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# Numerical Solution of Heat Equation using Modified Cubic B-spline Collocation Method

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**ABSTRACT**

In this paper, a collocation method is presented based on the Modified Cubic B-spline Method (MCBSM) for the numerical solution of the heat equation. The PDE is fully discretized by using the Modified Cubic B-spline basis collocation for spatial discretization and the finite difference method is used for the time discretization. A numerical example from PDE is used to evaluate the accuracy of the proposed method. The numerical results are evaluated in comparison to the exact solutions. The findings consistently indicate that the suggested technique provides good error estimates. We also discovered that our proposed method was unconditionally stable. Hence, based on the results and the efficiency of the method, the method is suitable for solving heat equation.

## 1. Introduction

Consider the following partial differential equation (PDE) [1]:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}\ , u(0,t) = 0, u(1,t) = 0, u(x,0) = g(x)\ , x \in [0,1] \tag{1}$$

Here $\alpha^2$ is a constant.

This problem belongs to a well-known second-order parabolic linear PDE [1-4]. The heat equation is very important in physics and engineering. It is a classical parabolic PDE in mathematics. It shows how heat conduction is distributed in a rod or each region over a given time. This PDE is of fundamental importance in the field of Thermal physics. In statistics and a more specific area

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probability theory, this equation is jointly studied with the Brownian motion using the Fokker – Planck equation [5].

The heat equation is used to simulate a variety of processes and is frequently used in financial mathematics to model options. The differential equation of the well-known Black – Scholes option pricing model may be converted into the heat equation, allowing for comparatively simple solutions using a well-known portion of mathematics [6]. It is also significant in Riemannian geometry, and therefore Richard S. Hamilton used it to establish the Ricci flow, which Grigori Perelman later used to answer the topological Poincare conjecture [7].

The more advanced forms of heat equation are wave equation, convection–diffusion problems and Burgers's, equation. Mittal and Jain [8, 9] proposed to approximate the solution of the linear and nonlinear Burgers' equation using modified cubic B-splines over finite elements. This PDE is also solved by Hadhoud *et al.,* [10] using non-polynomial B-spline and shifted Jacobi spectral collocation techniques but time step size used must be small enough for these solutions. Raslan and Ali [11] presented the tensor product schemes of Cubic B-splines of order 3 and 4 to solve some PDEs such as heat equation and MHD Duct flow problem but the meshing grid size are extremely large. Nonlinear Burgers–Fisher equation [12] was numerically solved by Singh *et al.,* using higher order cubic B-spline scheme which makes it computationally higher cost. Yaseen *et al.,* [13] numerically solved the generalized form of time-fractional diffusion equation using cubic trigonometric B-splines with collocation method but having trigonometric spline basis makes the subsequent process very difficult as the matrix system generated by this method is relatively very large. This type of PDE is also studied by Singh *et al.,* [14] in the form of reaction-diffusion equation using trigonometric B-spline with Neumann and Dirichlet boundary conditions. Jena and Senapati [15] presented solution of heat and advection-diffusion equation using improvised cubic B-spline collocation, finite element method and Crank-Nicolson technique but these methods have high arithmetic computations, lower accuracy, and complexity in computer programming. Goh and Ismail [16] used cubic b-spline collocation for the solution of heat and wave equation but to sustain accuracy smaller space steps are needed. Another paper from Goh *et al.,* [17] were presented on the solution of heat equation using cubic and higher order spline schemes. Mohebbi and Dehghan [18] studied the cubic B-spline collocation method and compact 4$^{th}$ order finite difference approximation method for the solution of heat equation. The implicit Crank – Nicolson method can be used to efficiently solve the heat equation [19, 20]. Many authors [21-25] have used higher order B-splines such as quintic, quartic, sextet and septic B-splines for the solutions of different PDEs but faced computational higher cost. Also, Riccati matrix delay differential equations has solved with variational iteration method in these papers [26-30].

In this paper, Modified Cubic B-spline Method (MCBSM) with collocation method is used to solve the heat equation (1). The MCBSM basis with a free parameter $\gamma$ is used to approximate the spatial derivative while finite difference method is used for the temporal derivative. The results are tested in comparison to the exact solution based on parametric value $\gamma = 2$ . The motivation behind suggesting MCBSM stems from its applicability to real-world problems in engineering, physics, biology, and other scientific disciplines. Its ability to capture complex behaviours and accurately represent experimental data makes it a valuable tool for modelling and simulation purposes. The benefits of using MCBSM is its free parameter $\gamma$ which gives more control on edges of curve solution globally enhancing smoothness, flexibility, and applicability to various scientific and engineering domains.

