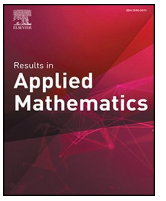
Results in Applied Mathematics 24 (2024) 100520

Contents lists available at ScienceDirect

Results in Applied Mathematics

journal homepage: www.elsevier.com/locate/results-in-applied-mathematics



Numerical solution of first kind Fredholm integral equations with semi-smooth kernel: A two-stage iterative approach

Mohana Sundaram Muthuvalu [a,*], Nor Aida Zuraimi Md Noar [b], Harry Setiawan [c], Isman Kurniawan [d], Shaher Momani [e,f]

[a] Department of Fundamental and Applied Sciences, Universiti Teknologi PETRONAS, Seri Iskandar 32610, Perak, Malaysia
[b] Centre for Mathematical Sciences, Universiti Malaysia Pahang Al-Sultan Abdullah, Lebuh Persiaran Tun Khalil Yaakob, Gambang Pahang 26300, Malaysia
[c] Communications Science Department, Universitas Islam Riau, Jl. Kaharuddin Nasution No.113, Marpoyan, Pekanbaru, Riau, Indonesia
[d] School of Computing, Telkom University, Jl. Telekomunikasi 1, Terusan Buahbatu, Kabupaten, Bandung, Jawa Barat, Indonesia
[e] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
[f] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE

ARTICLE INFO

Keywords:
First kind Fredholm integral equations
Semi-smooth kernel
Newton-cotes quadrature scheme
Dense linear system
Geometric Mean method
Complexity reduction approach

ABSTRACT

This paper examines two-stage iterative methods, specifically the Geometric Mean (GM) method and its variants, for solving dense linear systems associated with first-kind Fredholm integral equations with semi-smooth kernels. These equations, characterised by ill-posedness and sensitivity to input perturbations, are discretised using a composite closed Newton-Cotes quadrature scheme. The study evaluates the computational performance and accuracy of the standard GM method, also referred to as the Full-Sweep Geometric Mean (FSGM), in comparison with the Half-Sweep Geometric Mean (HSGM) and Quarter-Sweep Geometric Mean (QSGM) methods. Numerical experiments demonstrate significant reductions in computational complexity and execution time while maintaining high solution accuracy. The QSGM method achieves the best performance among the tested methods, highlighting its effectiveness in addressing computational challenges associated with first-kind Fredholm integral equations.

1. Introduction

Fredholm integral equations are fundamental in various fields of mathematical physics and applied sciences. This study focuses on the numerical solution of first-kind linear Fredholm integral equations with semi-smooth kernels, which are characterised by their ill-posed nature. Such equations exhibit sensitivity to small perturbations in input data, often resulting in dense, ill-conditioned systems upon discretisation. Addressing these challenges is essential for advancing theoretical frameworks and practical applications [8,16]. The general form of a first-kind linear Fredholm integral equation is expressed as:

* Corresponding author.
E-mail addresses: mohana.muthuvalu@utp.edu.my (M.S. Muthuvalu), aidaz@ump.edu.my (N.A.Z.M. Noar), harry.setiawan@comm.uir.ac.id (H. Setiawan), ismankrn@telkomuniversity.ac.id (I. Kurniawan).

https://doi.org/10.1016/j.rinam.2024.100520
Received 6 June 2024; Received in revised form 18 November 2024; Accepted 26 November 2024
Available online 6 December 2024

$$\int_{\alpha}^{\beta} K(x,t)y(t)dt = f(x),\ x \in [\alpha,\beta] \tag{1}$$

where $K(x,t)$ represents the kernel, $f(x)$ is the known function, and $y(t)$ is the unknown function to be determined. If operator κ is denoted by

$$\kappa : S \to T,\ \kappa(y(t)) = \int_{\alpha}^{\beta} K(x,t)y(t)dt \tag{2}$$

then the following definitions are satisfied.

Definition 1. [16]. A kernel $K(x,t)$ is called q- semi-smooth if

$$K(x,t) = \begin{cases} K_1(x,t) & if\ \alpha \le t \le x \\ K_2(x,t) & if\ x \le t \le \beta \end{cases} \tag{3}$$

where $K_{1,2}(x,t) \in C^{q}_{[\alpha,\beta]\ \times\ [\alpha,\beta]}$ for some $q > 1$.

Definition 2. [8]. Let $\kappa : S \to T$ be an operator from normed space S into a normed space T, the equation $\kappa y = f$ is called well-posed if κ is onto, one to one and the inverse operator $\kappa^{-1} : T \to S$ is continuous. Otherwise, the equation is called ill-posed.Semi-smooth kernels are prevalent in practical problems and demand specialised numerical techniques for accurate and stable solutions.

