Chapter

Perspective Chapter: A New Bivariate Inverted Nakagami Distribution - Properties and Applications

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Abstract

In this work, a new bivariate inverted Nakagami distribution that can be used to model real-world datasets has been investigated. The newly developed bivariate distribution's cumulative distribution function and probability density function are defined. The bivariate distribution derives from the Farlie Gumbel Morgenstern, and the marginal density functions are also determined. Some fundamental estimation techniques, such as maximum-likelihood estimation and inference functions for margins, are used to derive the parameters of its estimates. Applications to real-world datasets pertaining to kidney infection diseases and the UEFA Champions' League group stage for the seasons 2004–2005 and 2005–2006 help to assess the efficacy of the proposed distribution.

Keywords: bivariate inverted Nakagami, Farlie Gumbel Morgenstern, inverted Nakagami, marginal density functions, maximum likelihood estimation

1. Background

Over the past decades, many researchers have attempted to introduce new of probability distributions that provide better flexibility than the traditional ones. However, several of these distributions are inappropriate for modeling different characteristics of real data. Therefore, there is a need to develop more flexible distributions, particularly in practical domains including finance, environment, health, and engineering. This study proposed a novel multivariate probability distribution known as the bivariate inverted Nakagami distribution. This bivariate distribution was introduced from the inverse Nakagami distribution. The proposed distribution can serve as alternative to various current distributions, such as the traditional Nakagami and inverse Nakagami distributions and many others.

2. Introduction

The problem of developing novel families of continuous bivariate distributions is one of the significant and current research topics in probability and statistics. This is due to the limitations of the existing distributions that capture the true behavior of many real phenomena found in a broad variety of practical domains. The Nakagami distribution was introduced recently [1]. This distribution has been applied in ultrasound images [2], microwave hyperthermia [3], cataract stiffness [4], and many other applications. The Nakagami distribution is a probability distribution with two parameters that is related to the gamma distribution. This distribution can be used quite effectively in modeling many empirical datasets [5], especially in communications engineering and mobile radio [6–9]. Nakagami distribution has also found important applications in wind speed [10], medical sciences [11, 12], and hydrologic engineering [13–15]. Other important applications of this distribution are in medical image processing [16, 17], seismological analysis [18], and engineering [14]. This distribution has been used to model the hazard rate in reliability theories because of its memory less property. It has been shown that the Nakagami distribution is a more appropriate function to evaluate the reliability of electrical components compared to the Weibull and Gamma distributions [19].

Due to the successful use of Nakagami distribution in different fields, several researchers have explored the applicability of this distribution. For example, the Nakagami distribution was used [20] to evaluate the ablated region induced by focused ultrasound exposures at different acoustic power levels in transparent tissue-mimicking phantoms. Schwartz et al. [21] developed analytic and bootstrap bias-corrected maximum-likelihood estimators for the shape parameter of the Nakagami distribution. The relationship between the Nakagami distribution and other distributions such as the gamma distribution, the Rayleigh distribution, the Weibull distribution, the chi-square distribution, and the exponential distribution was explored [22]. The study suggested that through the gamma distribution, it is much easier to derive the moments of a Nakagami random variable. The Bayesian estimators of the scale parameter of the Nakagami distribution were derived [23]. The performance of the estimator was evaluated based on the relative posterior risk. The maximum-likelihood estimates for the Nakagami distribution have been compared with other estimators [24]. Recently, the Bayesian method of estimation is used in order to estimate the scale parameter of the Nakagami distribution by using Jeffreys', Extension of Jeffreys', and Quasi priors under three different loss functions [24]. Some of the distributional properties and reliability characteristics of this distribution are discussed [25].
The length-biased form of the Nakagami distribution was introduced by [26]. The new length-biased Nakagami distribution was applied [27] to generate a survival model.

The inverse Nakagami (inverted Nakagami) distribution is proposed [28]. This distribution is the reciprocal of the Nakagami model that plays an important role in the general areas of medical, communication engineering, hydrological sciences, and reliability systems. The proposed model is useful to describe devices that are subjected to high stress, providing a high failure rate after a short repair time.