In section two, the generalized derivation of the MCBSM collocation method has been provided. Section 3 is comprised of the stability of the proposed method. The solution of the problem (1) using

MCBSM based on the free parameter $\gamma$ and the numerical results are discussed to analyze errors in section 4. Finally, the concluding remarks are given in section 5.

## 2. Modified Cubic B-spline Collocation Method (MCBSM)

In the cubic B-splines collocation method, the approximate solution can be written as a linear combination of cubic B-spline basis functions for the approximation space.

Consider a mesh $0 = x_0 < x_1, \dots, x_{N-1} < x_N = P$ as a uniform partition of the solution domain $0 \leq x \leq P$ by the knots $x_i$ with $h = x_{i+1} - x_i = \frac{P}{N}$ , $i = 0,1, \dots, N-1$ .

Suppose that the proposed spline solution [31] to problem (1) is:

$$u(x,t) \cong U(x_i,t) = U_i^n = S(x) = \sum_{i=-1}^{N+1} d_i(t) B_i(x_i) \tag{2}$$

Here $d_i(t)$ are unknown time-dependent constants while $B_i(x)$ are the set of basis functions based on the definition of cubic B-spline Basis (CBS) for $i = 0,1,2, \dots N$ is:

$$B_i(x) = \begin{cases} \left(\frac{x-x_{i-2}}{h}\right)^3 , & if\ x \in [x_{i-2}, x_{i-1}] \\ 1 + 3\left(\frac{x-x_{i-1}}{h}\right) + 3\left(\frac{x-x_{i-1}}{h}\right)^2 + \left(\frac{x-x_{i-1}}{h}\right)^3 , & if\ x \in [x_{i-1}, x_i] \\ 1 + 3\left(\frac{x_{i+1}-x}{h}\right) + 3\left(\frac{x_{i+1}-x}{h}\right)^2 + \left(\frac{x_{i+1}-x}{h}\right)^3 , & if\ x \in [x_i, x_{i+1}] \\ \left(\frac{x_{i+2}-x}{h}\right)^3 , & if\ x \in [x_{i+2}, x_{i+1}] \\ 0 , & Otherwise, \end{cases} \tag{3}$$

where $\{B_{-1}, B_0, B_1, \dots, B_N, B_{N+1}\}$ are basis set over the interval $0 \leq x \leq P$.

Eq. (3) can be rewritten in a recurrence relation as:

$$B_i(x_j) = \begin{cases} 4, & if\ i = j, \\ 1, & if\ i - j = \pm 1, \quad i,j = -1,0,1, \dots, N, N+1 \\ 0, & if\ i - j = \pm 2 \end{cases} \tag{4}$$

and that $B_i(x) = 0$ for $x \leq x_{i-2}$ and $x \geq x_{i+2}$.

The first and second-order derivatives of $B_i(x_j)$ are given by:

$$B_i'(x_j) = \begin{cases} 0, & if\ i = j, \\ \pm\frac{3}{h}, & if\ i - j = \pm 1, \quad i,j = -1,0,1, \dots, N, N+1 \\ 0, & if\ i - j = \pm 2 \end{cases} \tag{5}$$

and

$$B_i''(x_j) = \begin{cases} -\frac{12}{h^2}, & if\ i = j, \\ \frac{6}{h^2}, & if\ i - j = \pm 1, \quad i,j = -1,0,1, \dots, N, N+1 \\ 0, & if\ i - j = \pm 2 \end{cases} \tag{6}$$

Table 1 summarized the value at each knot for $B_i(x), {B_i}'(x)$ and ${B_i}''(x)$.