A numerical approach for solving integral equations (1) is a crucial area of scientific research. In recent years, several effective discretization methods for Eq. (1) have been developed, as referenced in [2,3,5,8,10–12,15]. However, these discretization techniques often result in dense linear systems, which can become computationally prohibitive, particularly when direct methods are used as the order of the system increases. As a result, iterative methods provide a more efficient alternative for solving these systems.

The concept of half-sweep iteration was introduced by Abdullah [1] via the Explicit Decoupled Group (EDG) method for solving two-dimensional Poisson equations. This approach was later extended to quarter-sweep iteration concept through the Modified Explicit Group (MEG) method, resulting in enhanced computational efficiency [14]. Subsequently, numerous studies have explored the use of half- and quarter-sweep iteration concepts for solving a range of scientific problems, including two-dimensional Poisson equations [18–20], Fredholm integral equations [9–12], two-dimensional free-space wave propagation [4], and linear complementarity problems [6].

Among the various iterative methods, the Geometric Mean (GM) method stands out as an efficient numerical method for solving non-singular linear systems. In addition to the standard GM method, also known as the Full-Sweep Geometric Mean (FSGM) method, several variants have been proposed, including the Half-Sweep Geometric Mean (HSGM) [17] and Quarter-Sweep Geometric Mean (QSGM) [21] methods. These variants are derived by integrating the standard GM method with the concept of half- and quarter-sweep iterations, respectively. Previous studies have explored the effectiveness of the GM method and its variants for solving second-kind linear Fredholm integral equations [13]. However, the numerical approach for solving first-kind Fredholm integral equations differs significantly due to the inherent ill-posed nature of such equations. First-kind equations are more sensitive to perturbations in the input data, often resulting in dense, ill-conditioned systems upon discretisation. Consequently, iterative methods like FSGM, HSGM, and QSGM require careful tuning and may necessitate additional computational strategies, to ensure stability and accuracy. In contrast, second-kind equations are generally well-posed and yield better-conditioned systems, which are less sensitive to numerical instability.

This study extends the application of these methods to first-kind linear Fredholm integral equations, demonstrating the ability to handle the heightened numerical challenges posed by such systems. Specifically, the performance of the QSGM method is investigated in solving dense linear systems derived from the discretisation of first-kind Fredholm integral equations with a semi-smooth kernel, using a first-order composite closed Newton-Cotes quadrature scheme. The performance of the QSGM method is then compared with the FSGM and HSGM methods.

The remainder of this paper is organized as follows. In Section 2, the formulation of the full-, half-, and quarter-sweep approximation equations using the composite closed Newton-Cotes quadrature method is detailed. The following sections focus on the formulations and computational complexities of the FSGM, HSGM, and QSGM methods. Section 5 presents the numerical results from the simulations to evaluate the performance of the tested methods, and Section 6 concludes with final remarks.

2. Newton-Cotes quadrature approximation equations

To address the numerical solution of first-kind Fredholm integral equations, the composite closed Newton-Cotes quadrature scheme is applied for discretisation. Consider the interval $[\alpha,\beta]$, which is uniformly divided into N subintervals, yielding a discrete set of points $x_i = \alpha + ih$ $(i = 0,1,2,\cdots,N-2,N-1,N)$ and $t_j = \alpha + jh$ $(j = 0,1,2,\cdots,N-2,N-1,N)$. The constant step size, h, is defined as:

$$h = \frac{\beta - \alpha}{N}. \tag{4}$$

For simplicity in subsequent derivations, the following notations are introduced:

$$\left.\begin{array}{c} K_{i,j} = K(x_i, t_j) \\ y_j = y(t_j) \\ f_i = f(x_i) \end{array}\right\}. \tag{5}$$

As detailed in [9,10], the application of the composite closed Newton-Cotes quadrature method simplifies Eq. (1) to:

$$\sum_{j=0}^{N} w_j K_{i,j} \hat{y}_j = f_i, \ i = 0, 1, 2, \cdots, N-2, N-1, N. \tag{6}$$

The solution $\hat{y}$ approximates the exact solution y of (1), where w_j represents the weights of the quadrature method. The standard composite closed Newton-Cotes quadrature approximation equations, as defined in Eq. (6), are commonly referred to as the full-sweep composite closed Newton-Cotes quadrature approximation equations. As illustrated in Fig. 1, the finite grid networks depict the distribution of uniform node points used in formulating the half- and quarter-sweep composite closed Newton-Cotes quadrature approximation equations.