In many practical problems, multivariate lifetime data arise frequently, and in these situations, it is important to consider different multivariate models that could be used to model such multivariate lifetime data. Several authors, for example, [29–33], have considered the problem of proposing general multivariate models with given marginal distributions. There are very few multivariate distributions in the recent statistical literature. These include the bivariate Kumaraswamy distribution [34], the bivariate Poisson exponential-exponential distribution [35], and the bivariate alpha power exponential distribution [36]. For a bivariate model having given marginals to be useful in practical situations, it is important and desirable that the model can be handled with mathematical ease and that any parameter(s) incorporated in the model lends itself to some important physical representation, for example, the measure of location or scale or an association between components, etc. [37].

A random variable T is said to follow an inverted Nakagami distribution with shape parameters a and $b$if it's cumulative distribution function (cdf) and probability density function (pdf) are respectively given as

$$F(t;a,b) = 1 - \frac{\gamma\left(a, \frac{a}{bt^2}\right)}{\Gamma(a)}, \qquad a,b;t>0 \tag{1}$$

and

$$f(t;a,b) = \frac{2}{\Gamma(a)}\left(\frac{a}{b}\right)^a t^{-2a-1}\exp\left(-\frac{a}{bt^2}\right), \qquad a,b;t>0 \tag{2}$$

A bivariate Nakagami distribution with identical fading parameters was first presented in [1]. In Ref. [38], this restriction was raised and a bivariate Nakagami distribution with arbitrary fading parameters was derived. Recently, a bivariate Nakagami distribution with arbitrary correlation and fading parameters was studied [39]. The primary reason for this expansion was to derive the joint moment generating function, joint probability density function, joint cumulative distribution function, power correlation coefficient, and several statistics related to the signal-to-noise ratio at the output of the selection combiner, namely, outage probability, probability density function, mean, and among other expressions. A new multivariate Nakagami distribution with arbitrary correlation and fading parameters was introduced [40] to obtain the joint probability density function for the Nakagami distribution generated from correlated Gaussian random variables based on an arbitrary correlation matrix and different fading parameters.

Recently, many researchers considered the bivariate extension of the probability distributions, such as Yang et al. [41] presented a class of multivariate copulas whose two-dimensional marginals belong to the family of bivariate Fréchet copulas. Myrhaug and Leira [42] discussed the bivariate Fréchet distribution, which is obtained by transforming a bivariate Rayleigh distribution. Zheng et al. [43] discussed the bivariate Fréchet copula as a mixture of three simple structures co-monotonicity, independence, and counter-monotonicity. A copula is a convenient approach to describe a multivariate distribution with a dependence structure. Nelsen [44] introduced copulas as following; copula is a function that joins multivariate distribution functions with uniform [0, 1] margins. Sklar [45] introduced the pdf and cdf for the two dimension copula as follows, consider the two random variables T_1 and T_2, with distribution functions $F_1(t_1)$ and $F_2(t_2)$ respectively, then the cdf and pdf for bivariate copula are respectively given as

$$F(t_1,t_2) = C(F_1(t_1), F_2(t_2)), \tag{3}$$

$$f(t_1,t_2) = f_1(t_1)f_2(t_2)c(F_1(t_1), F_2(t_2)). \tag{4}$$

where $C(F_1(t_1), F_2(t_2))$ and $c(F_1(t_1), F_2(t_2))$ represents the copulas function for the cdf and pdf of the Bivariate function.

In this case, $F(t_i)$and$f(t_i)$ for $i = 1, 2$ represents the cdf and pdf of the Inverted Nakagami distribution.

Many copulas had been defined based on Eqs. (3) and (4) such as Farlie-Gumbel-Morgenstern (FGM), Ali-MikhailHaq (AMH), and Plackett. The FGM copula is one of the most popular parametric families of copulas, the family was first introduced [46]. Almetwally et al. [47] used the FGM copula to introduce the bivariate Weibull distribution. Ali et al. [37] proposed an AMH copula, and Kumar [48] discussed the correlation coefficient of the AMH copula by Spearman and Kendall. Almetwally and Muhammed [49] studied the bivariate extension of the Fréchet distribution based on FGM and AMH copula functions and discussed their statistical properties. The Plackett copula was introduced [50] to construct a class of bivariate distributions from given margins. This class contains the known boundary distributions and the members corresponding to independent random variables.