**Table 1**
Basis values at corresponding knots from Eq. (4-6)

| $x$ | $x_{i-2}$ | $x_{i-1}$ | $x_i$ | $x_{i+1}$ | $x_{i+2}$ |
|---|---|---|---|---|---|
| $B_i(x)$ | 0 | 1 | 4 | 1 | 0 |
| $h{B_i}'(x)$ | 0 | -3 | 0 | 3 | 0 |
| $h^2{B_i}''(x)$ | 0 | 6 | -12 | 6 | 0 |

The basis functions ought to disappear at the boundary of the curve when boundary conditions are given in the collocation technique. However, in the case of cubic B-splines the basis functions $B_{-1}, B_0, B_1, \dots, B_{N-1}, B_N, B_{N+1}$ are not disappearing on one of the boundary locations [12]. Consequently, the basis functions need to be adjusted to form a new set that vanishes on the boundaries where the boundary conditions are applied. To solve this, the modified term is introduced into Eq. (2) using a free parameter $\gamma$, given by equation below [8]:

$$\bar{B}_i(x) = \begin{cases} B_0(x) + \gamma B_{-1}(x) & for\ i = 0 \\ B_1(x) - B_{-1}(x) & for\ i = 1 \\ B_i(x) & for\ i = 2, \dots, N-2 \\ B_{N-1}(x) - B_{N+1}(x) & for\ i = N-1 \\ B_N(x) + \gamma B_{N+1}(x) & for\ i = N \end{cases} \tag{7}$$

The free parameter $\gamma = -4, 2$, is used to modify the end points. Therefore, we can rewrite Eq. (2) using modified basis from Eq. (7) as,

$$u(x,t) \cong \bar{S}(x) = \sum_{i=0}^{N} d_i \bar{B}_i(x) \tag{8}$$

Eq. (8) is equivalent to

$$\bar{S}(x) = d_0\bar{B}_0(x) + d_1\bar{B}_1(x) + \cdots + d_N\bar{B}_N(x). \tag{9}$$

Subsequently, Eq. (9) is used for the numerical computation of problem (1). The expression $\frac{\partial^2 u}{\partial x^2}$ in (1) is substituted with the corresponding Spline function and expressed in the matrix form. The unknowns $d_i$ , $i = 0, \dots, N$ are finally solved by fulfil the collocation equations as well as the boundary conditions.

## 3. Implementation of MCBSM

Eq. (1) can be rearranged as:

$$-u_{xx} + u_t = 0, \alpha^2 = 1 \tag{10}$$

Suppose $L$ is a linear differential operator for the solution of (10). The proposed solution must satisfy the linear differential operator properties such as:

$$L\big(u(x)\big) = X(x) \tag{11}$$

Using differential operator properties and Taylor series, Eq. (11) expands to

$$\left(C_0 D^{(n)} + C_1 D^{(n-1)} + C_2 D^{(n-2)} + \cdots + C_n\right)u(x) = X(x) \tag{12}$$

Here D is the $nth$ order differentiation operator. In the case of heat equation, we have order $n = 2$ derivative of spatial value $x$. Therefore, the corresponding MCBSM associates with Eq. (11) is,

$$L\left(\bar{S}(x)\right) = X\left(x_j\right), 0 \le x_j \le N, \tag{13}$$

with boundary conditions

$$\bar{S}(0) = \beta_0,\ \bar{S}(1) = \beta_1 \tag{14}$$

Expanding (13) according to Eq. (12), we have:

$$-\bar{S}''\left(x_j\right) + T\left(x_j\right)\bar{S}\left(x_j\right) = X(x_j) \tag{15}$$

Here $C_0 D^{(2)} u(x) = -\bar{S}''\left(x_j\right), C_n U = T\left(x_j\right)\bar{S}\left(x_j\right),\ X(x) = X(x_j)$ and $C_n = T\left(x_j\right)$ is positive variable coefficient comes due to linear differential operator properties.

$$-\sum_{i,j=0}^{N} d_i \bar{B}_i{}''\left(x_j\right) + T\left(x_j\right)\sum_{i,j=0}^{N} d_i \bar{B}_i\left(x_j\right) = X(x_j) \tag{16}$$

Eq. (16) is expanded as:

$$\begin{aligned} &-\left[d_{j-1}\bar{B}_{j-1}{}''\left(x_j\right) + d_j\bar{B}_j{}''\left(x_j\right) + d_{j+1}\bar{B}_{j+1}{}''\left(x_j\right)\right] \\ &\qquad + T\left(x_j\right)\left[d_{j-1}\bar{B}_{j-1}\left(x_j\right) + d_j\bar{B}_j\left(x_j\right) + d_{j+1}\bar{B}_{j+1}\left(x_j\right)\right] = X\left(x_j\right), \forall\, j = 0,1,2,\ldots,N. \end{aligned}$$