As shown in Fig. 1, the half- and quarter-sweep iterative methods calculate approximate values only at specific types of node points ● until the convergence criterion is met. Once convergence is achieved, the approximate solutions for the remaining node points are computed directly [1,13,14].

By incorporating the half- and quarter-sweep iteration concepts, the generalized composite closed Newton-Cotes quadrature approximation equations are formulated for full-, half-, and quarter-sweep approaches, as follows:

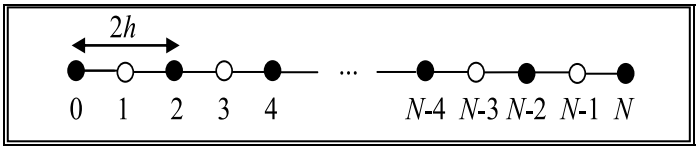
$$\sum_{j=0,p,2p}^{N} w_j K_{i,j} \hat{y}_j = f_i, \ i = 0, p, 2p, \cdots, N-2p, N-p, N. \tag{7}$$

The parameter p, corresponding to one, two, and four, signifies the full-sweep, half-sweep, and quarter-sweep composite closed Newton-Cotes quadrature approximation equations, respectively. Additionally, the approximation equation in Eq. (7) can be reformulated in matrix form as:

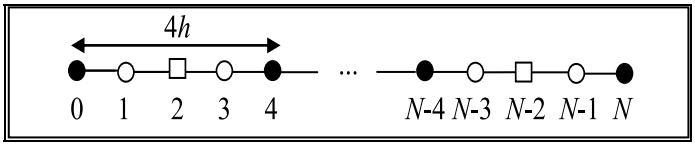
$$A\hat{y} = f \tag{8}$$

where

$$A = \begin{bmatrix} w_0K_{0,0} & w_pK_{0,p} & w_{2p}K_{0,2p} & \cdots & w_{N-2p}K_{0,N-2p} & w_{N-p}K_{0,N-p} & w_NK_{0,N} \\ w_0K_{p,0} & w_pK_{p,p} & w_{2p}K_{p,2p} & \cdots & w_{N-2p}K_{p,N-2p} & w_{N-p}K_{p,N-p} & w_NK_{p,N} \\ w_0K_{2p,0} & w_pK_{2p,p} & w_{2p}K_{2p,2p} & \cdots & w_{N-2p}K_{2p,N-2p} & w_{N-p}K_{2p,N-p} & w_NK_{2p,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_0K_{N-2p,0} & w_pK_{N-2p,p} & w_{2p}K_{N-2p,2p} & \cdots & w_{N-2p}K_{N-2p,N-2p} & w_{N-p}K_{N-2p,N-p} & w_NK_{N-2p,N} \\ w_0K_{N-p,0} & w_pK_{N-p,p} & w_{2p}K_{N-p,2p} & \cdots & w_{N-2p}K_{N-p,N-2p} & w_{N-p}K_{N-p,N-p} & w_NK_{N-p,N} \\ w_0K_{N,0} & w_pK_{N,p} & w_{2p}K_{N,2p} & \cdots & w_{N-2p}K_{N,N-2p} & w_{N-p}K_{N,N-p} & w_NK_{N,N} \end{bmatrix}_{\left(\frac{N}{p}+1\right)\times\left(\frac{N}{p}+1\right)}$$



(a)



(b)

Fig. 1. a) and b) show distribution of uniform node points for the half- and quarter-sweep cases respectively.

$$\widehat{y} = \begin{bmatrix} \widehat{y}_0 \\ \widehat{y}_p \\ \widehat{y}_{2p} \\ \vdots \\ \widehat{y}_{N-2p} \\ \widehat{y}_{N-p} \\ \widehat{y}_N \end{bmatrix} \text{ and } f = \begin{bmatrix} f_0 \\ f_p \\ f_{2p} \\ \vdots \\ f_{N-2p} \\ f_{N-p} \\ f_N \end{bmatrix}.$$

From Eq. (8), it is noticeable that the application of half- and quarter-sweep iteration concepts reduces the size of the original matrix from $(N+1) \times (N+1)$ to $\left(\frac{N}{2}+1\right) \times \left(\frac{N}{2}+1\right)$ and $\left(\frac{N}{4}+1\right) \times \left(\frac{N}{4}+1\right)$, respectively. This reduction indirectly decreases the computational complexity of the iterative methods during the iteration process. For the first-order composite closed Newton-Cotes quadrature method, the quadrature weights, w_j, satisfy the following relation:

$$w_j = \begin{cases} \frac{1}{2}ph, & j = 0, N \\ ph, & otherwise \end{cases}. \tag{9}$$