The need for an accurate and effective estimating method for real life data using probability distribution is of great importance. This chapter presents a novel bivariate inverted Nakagami, which provides greater accuracy and flexibility in fitting real life data in a broad variety of practical domains.

In this chapter, we examine and describe the statistical characteristics of the novel bivariate inverted Nakagami distribution based on the FGM copula function. Different estimation techniques are used to estimate the parameters for the bivariate-inverted Nakagami distribution.

2.1 Motivation of the chapter

The bivariate probability distributions are set of distributions proposed to foster new hybridized probability distributions with the intent of expanding the modeling capacity of classical probability distributions. This work attempts to improve the classical Nakagami and inverse Nakagami of distributions for modelimg real life data.

2.2 Challenges of the topic

The Nakagami and inverse distributions seem to be flexible but has not been fully explored in statistical literature and several of their properties have not been studied.

2.3 Significance/implication

This research work developed a bivariate distribution capable of handling skewness and leptokurtic behavior in most datasets in different fields such as medicine, engineering, finance and economics. It also shown that noticeable improvements are made when the bivariate inverted Nakagami distribution is used and tested among the traditional Nakagami models.

The remaining section of this chapter is structured as follows: In Section 2, a bivariate-inverted Nakagami distribution has been identified. Section 3 discusses parameter estimation techniques for the bivariate inverted Nakagami distributions. Applications to two real-world datasets are provided in Section 4, and Section 5 addresses the conclusion of a few remarks for the bivariate-inverted Nakagami model.

3. Bivariate-inverted Nakagami distribution

Bivariate-inverted Nakagami (BIN) distribution can be obtained based on copula function by considering the cdf and pdf defined respectively in Eqs. (3) and (4) presented as

$$F(t_1, t_2) = C\left(1 - \gamma_1\left(a_1, \frac{a_1}{b_1 {t_1}^2}\right),\ 1 - \gamma_2\left(a_2, \frac{a_2}{b_2 {t_2}^2}\right)\right) \tag{5}$$

which is the cdf of the BIN distribution, where $\gamma_i\left(a_i, \frac{a_i}{b_i {t_i}^2}\right) = \frac{\gamma\left(a_i, \frac{a_i}{b_i {t_i}^2}\right)}{\Gamma(a_i)}$ for $i = 1, 2$. The pdf corresponding to Eq. (5) is obtained as

$$f(t_1, t_2) = \frac{4}{\Gamma(a_1)\Gamma(a_2)}\left(\frac{a_1}{b_1}\right)^{a_1}\left(\frac{a_2}{b_2}\right)^{a_2} {t_1}^{-2a_1-1} {t_2}^{-2a_2-1} \exp\left(-\frac{a_1}{b_1 {t_1}^2}\right)\exp\left(-\frac{a_2}{b_2 {t_2}^2}\right) c\left(1 - \gamma_1\left(a_1, \frac{a_1}{b_1 {t_1}^2}\right), 1 - \gamma_2\left(a_2, \frac{a_2}{b_2 {t_2}^2}\right)\right) \tag{6}$$

According to Refs. [45–47], the cdf and pdf presented in Eqs. (3) and (4) can be defined as

$$C(\theta, \lambda) = \theta\lambda\{1 + \alpha(1 - \theta)(1 - \lambda)\} \tag{7}$$

and

$$c(\theta, \lambda) = \{1 + \alpha(1 - 2\theta)(1 - 2\lambda)\}, \tag{8}$$

which is the FGM copula class, where $\theta = F_1(t_1)$, $\lambda = F_2(t_2)$; then $v, w \in \mathrm{I}$ for $\mathrm{I} = [0, 1]$ and $\alpha \in [-1, 1]$, and this serves as the dependence parameter, likewise an independence parameter if $\alpha = 0$.