Rearranging the terms according to $d_j$ as:

$$\begin{aligned} &d_{j-1}\left[-\bar{B}_{j-1}{}''\left(x_j\right) + T\left(x_j\right)\bar{B}_{j-1}\left(x_j\right)\right] \\ &+d_j\left[-\bar{B}_j{}''\left(x_j\right) + T(x_j)\bar{B}_j\left(x_j\right)\right] + d_{j+1}\left[-\bar{B}_{j+1}{}''\left(x_j\right) + T(x_j)\bar{B}_{j+1}\left(x_j\right)\right] = X\left(x_j\right), \\ &\forall\, j = 0,1,2,\ldots,N. \end{aligned} \tag{17}$$

Using table 1, Eq. (17) is simplified to:

$$\left(-6 + T_j h^2\right)d_{j-1} + \left(12 + 4T_j h^2\right)d_j + \left(-6 + T_j h^2\right)d_{j+1} = h^2 X_j, \forall\, j = 0,1,2,\ldots,N. \tag{18}$$

where $T\left(x_j\right) = T_j$ and $X\left(x_j\right) = X_j$.

For the given boundary conditions in (14), the MCBSM is equivalent to:

$$\bar{S}\left(x_j\right) = \bar{S}(x_0) = \beta_0$$

Expand the $\bar{S}\left(x_j\right)$ according to (9)

$$d_{-1}\bar{B}_{-1}(x_0) + d_0\bar{B}_0(x_0) + d_1\bar{B}_1(x_0) + \cdots + d_N\bar{B}_N(x_0) + d_{N+1}\bar{B}_{N+1}(x_0) = \beta_0$$

Using table 1,

$0d_{-1} + 6d_0 + 0d_1 = \beta_0$

and therefore

$$6d_0 = \beta_0 \tag{19}$$

The same approach in (19) is applied to the second boundary condition in (14)

$$6d_N = \beta_1 \tag{20}$$

Eq. (18-20) generates a $(N+3) \times (N+3)$ trigonal system with $(N+3)$ unkonwns,

$d_N = (d_0, \dots, d_N)^t$.

and from the (17) and (18),

$$36d_0 = X_0h^2 - \beta_0(-6 + T_0h^2) \tag{21}$$

Again eliminating $d_{N+1}$ from the (17) and (19), we find

$$36d_N = X_Nh^2 - \beta_1(-6 + T_Nh^2) \tag{22}$$

From (17) we have,

$$\left.\begin{array}{r} j = 1,\ (-6 + T_1h^2)d_0 + (12 + 4T_1h^2)d_1 + (-6 + T_1h^2)d_2 = h^2X_1, \\ j = 2,\ (-6 + T_2h^2)d_1 + (12 + 4T_2h^2)d_2 + (-6 + T_2h^2)d_3 = h^2X_2, \\ \vdots \\ j = i,\ \left(-6 + T_jh^2\right)d_{j-1} + \left(12 + 4T_jh^2\right)d_j + \left(-6 + T_jh^2\right)d_{j+1} = h^2X_j, \\ \vdots \\ j = N-1, (-6 + T_{N-1}h^2)d_{N-2} + (12 + 4T_{N-1}h^2)d_{N-1} + (-6 + T_{N-1}h^2)d_N = h^2X_{N-2} \end{array}\right\} \tag{23}$$

Eq. (21), (22) and (23) led to the system of $(N+1)$ linear equations $Ax_N = C_N$ in the $(N+1)$ unknowns $x_N = (d_0, d_1, \dots, d_N)^t$ of the form:

$$\begin{bmatrix} 36 & 0 & 0 & 0 & 0 & 0 \\ \delta & \omega & \delta & 0 & 0 & 0 \\ 0 & \delta & \omega & \delta & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \delta & \omega & \delta \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \\ d_N \end{bmatrix} = \begin{bmatrix} X_0h^2 - \beta_0\delta \\ X_1h^2 \\ X_2h^2 \\ \vdots \\ X_{N-1}h^2 \\ X_Nh^2 - \beta_1\delta \end{bmatrix} \tag{24}$$

where $\delta = -6 + Th^2$ , $\omega = 12 + 4Th^2$. Since $T(x) > 0$, so obviously $A$ has strictly dominance property and therefore $A$ is non-singular. The system $Ax_N = C_N$ can be solved for $d_0, d_1, \dots, d_N$ using Thomas Method.