3. Geometric mean iterative methods

The FSGM, HSGM, and QSGM methods are fundamentally two-stage iterative methods. The iteration process involves solving two independent systems, with $\widehat{y^1}$ and $\widehat{y^2}$ representing intermediate solutions for $\widehat{y}$. To derive the formulation for all three GM methods, the following splitting is considered:

$$A = D - L - U \tag{10}$$

where

$$D = \begin{bmatrix} w_0K_{0,0} & & & & & & & \\ & w_pK_{p,p} & & & & & & \\ & & w_{2p}K_{2p,2p} & & 0 & & & \\ & & & \ddots & & & & \\ & & 0 & & w_{N-2p}K_{N-2p,N-2p} & & & \\ & & & & & w_{N-p}K_{N-p,N-p} & & \\ & & & & & & w_NK_{N,N} \end{bmatrix},$$

$$-L = \begin{bmatrix} & & & & & & \\ w_0K_{p,0} & & & & & & \\ w_0K_{2p,0} & w_pK_{2p,p} & & & 0 & & \\ \vdots & \vdots & \ddots & & & & \\ w_0K_{N-2p,0} & w_pK_{N-2p,p} & \cdots & w_{N-3p}K_{N-2p,N-3p} & & & \\ w_0K_{N-p,0} & w_pK_{N-p,p} & \cdots & w_{N-3p}K_{N-p,N-3p} & w_{N-2p}K_{N-p,N-2p} & & \\ w_0K_{N,0} & w_pK_{N,p} & \cdots & w_{N-3p}K_{N,N-3p} & w_{N-2p}K_{N,N-2p} & w_{N-p}K_{N,N-p} \end{bmatrix}$$

and

$$-U = \begin{bmatrix} w_pK_{0,p} & w_{2p}K_{0,2p} & w_{3p}K_{0,3p} & \cdots & w_{N-p}K_{0,N-p} & w_NK_{0,N} \\ & w_{2p}K_{p,2p} & w_{3p}K_{p,3p} & \cdots & w_{N-p}K_{p,N-p} & w_NK_{p,N} \\ & & w_{3p}K_{2p,3p} & \cdots & w_{N-p}K_{2p,N-p} & w_NK_{2p,N} \\ & & & \ddots & \vdots & \vdots \\ & & 0 & & w_{N-p}K_{N-2p,N-p} & w_NK_{N-2p,N} \\ & & & & & w_NK_{N-p,N} \\ & & & & & \end{bmatrix}.$$

Therefore, for non-singular $(D-\omega L)$ and $(D-\omega U)$ matrices, the general formulation for the FSGM, HSGM and QSGM methods is defined as follows:

$$\left.\begin{aligned} (D-\omega L)\widehat{y^1} &= [(1-\omega)D + \omega U]\widehat{y}^{(k)} + \omega f \\ (D-\omega U)\widehat{y^2} &= [(1-\omega)D + \omega L]\widehat{y}^{(k)} + \omega f \\ \widehat{y}^{(k+1)} &= \left(\widehat{y^1} \circ \widehat{y^2}\right)^{\frac{1}{2}} \end{aligned}\right\} \tag{11}$$

where ω, $\circ$, $(\cdot)^{\frac{1}{2}}$ and k denotes the acceleration parameter, Hadamard product, Hadamard power and k-th iteration, respectively.

The iterative scheme for the FSGM, HSGM, and QSGM methods, applied to solve the linear systems in Eq. (8), take the following form:

$$\widehat{y}^{(k+1)} = T_{FSGM}\widehat{y}^{(k)} + c_{FSGM}f, \tag{12}$$

$$\widehat{y}^{(k+1)} = T_{HSGM}\widehat{y}^{(k)} + c_{HSGM}f \tag{13}$$

and

$$\widehat{y}^{(k+1)} = T_{QSGM}\widehat{y}^{(k)} + c_{QSGM}f \tag{14}$$

respectively, where

$$T_{FSGM} = T_{HSGM} = T_{QSGM} = \left[\left((D-\omega L)^{-1}((1-\omega)D+\omega U)\right)\left((D-\omega U)^{-1}((1-\omega)D+\omega L)\right)\right]^{\frac{1}{2}}$$

and

$$c_{FSGM} = c_{HSGM} = c_{QSGM} = \left[\omega^2(D-\omega L)^{-1}(D-\omega U)^{-1}\right]^{\frac{1}{2}}.$$