As defined in Eqs. (7) and (8), the cdf and pdf of the new bivariate-inverted Nakagami distribution can be obtained from Eqs. (5) and (7) as

$$F(t_1, t_2) = \left\{1 - \gamma_1\left(a_1, \frac{a_1}{b_1 {t_1}^2}\right)\right\}\left\{1 - \gamma_2\left(a_2, \frac{a_2}{b_2 {t_2}^2}\right)\right\}\left\{1 + \alpha\left(\gamma_1\left(a_1, \frac{a_1}{b_1 {t_1}^2}\right)\right)\left(\gamma_2\left(a_2, \frac{a_2}{b_2 {t_2}^2}\right)\right)\right\} \tag{9}$$

and

$$\begin{aligned} f(t_1, t_2) = {} & \frac{4}{\Gamma(a_1)\Gamma(a_2)}\left(\frac{a_1}{b_1}\right)^{a_1}\left(\frac{a_2}{b_2}\right)^{a_2} {t_1}^{-2a_1-1} {t_2}^{-2a_2-1} \exp\left(-\frac{a_1}{b_1 {t_1}^2}\right)\exp\left(-\frac{a_2}{b_2 {t_2}^2}\right) \\ & \times\left\{1 + \alpha\left(1 - 2\left(1 - \gamma_1\left(a_1, \frac{a_1}{b_1 {t_1}^2}\right)\right)\right)\left(1 - 2\left(1 - \gamma_2\left(a_2, \frac{a_2}{b_2 {t_2}^2}\right)\right)\right)\right\} \end{aligned} \tag{10}$$

4. Parameter estimation of the copula-based model

In this section, the maximum-likelihood estimation (MLE) and Inference functions for margins (IMF) are employed in estimating the parameters of the bivariate-inverted Nakagami distribution.

4.1 Estimation using maximum-likelihood method

To obtain the parameters of the BIN distribution using maximum-likelihood method, the likelihood function of Eq. (10) can be expressed as

$$L = \left(\frac{4}{\Gamma(a_1)\Gamma(a_2)}\left(\frac{a_1}{b_1}\right)^{a_1}\left(\frac{a_2}{b_2}\right)^{a_2}\right)^n \prod_{i=1}^{n}\left(t_{1i}{}^{-2a_1-1}t_{2i}{}^{-2a_2-1}\right)$$
$$\times \exp\left(-\sum_{i=1}^{n}\left(\frac{a_1}{b_1 t_{1i}{}^2}\right)\right)\exp\left(-\sum_{i=1}^{n}\left(\frac{a_2}{b_2 t_{2i}{}^2}\right)\right)$$
$$\times \prod_{i=1}^{n}\left\{1+\alpha\left(1-2\left(1-\gamma_1\left(a_1,\frac{a_1}{b_1 t_{1i}{}^2}\right)\right)\right)\left(1-2\left(1-\gamma_2\left(a_2,\frac{a_2}{b_2 t_{2i}{}^2}\right)\right)\right)\right\} \tag{11}$$

The log-likelihood function corresponding to Eq. (11) can be presented as

$$\ell = n\log\left(\frac{4}{\Gamma(a_1)\Gamma(a_2)}\left(\frac{a_1}{b_1}\right)^{a_1}\left(\frac{a_2}{b_2}\right)^{a_2}\right) - (2a_1+1)\sum_{i=1}^{n}(t_{1i}) - (2a_2+1)\sum_{i=1}^{n}(t_{2i}) \tag{12}$$
$$-\frac{a_1}{b_1}\sum_{i=1}^{n}\left(\frac{1}{t_{1i}{}^2}\right) - \frac{a_2}{b_2}\sum_{i=1}^{n}\left(\frac{1}{t_{2i}{}^2}\right) + \sum_{i=1}^{n}\log$$
$$\left(1+\alpha\left(1-2\left(1-\gamma_1\left(a_1,\frac{a_1}{b_1 t_{1i}{}^2}\right)\right)\right)\left(1-2\left(1-\gamma_2\left(a_2,\frac{a_2}{b_2 t_{2i}{}^2}\right)\right)\right)\right)$$