## 4. Stability Analysis

For the solution of (1) at time level $t_{j+1}th$ [17], consider:

$$(U_t)_i^k + (1-\theta)f_i^k + \theta f_i^{k+1} = 0, \tag{25}$$

Here $f_i^k = -\alpha^2(U_{xx})_i^k$ and for the successive time levels, we used $k$ and $k+1$, $k = 0,1,2,\ldots$. Now, using a first-order accurate forward difference technique to discretize the time derivative and rewriting the equation,

$$U_i^{k+1} + \theta\Delta t f_i^{k+1} = U_i^k - (1-\theta)\,\Delta t f_i^k \tag{26}$$

Here we use $\Delta t$ as the time step. For $\theta = 0$ system becomes an explicit technique, for $\theta = 1$ it becomes implicit technique, and for $\theta = 0.5$ it becomes a mixed technique of Crank-Nicolson.

Here we used Von Neumann stability method to analyze the stability of the said method [9, 15-17][9, 15-17]. Consider the Fourier mode solution at a given point $x_m$

$$C_m^k = \sigma^k\, exp(i\delta mh) \tag{27}$$

where $i = \sqrt{-1}$ and $\delta$ is the mode number.

By substituting $U_i^k = \sum_{i=-1}^{N+1} d_i \bar{B}_i(x)$ into (26) and rearranging the equation, we have,

$$P_A C_{m-2}^{k+1} + P_B C_{m-1}^{k+1} = Q_A C_{m-2}^k + Q_B C_{m-1}^k \tag{28}$$

where,

$$P_A = \frac{1}{6} - \frac{\theta\Delta t\alpha^2}{h^2}, P_B = \frac{4}{6} + \frac{2\theta\Delta t\alpha^2}{h^2}, Q_A = \frac{1}{6} + \frac{(1-\theta)\Delta t\alpha^2}{h^2}, Q_B = \frac{4}{6} - \frac{2(1-\theta)\Delta t\alpha^2}{h^2}$$

Applying the Eq. (18) into (19) and evaluating the equation we have,

$$\gamma = \frac{S}{T} \tag{29}$$

where,

$$S = \frac{2+cos\delta h}{3} - \frac{2(1-\theta)\Delta t\alpha^2}{h^2}[1 - cos\delta h]$$

And

$$T = \frac{2+cos\delta h}{3} + \frac{2\theta\Delta t\alpha^2}{h^2}[1 - cos\delta h]$$

This implies that,

$$|\gamma| = |\frac{S}{T}| \leq 1$$

Thus, this shows that the presented numerical scheme for the heat equation is unconditionally stable.

## 5. Numerical Results

Eq. (1) is given below:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}\ ,\ u(0,t) = 0, u(1,t) = 0, u(x,0) = g(x)$$

Using notation for u to the mesh point as $p(x_j, t_n)$ by

$$U_p = U(x_j, t_n) = U_j^n$$

Using the forward difference relation for $u_t$ is

$$u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \tag{30}$$

Substituting $p = u_j$ in above equation we have,

$$\frac{p^{n+1}(x) - p^n(x)}{\Delta t} = \alpha^2 \frac{\partial^2 p^{n+1}(x)}{\partial x^2},$$

$$-\Delta t \alpha^2 p_{xx}^{n+1} + p^{n+1} = p^n$$

At $t = 0, n = 0, t_0$

$$-\Delta t \alpha^2 p_{xx}^1 + p^1 = p^0 \tag{31}$$

Using Eq. (4), (5) and (7), the following results are deduced:

$$p_{xx}^1 = \sum_{i=-1}^{N+1} d_i \bar{B}_i{}''(x_j)\ ,\ p^1 = \sum_{i=-1}^{N+1} d_i \bar{B}_i(x_j),\ p^0 = u(x_j, 0) = g(x_j)$$

Using these results in Eq. (29), we have:

$$-\Delta t \alpha^2 \sum_{i,j=0}^{N} d_i \bar{B}_i{}''(x_j) + \sum_{i,j=0}^{N} d_i \bar{B}_i(x_j) = g(x_j)$$