It is already noted that the successive approximations (12), (13) and (14) for $k = 0, 1, 2, \cdots$ converge for each $c_{FSGM} \in \mathbb{C}^{N+1}$, $c_{HSGM} \in \mathbb{C}^{\frac{N}{2}+1}$ and $c_{QSGM} \in \mathbb{C}^{\frac{N}{4}+1}$, respectively and, each $\widehat{y}^{(0)} \in \mathbb{C}^{N+1}$, $\widehat{y}^{(0)} \in \mathbb{C}^{\frac{N}{2}+1}$ and $\widehat{y}^{(0)} \in \mathbb{C}^{\frac{N}{4}+1}$, respectively if and only if the spectral radius of the iteration matrices i.e. T_{FSGM}, T_{HSGM} and T_{QSGM} is less than one, that is, $\rho(T_{FSGM}) < 1$, $\rho(T_{HSGM}) < 1$ and $\rho(T_{QSGM}) < 1$ (Theorem 4.1

Algorithm 1
FSGM, HSGM and QSGM schemes.

i. Set $\widehat{y}^{(0)} = \widehat{y^1} = \widehat{y^2}$, ε, ω, w, K and f.
ii. Iteration cycle
  a. Stage 1
    1. Level 1
    for $i = 0, p, 2p, \cdots, N-2p, N-p, N$
    Compute
    $$\widehat{y^1}_i \leftarrow (1-\omega)\widehat{y}_i^{(k)} + \frac{\omega}{w_i K_{i,i}}\left(f_i - \sum_{j=0,\ p,\ 2p}^{i-p} w_j K_{i,j}\widehat{y^1}_j \quad - \sum_{j=i+p,\ i+2p,\ i+3p}^{N} w_j K_{i,j}\widehat{y}_j^{(k)}\right)$$
  b. Stage 2
    1. Level 2
    for $i = N, N-p, N-2p, \cdots, 2p, p, 0$
    Compute
    $$\widehat{y^2}_i \leftarrow (1-\omega)\widehat{y}_i^{(k)} + \frac{\omega}{w_i K_{i,i}}\left(f_i - \sum_{j=0,\ p,\ 2p}^{i-p} w_j K_{i,j}\widehat{y}_j^{(k)} \quad - \sum_{j=i+p,\ i+2p,\ i+3p}^{N} w_j K_{i,j}\widehat{y^2}_j\right)$$
    2. **for** $i = 0, p, 2p, \cdots, N-2p, N-p, N$
    Compute
    $$\widehat{y}_i^{(k+1)} \leftarrow \begin{cases} \left(\widehat{y^1}_i\ \widehat{y^2}_i\right)^{\frac{1}{2}} & if \quad \widehat{y^1}_i \geq 0 \wedge \widehat{y^2}_i \geq 0 \qquad (Case1) \\ -\left(\widehat{y^1}_i\ \widehat{y^2}_i\right)^{\frac{1}{2}} & if \quad \widehat{y^1}_i < 0 \wedge \widehat{y^2}_i < 0 \qquad (Case2) \\ \widehat{y^1}_i - \left(\left|\widehat{y^1}_i\ \widehat{y^2}_i\right|\right)^{\frac{1}{2}} & if \quad \widehat{y^1}_i > 0 \wedge \widehat{y^2}_i < 0 \quad (Case3) \\ \widehat{y^2}_i - \left(\left|\widehat{y^1}_i\ \widehat{y^2}_i\right|\right)^{\frac{1}{2}} & if \quad \widehat{y^1}_i < 0 \wedge \widehat{y^2}_i > 0 \quad (Case4) \end{cases}$$
iii. Check the convergence criterion. If satisfied, proceed to Step (v) for the FSGM method or Step (iv) for the HSGM and QSGM methods. Otherwise, repeat the iteration cycle (i.e., return to Step (ii)).
iv. Compute the remaining points
  a. HSGM method
  $$\widehat{y}_i = \begin{cases} \frac{3}{8}\widehat{y}_{i-1} + \frac{3}{4}\widehat{y}_{i+1} - \frac{1}{8}\widehat{y}_{i+3}, & i = 1, 3, 5, \cdots, N-3 \\ \frac{3}{4}\widehat{y}_{i-1} + \frac{3}{8}\widehat{y}_{i+1} - \frac{1}{8}\widehat{y}_{i-3}, & i = N-1 \end{cases}$$
  b. QSGM method
  $$\widehat{y}_i = \begin{cases} \frac{3}{8}\widehat{y}_{i-2} + \frac{3}{4}\widehat{y}_{i+2} - \frac{1}{8}\widehat{y}_{i+6}, & i = 2, 6, 10, \cdots, N-6 \\ \frac{3}{4}\widehat{y}_{i-2} + \frac{3}{8}\widehat{y}_{i+2} - \frac{1}{8}\widehat{y}_{i-6}, & i = N-2 \\ \frac{3}{8}\widehat{y}_{i-1} + \frac{3}{4}\widehat{y}_{i+1} - \frac{1}{8}\widehat{y}_{i+3}, & i = 1, 3, 5, \cdots, N-3 \\ \frac{3}{4}\widehat{y}_{i-1} + \frac{3}{8}\widehat{y}_{i+1} - \frac{1}{8}\widehat{y}_{i-3}, & i = N-1 \end{cases}$$
v. Stop