Now, we can derive the parameters of bivariate-inverted Nakagami distribution by differentiating Eq. (12) partially with respect to parameters $a_1, a_2, b_1, b_2,$ and α obtained as

$$\frac{\partial\ell}{\partial a_1} = -n\psi(a_1) + n\left(1+\log\left(\frac{a_1}{b_1}\right)\right) - 2\sum_{i=1}^{n}\log(t_{1i}) - \frac{1}{b_1}\sum_{i=1}^{n}\left(\frac{1}{t_{1i}{}^2}\right) \tag{13}$$
$$+\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial a_1}(1+\alpha M_1 M_2)}{1+\alpha M_1 M_2}\right)$$

$$\frac{\partial\ell}{\partial a_2} = -n\psi(a_2) + n\left(1+\log\left(\frac{a_2}{b_2}\right)\right) - 2\sum_{i=1}^{n}\log(t_{2i}) - \frac{1}{b_2}\sum_{i=1}^{n}\left(\frac{1}{t_{2i}{}^2}\right) \tag{14}$$
$$+\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial a_2}(1+\alpha M_1 M_2)}{1+\alpha M_1 M_2}\right)$$

$$\frac{\partial\ell}{\partial b_1} = -\frac{n a_1}{b_1} + \frac{a_1}{b_1{}^2}\sum_{i=1}^{n}\left(\frac{1}{t_{1i}{}^2}\right) + \sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial b_1}(1+\alpha M_1 M_2)}{1+\alpha M_1 M_2}\right) \tag{15}$$

$$\frac{\partial \ell}{\partial b_2} = -\frac{na_2}{b_2} + \frac{a_2}{b_2{}^2}\sum_{i=1}^{n}\left(\frac{1}{t_{2i}{}^2}\right) + \sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial b_2}(1+\alpha M_1 M_2)}{1+\alpha M_1 M_2}\right) \tag{16}$$

$$\frac{\partial \ell}{\partial \alpha} = -\sum_{i=1}^{n}\left(\frac{1}{1+\alpha M_1 M_2}\right)\frac{\partial}{\partial \alpha}(1+\alpha M_1 M_2) \tag{17}$$

Simplifying Eqs. (13)–(17) and then equating to zero will yield the estimates of the parameters of the bivariate-inverted Nakagami distribution.

4.2 Estimation using inference functions for margins

Estimation using inference function for margins can be obtained by considering marginal density functions of the bivariate-inverted Nakagami distribution. The marginal density functions of the bivariate Maxwell distribution can be derived as:

4.2.1 Marginal density function of T_1

The marginal density function of T_1 can be defined as

$$f_1(t_1) = \int_{-\infty}^{\infty} f(t_1, t_2) dt_2 \tag{18}$$

where $f(t_1, t_2)$ is defined in Eq. (10). Substituting Eq. (10) into Eq. (18) gives

$$f_1(t_1) = \frac{4}{\Gamma(a_1)\Gamma(a_2)}\left(\frac{a_1}{b_1}\right)^{a_1}\left(\frac{a_2}{b_2}\right)^{a_2}\int_0^{\infty} t_1{}^{-2a_1-1} t_2{}^{-2a_2-1}\exp\left(-\frac{a_1}{b_1 t_1{}^2}\right)\exp\left(-\frac{a_2}{b_2 t_2{}^2}\right)$$
$$\left\{1+\alpha\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_1{}^2}\right)\right)\right)\left(1-2\left(1-\gamma_2\left(a_2, \frac{a_2}{b_2 t_2{}^2}\right)\right)\right)\right\}dt_2 \tag{19}$$

Let

$$A = 1-\gamma_2\left(a_2, \frac{a_2}{b_2 t_2{}^2}\right), \quad \Rightarrow \quad dt_2 = \frac{\Gamma(a_2) b_2{}^{a_2} t_2{}^{2a_2+1}}{2a_2{}^{a_2} e^{-\frac{a_2}{b_2 t_2{}^2}}} dA \tag{20}$$