Using expansion of these series,

$$-\Delta t \alpha^2 \left[d_{j-1} \bar{B}_{j-1}{}''(x_j) + d_j \bar{B}_j{}''(x_j) + d_{j+1} \bar{B}_{j+1}{}''(x_j)\right] + \left[d_{j-1} \bar{B}_{j-1}(x_j) + d_j \bar{B}_j(x_j) + d_{j+1} \bar{B}_{j+1}(x_j)\right] = g(x_j), \forall\ j = 0,1,2, \dots, N. \tag{32}$$

$$d_{j-1}\left[-\Delta t \alpha^2 \bar{B}_{j-1}{}''(x_j) + \bar{B}_{j-1}(x_j)\right] + d_j\left[-\Delta t \alpha^2 \bar{B}_j{}''(x_j) + \bar{B}_j(x_j)\right] + d_{j+1}\left[-\Delta t \alpha^2 \bar{B}_{j+1}{}''(x_j) + \bar{B}_{j+1}(x_j)\right] = g(x_j), \forall\ j = 0,1,2, \dots, N.$$

Using Eq. (3)-(5) and table 1, we get

$$(-6\Delta t\alpha^2 + h^2)d_{j-1} + (12\Delta t\alpha^2 + 4h^2)d_j + (-6\Delta t\alpha^2 + h^2)d_{j+1} = h^2 g_j,$$
$$\forall\, j = 0,1,2,\dots,N. \tag{33}$$

where $g(x_j) = g_j$.

$j = 1, (-6\Delta t\alpha^2 + h^2)d_0 + (12\Delta t\alpha^2 + 4h^2)d_1 + (-6\Delta t\alpha^2 + h^2)d_2 = h^2 g_1,$

$j = 2, (-6\Delta t\alpha^2 + h^2)d_1 + (12\Delta t\alpha^2 + 4h^2)d_2 + (-6\Delta t\alpha^2 + h^2)d_3 = h^2 g_2,$

$\vdots$

$j = i, (-6\Delta t\alpha^2 + h^2)d_{j-1} + (12\Delta t\alpha^2 + 4h^2)d_j + (-6\Delta t\alpha^2 + h^2)d_{j+1} = h^2 g_j,$

$\vdots$

$j = N-1, (-6\Delta t\alpha^2 + h^2)d_{N-2} + (12\Delta t\alpha^2 + 4h^2)d_{N-1} + (-6\Delta t\alpha^2 + h^2)d_N = h^2 g_{N-1}$

These equations led to the system of $(N + 1)$ linear equations $Ax_N = C_N$ in the $(N + 1)$ unknowns $x_N = (d_0, d_1, \dots, d_N)^t$ of the form:

$$\begin{bmatrix} 36\Delta t & 0 & 0 & 0 & 0 & 0 \\ \delta & \omega & \delta & 0 & 0 & 0 \\ 0 & \delta & \omega & \delta & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \delta & \omega & \delta \\ 0 & 0 & 0 & 0 & 0 & 36\Delta t \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-1} \\ d_N \end{bmatrix} = \begin{bmatrix} g_0 h^2 \\ g_1 h^2 \\ g_2 h^2 \\ \vdots \\ g_{N-1} h^2 \\ g_N h^2 \end{bmatrix} \tag{34}$$

where $\delta = -6\Delta t\alpha^2 + h^2$ , $\omega = 12\Delta t\alpha^2 + 4h^2$. The system is strictly dominant and hence nonsingular. Finally, the system $Ax_N = C_N$ are solved for $d_0, d_1, \dots, d_N$ of the spline function, $\bar{S}(x)$.

**Problem 1 [1]**

We have exact solution of above equation with thermal diffusivity constant $\alpha^2 = 1$ as:

$$u(x,t) = \sin(2\pi x).\, exp(-4\alpha^2\pi^2 t), \qquad g(x) = \sin(2\pi x)$$

Table 2 shows the numerical results of the proposed MCBSM for the step size, $h = 0.05$. The computational work is done by MATLAB program. Here Table 2 consists of errors generated by the proposed numerical method of problem 1 and shows better approximations in comparison of table given in [1] while Table 3 shows absolute errors of problem 2 [17]. From problem 1, Figure 1 shows 3d image of exact data values while Figure 2 displays numerical data values at $\mathrm{h} = 0.05$; $\Delta t = 0.001$, and $\gamma = 2$. Figure 3 shows comparison of errors between proposed and cubic B-spline method [1]. In Figures 4 and 5, exact and numerical data values of problem 2 are shown in terms of surfaces. From tables and figures MCBSM is working better in comparison of results given in [1].