in [7]). It is also palpable that the product of both $\widehat{y^1}$ and $\widehat{y^2}$ must be nonnegative values. Furthermore, slight modifications are required in calculating $\widehat{y}^{(k+1)}$ when $\widehat{y^1}$ or/and $\widehat{y^2}$ are negative value(s). Such modifications are explained in Algorithm 1.

By setting $\widehat{y}_i = \widehat{y}_j$, the algorithms for the FSGM, HSGM and QSGM methods, corresponding to the full-sweep, half-sweep and quarter-sweep composite closed Newton-Cotes quadrature approximation equations, respectively, are outlined in Algorithm 1 to solve Eq. (1). The GM algorithms are executed by alternately using all equations at Levels 1 and 2 until the solution satisfies a specified convergence criterion, i.e. the maximum norm $\| \widehat{y}^{(k+1)} - \widehat{y}^{(k)} \| \leq \varepsilon$ where ε is the convergence criterion. The GM methods are distinguished by well-defined substructures within their overall mathematical framework, allowing these components to be executed concurrently. This characteristic makes the GM methods particularly suitable for implementation on a multiprocessor system. Following the iteration process, additional computations are required for the HSGM and QSGM methods to calculate the remaining points. In this study, the second-order Lagrange interpolation technique [13] is employed to compute these remaining points.

4. Computational complexity analysis

This section presents a computational complexity analysis based on arithmetic operations. To evaluate the arithmetic complexity of the FSGM, HSGM, and QSGM methods with the associated composite closed Newton-Cotes quadrature approximation equations for solving Eq. (1), an estimate of the computational work required by the iterative methods has been carried out. This estimate considers the arithmetic operations performed during each iteration. According to Algorithm 1, four distinct cases for the GM methods can be identified. In this analysis, the kernel value K, the function f and weights w_j are precomputed and stored in advance.

From Algorithm 1, it is observed that the total number of arithmetic operations for Case 1 is $\frac{6N}{p} + 14$ for each node point in the solution domain. For Cases 2, 3, and 4, the required operations are $\frac{6N}{p} + 15$. The value of p which corresponds to one, two, and four, represents the FSGM, HSGM, and QSGM methods, respectively. For the HSGM and QSGM methods, the iteration process is performed only on $\left(\frac{N}{2}+1\right)$ and $\left(\frac{N}{4}+1\right)$ mesh points, respectively. Consequently, an additional eight arithmetic operations are needed to compute the remaining points after convergence, using second-order Lagrange interpolation. The total arithmetic operations involved in the FSGM, HSGM, and QSGM methods are summarized in Table 1.

5. Numerical tests

To evaluate the performance of the FSGM, HSGM, and QSGM iterative methods, several numerical tests were conducted on two first-kind linear Fredholm integral equations with semi-smooth kernels.

Test Problem 1 [16]

Consider the following linear Fredholm integral equations of the first kind:

$$\int_0^1 K(x,t)y(t)dt = \frac{1}{6}\left(x^3 - x\right), \ x \in [0,1] \tag{15}$$

with the kernel defined as:

$$K(x,t) = \begin{cases} t(x-1), & t < x \\ x(t-1), & x \leq t \end{cases}.$$

The exact solution to this problem is:

$$y(x) = x.$$

Table 1
Total computing operations involved for the FSGM, HSGM and QSGM methods.