Inserting Eq. (20) into Eq. (19) becomes

$$\begin{aligned} f_1(t_1) &= \frac{2}{\Gamma(a_1)}\left(\frac{a_1}{b_1}\right)^{a_1} t_1{}^{-2a_1-1}\exp\left(-\frac{a_1}{b_1 t_1{}^2}\right)\int_0^1\left\{1+\alpha\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_1{}^2}\right)\right)\right)(1-2A)\right\}dA \\ &= f(t_1; a_1, b_1) + \alpha\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_1{}^2}\right)\right)\right) f(t_1; a_1, b_1)\int_0^1\{(1-2A)\}dA \\ &= f(t_1; a_1, b_1) + \alpha\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_1{}^2}\right)\right)\right) f(t_1; a_1, b_1)\{0\} \end{aligned}$$
$$f_1(t_1) = \frac{2}{\Gamma(a_1)}\left(\frac{a_1}{b_1}\right)^{a_1} t_1{}^{-2a_1-1}\exp\left(-\frac{a_1}{b_1 t_1{}^2}\right) \tag{21}$$

which is the marginal density function of T_1. Hence, the marginal density function of T_2 can be presented as

$$f_2(t_2) = \frac{2}{\Gamma(a_2)}\left(\frac{a_2}{b_2}\right)^{a_2} t_2{}^{-2a_2-1}\exp\left(-\frac{a_2}{b_2 t_2{}^2}\right) \tag{22}$$

The parameter estimations of the marginal densities of T_1 and T_2 using inference function for margins can be obtained from Eqs. (21) and (22), and then the log-likelihood function of these equations can be presented as

$$\ell_{T_{1i}} = n\log(2) - n\log(\Gamma(a_1)) + na_1\log(a_1) - na_1\log(b_1) - (2a_1+1)\sum_{i=1}^{n}\log(t_{1i}) - \frac{a_1}{b_1}\sum_{i=1}^{n}\left(\frac{1}{t_{1i}{}^2}\right) \tag{23}$$

and

$$\ell_{T_{2i}} = n\log(2) - n\log(\Gamma(a_2)) + na_2\log(a_2) - na_2\log(b_2) - (2a_2+1)\sum_{i=1}^{n}\log(t_{2i}) - \frac{a_2}{b_2}\sum_{i=1}^{n}\left(\frac{1}{t_{2i}{}^2}\right) \tag{24}$$

Maximizing Eqs. (23) and (24) over parameters $a_j$and b_j for $j = 1, 2$, and then equating to zero we can have.

$$\frac{\partial\ell_{T_{ji}}}{\partial a_j} = -n\psi(a_j) + n(1 + \log(a_j)) - n\log(b_j) - 2\sum_{i=1}^{n}(t_{ji}) - \frac{1}{b_j}\sum_{i=1}^{n}\left(\frac{1}{t_{ji}{}^2}\right) = 0 \tag{25}$$

Eq. (25) is nonlinear and it cannot be derived numerically, the statistical software such as R, MATLAB, and so on could be employed effectively in estimating the parameters of the marginal density functions of of T_1 and T_2 Furthermore, the parameter αin Eq. (10) can be obtained using the following ways:

$$\ell_\alpha = \sum_{i=1}^{n}\log\left(1+\alpha\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_{1i}{}^2}\right)\right)\right)\left(1-2\left(1-\gamma_2\left(a_2, \frac{a_2}{b_2 t_{2i}{}^2}\right)\right)\right)\right) \tag{26}$$

The partial derivative with respect to parameter α in Eq. (26), and equating zero it becomes

$$\frac{\partial\ell_\alpha}{\partial\alpha} = \sum_{i=1}^{n}\left(\frac{\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_{1i}{}^2}\right)\right)\right)\left(1-2\left(1-\gamma_2\left(a_2, \frac{a_2}{b_2 t_{2i}{}^2}\right)\right)\right)}{1+\alpha\left(1-2\left(1-\gamma_1\left(a_1, \frac{a_1}{b_1 t_{1i}{}^2}\right)\right)\right)\left(1-2\left(1-\gamma_2\left(a_2, \frac{a_2}{b_2 t_{2i}{}^2}\right)\right)\right)}\right) = 0 \tag{27}$$

equating (27), and then simplifying for α gives the estimate of the parameter of the bivariate-inverted Nakagami distribution using the maximum-likelihood estimation.

5. Application to real-life datasets

The effectiveness of the new bivariate-inverted Nakagami distribution based on the two datasets is evaluated through an application to real-world datasets. The first dataset for kidney infection diseases is given in [48] and involves 30 observations. The second dataset for the UEFA Champions' League group stage for the seasons 2004–2005 and 2005–2006 is presented [49].