**Table 2**
Comparison of Numerical and Exact values with corresponding error values.

| $x$ | Approximated values | Exact values | Absolute Error | Absolute Error[1] |
|---|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0 |
| 0.05 | 0.0062348614 | 0.0059628855 | 0.0012914192 | 0.0185 |
| 0.10 | 0.0118594112 | 0.0113420823 | 0.0024564253 | 0.0353 |
| 0.15 | 0.0163230792 | 0.0156110370 | 0.0033809793 | 0.0486 |
| 0.20 | 0.0191889304 | 0.0183518746 | 0.0039745796 | 0.0571 |
| 0.25 | 0.0201764354 | 0.0192963029 | 0.0041791203 | 0.06 |
| 0.30 | 0.0191889304 | 0.0183518746 | 0.0039745796 | 0.0571 |
| 0.35 | 0.0163230792 | 0.0156110370 | 0.0033809793 | 0.0486 |
| 0.40 | 0.0118594112 | 0.0113420823 | 0.0024564253 | 0.0353 |
| 0.45 | 0.0062348614 | 0.0059628855 | 0.0012914192 | 0.0185 |
| 0.50 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0 |
| 0.55 | -0.0062348614 | -0.0059628855 | 0.0012914192 | 0.0185 |
| 0.60 | -0.0118594112 | -0.0113420823 | 0.0024564253 | 0.0353 |
| 0.65 | -0.0163230792 | -0.0156110370 | 0.0033809793 | 0.0486 |
| 0.70 | -0.0191889304 | -0.0183518746 | 0.0039745796 | 0.0571 |
| 0.75 | -0.0201764354 | -0.0192963029 | 0.0041791203 | 0.06 |
| 0.80 | -0.0191889304 | -0.0183518746 | 0.0039745796 | 0.0571 |
| 0.85 | -0.0163230792 | -0.0156110370 | 0.0033809793 | 0.0486 |
| 0.90 | -0.0118594112 | -0.0113420823 | 0.0024564253 | 0.0353 |
| 0.95 | -0.0062348614 | -0.0059628855 | 0.0012914192 | 0.0185 |
| 1 | 0.0000000000 | 0.0000000000 | 0.0000000000 | 0 |

