdot \sqrt{\frac{2}{q}} \sin \frac{(2j-1)\pi}{q} y, \\ \gamma_{2i-1,2j-1} &= \varepsilon_1 \cdot \pi^2 \left[\left(\frac{2i-1}{p}\right)^2 + \left(\frac{2j-1}{q}\right)^2 \right]; \end{aligned}$$

if $k = 2i - 1, m = 2j, i, j = 1, 2, \dots$, then $z_{2i-1,2j}(p-x, q-y) = -z_{2i-1,2j}(x, y)$

and therefore

$$w_{2i-1,2j}(x, y) = -\sqrt{\frac{2}{p}} \sin \frac{(2i-1)\pi}{p} x \cdot \sqrt{\frac{2}{q}} \sin \frac{2j\pi}{q} y, \gamma_{2i-1,2j} = \varepsilon_2 \cdot \pi^2 \left[\left(\frac{2i-1}{p}\right)^2 + \left(\frac{2j}{q}\right)^2 \right];$$

if $k = 2i, m = 2j - 1, i, j = 1, 2, \dots$, then $z_{2i,2j-1}(p-x, q-y) = -z_{2i,2j-1}(x, y)$

and hence,

$$w_{2i,2j-1}(x, y) = -\sqrt{\frac{2}{p}} \sin \frac{2i\pi}{p} x \cdot \sqrt{\frac{2}{q}} \sin \frac{(2j-1)\pi}{q} y, \gamma_{2i,2j-1} = \varepsilon_2 \cdot \pi^2 \left[\left(\frac{2i}{p}\right)^2 + \left(\frac{2j-1}{q}\right)^2 \right];$$

Corollary 2.3. Let the condition be satisfied $\varepsilon_1 > 0, \varepsilon_2 > 0$. Then the system of eigenfunctions of the problem (7),(8) is complete in space $L_2(\Pi)$.

Theorem 4. The eigenfunctions of the problem (1)-(4)

$$u_{n,k,m}(t, x, y) = \sqrt{\frac{8}{pqT}} \sin\left(n + \frac{1}{2}\right) \frac{\pi t}{T} \sin \frac{k\pi}{p} x \cdot \sin \frac{m\pi}{q} y, \quad n = 0, 1, \dots, k, m = 1, 2, \dots$$

corresponding to the eigenvalues $\lambda_{n,k,m} = \mu_n - \gamma_{k,m}$, where $\mu_n = (-1)^n \left(n + \frac{1}{2} \right) \frac{\pi}{T}$, $n = 0, 1, 2, \dots$, and

$$\gamma_{k,m} = \varepsilon \cdot \pi^2 \left[\left(\frac{k}{p} \right)^2 + \left(\frac{m}{q} \right)^2 \right], \text{ with } \varepsilon = \begin{cases} \varepsilon_1, & \text{if } k = 2i - 1, m = 2j - 1 \\ \varepsilon_2, & \text{if } k = 2i - 1, m = 2j \text{ or } k = 2i, m = 2j - 1 \end{cases}$$

form a complete orthonormal system in $L_2(G)$.

3. On the existence and uniqueness of the regular solution

Let consider a linear operator Λ

$$\Lambda u = \frac{\partial u}{\partial t}(T - t, x, y) + Lu(t, x, y) \tag{18}$$

corresponding to the (1) with the domain of definition

$$D_\Lambda = \left\{ u(t, x, y) \in C(\overline{\Omega}) : \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} \in C(\overline{\Omega}); u|_{t=0} = 0, u|_{\partial G} = 0 \right\}.$$

The operator Λ with the domain of definition D_Λ is symmetric, consequently $\ker(\Lambda) = \{0\}$, which guarantees the existence of the inverse Λ^{-1} . Let $u \in D_\Lambda, f \in L_2(G)$, and

$$\Lambda u = f.$$

We represent Λu and f by eigenfunctions $\{u_{n,k,m}(t, x, y), n = 0, 1, \dots, k, m = 1, 2, \dots\}$:

$$\begin{aligned} \Lambda u &= \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (\Lambda u, u_{n,k,m}) u_{n,k,m}(t, x, y) \\ f &= \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (f, u_{n,k,m}) u_{n,k,m}(t, x, y) \end{aligned}$$

By comparing the coefficients of both sides

$$(\Lambda u, u_{n,k,m}) = (f, u_{n,k,m})$$

Using the symmetricity of the operator Λ we obtain

$$(\Lambda u, u_{n,k,m}) = (u, \Lambda u_{n,k,m}) = \lambda_{n,k,m} (u, u_{n,k,m})$$

Hence

$$(u, u_{n,k,m}) = \frac{(f, u_{n,k,m})}{\lambda_{n,k,m}}.$$

Finally, the inverse operator has a following form:

$$u(t, x, y) = \Lambda^{-1} f = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(f, u_{n,k,m})}{\lambda_{n,k,m}} u_{n,k,m}(t, x, y)$$

Theorem 5. In order that there exist unique regular solution of the problem (1)-(4), it is necessary and sufficient that $\lambda_{n,k,m} \neq 0, k, m = 1, 2, \dots, n = 0, 1, \dots$.

It is not difficult to see that if the conditions of the Theorem 5 are satisfied then for any $f \in L_2(G)$ there exists a regular solution of the (1)-(4), which has the following form

$$u(t, x, y) = \Lambda^{-1} f = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(f, u_{n,k,m})}{\lambda_{n,k,m}} u_{n,k,m}(t, x, y)$$

To satisfy the condition that the regular solution $u \in L_2(G)$, we have to require that

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left| \frac{(f, u_{n,k,m})}{\lambda_{n,k,m}} \right|^2 < \infty.$$

4. Conclusion

The necessary and sufficient conditions for regular solution to exist and to be unique are obtained by using the properties of the orthonormal system eigenfunction of the mixed heat problem with involution in all arguments. The properties of the second order partial differential operators with involution are studied and applied to solve mixed heat equation with involution.

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