Case	Total Operations		
	FSGM	HSGM	QSGM
1	$(6N^2 + 20N + 14)k$	$\left(\frac{3}{2}N^2 + 10N + 14\right)k + 4N$	$\left(\frac{3}{8}N^2 + 5N + 14\right)k + 6N$
2	$(6N^2 + 21N + 15)k$	$\left(\frac{3}{2}N^2 + \frac{21}{2}N + 15\right)k + 4N$	$\left(\frac{3}{8}N^2 + \frac{21}{4}N + 15\right)k + 6N$
3	$(6N^2 + 21N + 15)k$	$\left(\frac{3}{2}N^2 + \frac{21}{2}N + 15\right)k + 4N$	$\left(\frac{3}{8}N^2 + \frac{21}{4}N + 15\right)k + 6N$
4	$(6N^2 + 21N + 15)k$	$\left(\frac{3}{2}N^2 + \frac{21}{2}N + 15\right)k + 4N$	$\left(\frac{3}{8}N^2 + \frac{21}{4}N + 15\right)k + 6N$

(k is the number of iterations).

Test Problem 2 [16]

Consider the following linear Fredholm integral equations of the first kind

$$\int_0^1 K(x,t)y(t)dt = e^x + (1-e)x - 1,\ x \in [0,1] \tag{16}$$

with the kernel defined as:

$$K(x,t) = \begin{cases} t(x-1), & t \le x \\ x(t-1), & x < t \end{cases}.$$

The exact solution to this problem is:

$$y(x) = e^x.$$

For the numerical tests, three evaluation parameters were measured i.e. the number of iterations, CPU time (in seconds) and accuracy. Accuracy was measured by the maximum absolute error, defined as $\| e_N\| = \max_{a \le x \le \beta} |y(x) - \widehat{y}(x)|$. The convergence test employed a tolerance error, $\varepsilon = 10^{-10}$, and simulations were conducted on several different values of N. Additionally, the experimental values of ω were determined within $a \pm 0.01$ range by running Algorithm 1 for different values ω and selecting those that minimized the number of iterations. For comparison, numerical results from the standard Gauss-Seidel (GS) method with the standard composite closed Newton-Cotes quadrature approximation equations were also included for solving test problems 1 and 2. All methods were implemented on a computer with an Intel(R) Core(TM) 2 CPU 1.66 GHz processor, and the algorithm was coded in C programming language. The numerical results for the proposed methods on test problems 1 and 2 are provided in Tables 2 and 3, respectively. Additionally, Table 4 summarizes the reduction percentages in terms of the number of iterations and CPU time for the FSGM, HSGM, and QSGM methods in comparison with the GS method.

The numerical results in Tables 2 and 3 demonstrate the performance of the proposed methods (FSGM, HSGM, and QSGM) in terms of number of iterations, CPU time, and maximum absolute error. These results align with the computational complexity estimates outlined in Table 1. Specifically, Table 1 highlights the reduced arithmetic operations for HSGM and QSGM methods compared to FSGM. This reduction directly translates to fewer iterations and hierarchical decrease in CPU time from FSGM to HSGM to QSGM, as observed in Tables 2 and 3, matches the reduction in computational operations pre iteration as outlined in Table 1. Furthermore,

Table 2
Comparison of number of iterations, CPU time and maximum absolute error for the tested iterative methods (Test Problem 1).

Number of iterations				
N	Methods			
	GS	FSGM	HSGM	QSGM
240	303	106 ($\omega = 1.81$)	104 ($\omega = 1.80$)	102 ($\omega = 1.80$)
480	375	110 ($\omega = 1.81$)	106 ($\omega = 1.81$)	104 ($\omega = 1.80$)
960	451	110 ($\omega = 1.82$)	110 ($\omega = 1.81$)	106 ($\omega = 1.81$)
1920	540	114 ($\omega = 1.83$)	110 ($\omega = 1.82$)	110 ($\omega = 1.81$)
3840	637	127 ($\omega = 1.85$)	114 ($\omega = 1.83$)	110 ($\omega = 1.82$)
7680	743	153 ($\omega = 1.87$)	127 ($\omega = 1.85$)	114 ($\omega = 1.83$)

CPU time (in seconds)				
N	Methods			
	GS	FSGM	HSGM	QSGM
240	4.29	2.97	1.98	1.04
480	13.39	9.88	3.20	2.23
960	52.31	35.41	11.24	4.12
1920	183.39	117.27	48.16	15.92
3840	595.79	338.40	154.80	60.55
7680	1974.28	1064.13	520.34	188.19