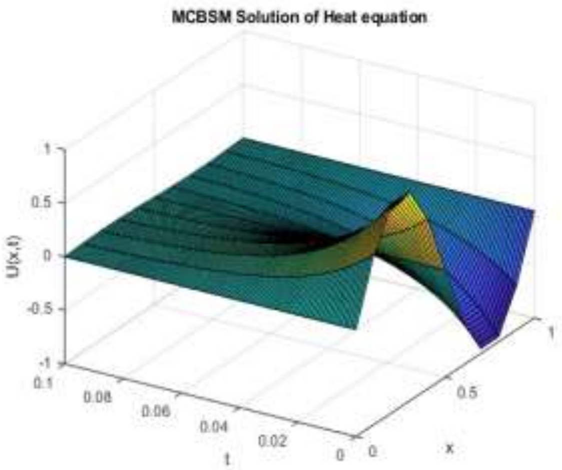


**Fig. 1.** Numerical Solution of Heat Equation at $\mathrm{h} = 0.05;\ \Delta\mathrm{t} = 0.001, \gamma = 2$

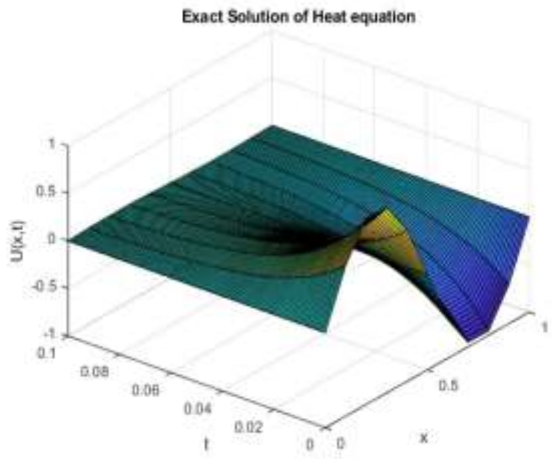


**Fig. 2.** Exact Solution of Heat Equation at $\mathrm{h} = 0.05; \Delta\mathrm{t} = 0.001, \gamma = 2$

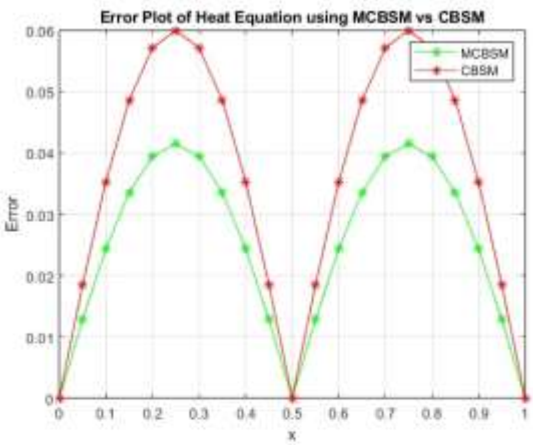


**Fig. 3.** Error plot of Heat Equation at $h = 0.05;\ \Delta t = 0.001, \gamma = 2$

**Problem 2 [17]**

We have exact solution of above equation with thermal diffusivity constant $\alpha^2 = 1$ as:

$$u(x,t) = \cos\left(\frac{\pi x}{2}\right).exp\left(-\frac{\alpha^2\pi^2 t}{4}\right),\ g(x) = \cos\left(\frac{\pi x}{2}\right)$$

Table 3 shows the numerical results of the proposed MCBSM for the step size, $h = 0.1$.

**Table 3**
Comparison of Numerical and Exact values with corresponding error values

| $x$ | Approximated values | Exact values | Absolute Error |
|---|---|---|---|
| 0 | 0.0000000000 | 0.0000000000 | 0.0000000000 |
| 0.1 | 0.203016756069411 | 0.771724092658820 | 0.568707336589408 |
| 0.2 | 0.335123184303920 | 0.743102046403511 | 0.407978862099592 |
| 0.3 | 0.404989242637741 | 0.696182361551451 | 0.291193118913710 |
| 0.4 | 0.425801671283474 | 0.632120356461202 | 0.206318685177728 |
| 0.5 | 0.407980688824566 | 0.552493450307692 | 0.144512761483126 |
| 0.6 | 0.360035210798434 | 0.459262321786972 | 0.0992271109885374 |
| 0.7 | 0.289174020478991 | 0.354722630699596 | 0.0655486102206052 |
| 0.8 | 0.201741302655282 | 0.241448491187480 | 0.0397071885321980 |
| 0.9 | 0.103528158961270 | 0.122229088500728 | 0.0187009295394575 |
| 1 | 0.0000000000 | 0.0000000000 | 0.0000000000 |

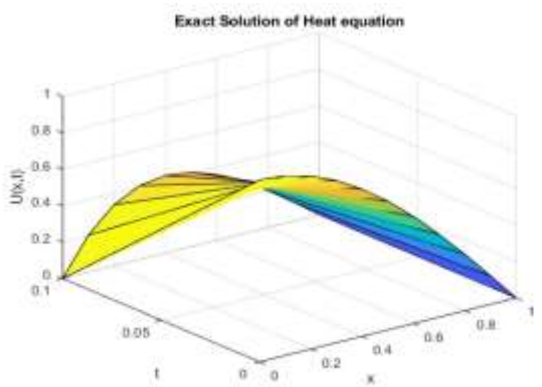


**Fig. 4.** Exact Solution of Heat Equation at $h = 0.1; \Delta t = 0.1, \gamma = 2$

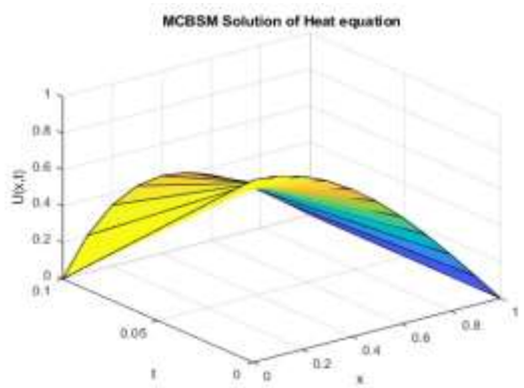


**Fig. 5.** Numerical Solution of Heat Equation at $h = 0.1; \Delta t = 0.1, \gamma = 2$

## 6. Conclusions

In this paper, the Modification of Cubic B-Spline Basis function using a free parameter together with collocation technique is utilized for the solution of the one-dimensional heat equation. A generalized method is developed to incorporate the usage of free parameter $\gamma$. For the results, MATLAB was employed to present results and graphs, facilitating a comparison between numerical and exact solutions. A numerical experiment was carried out to showcase the computational viability and efficiency of the proposed method. The numerical results show that the proposed method is capable in solving the PDE of the heat equation.

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### Conflicts of Interest
The authors declare no conflict of interest.

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