Table 1 presents the mean estimate (estimate), standard error (std. error), T-square (t) and probability (p) values for the kidney infection diseases and the group stage of the UEFA Champion's League.

The estimates, standard error, t, and p values for the MLE and IFM approaches are shown in **Table 1**. Based on standard error values, the IFM estimates of the parameters are generally superior to the corresponding MLE estimates. With the exception of the parameters b_2 and α from the MLE and IFM, respectively, both estimation techniques are statistically significant. This determines whether the FGM copula is appropriate for the first dataset.

Table 2 shows that the findings of the IFM are superior to those of the MLE, and the corresponding p values for each parameter are statistically significant as well. This demonstrates that the FGM copula is appropriate for the second datasets.

The copula goodness of fit measures for kidney disease infection and the group stage of the UEFA Champions League are presented in **Tables 3** and **4**. The copula goodness and fit measures of the IMF and MLE technique of estimations are measured by the log-likelihood (log-lik), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). The best technique should be defined as having a maximum log-lik value and a minimum AIC, BIC, and HQIC value.

Method	Parameter	Estimate	Std. error	t	p
	a_1	0.2041	0.0278	7.4060	0.0000
	b_1	1.3083	0.5684	2.3020	0.0214
MLE	a_2	0.2654	0.0499	5.3130	0.0000
	b_2	0.2234	0.1581	1.4130	0.1576
	α	10.6062	4.6728	2.2700	0.0232
	a_1	0.2251	0.0449	5.0090	0.0000
	b_1	0.0122	0.0050	2.4250	0.0153
IFM	a_2	0.2916	0.0593	4.9150	0.0000
	b_2	0.0042	0.0015	2.8300	0.0047
	α	1.0586	0.6184	1.7120	0.0869

Table 1.
Goodness-of-fit measures for the kidney infection diseases.

Method	Parameter	Estimate	Std. error	t	p
	a_1	0.2506	0.0323	7.7470	0.0000
	b_1	0.0084	0.0021	3.9820	0.0000
MLE	a_2	0.3387	0.0601	5.6370	0.0000
	b_2	0.0217	0.0092	2.3630	0.0181
	α	3.4233	1.1019	3.1070	0.0018
	a_1	0.3085	0.0568	5.4350	0.0000
	b_1	0.0079	0.0023	3.4660	0.0005
IFM	a_2	0.3202	0.0591	5.4180	0.0000
	b_2	0.0161	0.0048	3.3350	0.0009
	α	2.3231	0.6054	3.8370	0.0001

Table 2.
Goodness of fit measures for the group stage of the UEFA Champion's league.

Copula	Log-lik	AIC	BIC	HQIC
IFM	−348.2453	706.4906	713.4966	708.7319
MLE	−350.8327	711.6654	718.6714	713.9067

Table 3.
Copula goodness-of-fit measures results for the first dataset.

Copula	Log-lik	AIC	BIC	HQIC
IFM	−371.2927	752.5854	760.6400	755.4250
MLE	−371.4730	752.9460	761.0006	755.7856

Table 4.
Copula goodness-of-fit measures results for the second dataset.

Table 3 shows that the IFM provides a minimal value for the AIC, BIC, and HQIC and a maximum value for log-lik. This proved that the IMF's approach to finding the bivariate-inverted Nakagami distribution's parameters was superior.

Table 4 shows the findings of the IFM and MLE, and it is evident from this table that the estimation using the IFM gave the best results, having a higher log-lik value and with the smallest AIC, BIC, and HQIC values. **Table 4** shows the findings of the IFM and MLE, and it is evident from this table that the estimation using the IFM gave the best results, having a higher log-lik value and with the smallest AIC, BIC, and HQIC values.

6. Conclusion

This chapter introduces a brand-new bivariate inverted Nakagami distribution, along with its characteristics and practical applications. The new bivariate distribution's cdf, pdf, and marginal density functions are specified. The model parametrs were estimated using a variety of estimation techniques. To demonstrate the effectiveness of the novel distribution, two datasets are taken into account. The findings indicate that the IFM produced the most accurate method for estimating the bivariate-inverted Nakagami distribution's parameters.

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Conflict of interest

The authors claim to have no conflicts of interest.

Declarations

We certify that all authors have reviewed and approved this chapter.

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