$\| e_N\|$				
N	Methods			
	GS	FSGM	HSGM	QSGM
240	6.847871×10^{-10}	4.756342×10^{-10}	5.436210×10^{-10}	5.197360×10^{-10}
480	7.368467×10^{-10}	5.863278×10^{-10}	4.756342×10^{-10}	5.436210×10^{-10}
960	9.065918×10^{-10}	5.529916×10^{-10}	5.863278×10^{-10}	4.881404×10^{-10}
1920	8.796745×10^{-10}	3.302765×10^{-10}	5.529916×10^{-10}	5.863278×10^{-10}
3840	9.369449×10^{-10}	2.407165×10^{-10}	3.302765×10^{-10}	5.529916×10^{-10}
7680	9.687818×10^{-10}	9.615541×10^{-10}	2.407259×10^{-10}	3.302799×10^{-10}

Table 3
Comparison of number of iterations, CPU time and maximum absolute error for the tested iterative methods (Test Problem 2).

Number of iterations				
N	Methods			
	GS	FSGM	HSGM	QSGM
240	315	116 ($\omega = 1.88$)	112 ($\omega = 1.87$)	110 ($\omega = 1.85$)
480	388	117 ($\omega = 1.88$)	116 ($\omega = 1.88$)	112 ($\omega = 1.87$)
960	470	118 ($\omega = 1.90$)	117 ($\omega = 1.88$)	116 ($\omega = 1.88$)
1920	559	118($\omega = 1.90$)	118 ($\omega = 1.90$)	117 ($\omega = 1.88$)
3840	657	130($\omega = 1.94$)	118($\omega = 1.90$)	118 ($\omega = 1.90$)
7680	769	168 ($\omega = 1.94$)	130 ($\omega = 1.94$)	118 ($\omega = 1.90$)
CPU time (in seconds)				
N	Methods			
	GS	FSGM	HSGM	QSGM
240	4.44	3.22	2.23	1.14
480	14.83	10.49	3.40	2.51
960	58.32	37.63	12.00	4.75
1920	195.66	119.82	50.66	16.73
3840	633.92	348.22	161.10	68.07
7680	2159.60	1092.15	530.17	198.83
$\|\| e_N \|\|$				
N	Methods			
	GS	FSGM	HSGM	QSGM
240	3.916332×10^{-06}	3.916175×10^{-06}	1.740409×10^{-05}	6.846364×10^{-05}
480	9.810628×10^{-07}	9.808836×10^{-07}	4.387394×10^{-06}	1.740409×10^{-05}
960	2.453851×10^{-07}	2.452189×10^{-07}	1.101227×10^{-06}	4.387394×10^{-06}
1920	6.290920×10^{-08}	6.117839×10^{-08}	2.756174×10^{-07}	1.101227×10^{-06}
3840	2.602358×10^{-08}	1.681454×10^{-08}	6.873765×10^{-08}	2.756174×10^{-07}
7680	4.977974×10^{-08}	6.931611×10^{-09}	1.809371×10^{-08}	6.873765×10^{-08}

Table 4
Percentage reduction in the number of iterations and CPU time for the FSGM, HSGM, and QSGM methods compared to the GS method.

Test Problem	Methods	Number of iterations (%)	CPU time (%)
1	FSGM	65.01 – 80.07	26.21 – 46.11
	HSGM	65.67 – 82.91	53.84 – 78.52
	QSGM	66.33 – 84.66	75.75 – 92.13
2	**FSGM**	63.17 – 80.22	27.47 – 49.43
	HSGM	64.44 – 83.10	49.77 – 79.43
	QSGM	65.07 – 84.66	74.32 – 91.86

Table 4 summarises the reduction percentages in both number of iterations and CPU time relative to the GS method, supporting the connection between the theoretical complexity in Table 1 and the observed numerical results in Tables 2 and 3.

6. Conclusions

This paper examines the effectiveness of the QSGM method for solving dense linear systems resulting from the discretization of first kind linear Fredholm integral equations using a first-order composite closed Newton-Cotes quadrature scheme. Based on the numerical results obtained, it clearly shows that

- Application of the GM methods reduce number of iterations and computational time compared to the GS method.
- QSGM method has the least number of iterations and computes with the fastest time for all considered mesh sizes.
- In terms of accuracy, the approximation solution obtained by using the QSGM method is comparable with the solutions generated via the GS, FSGM and HSGM methods.

Finally, it can be concluded that the QSGM method outperforms both the FSGM and HSGM methods, as well as the GS method, in terms of iterations and CPU time. This is primarily due to the reduction in computational complexity, as the QSGM method only considers approximately a quarter of the interior node points in the solution domain during the iteration process.

Funding

This research was funded by Joint Research Project (Universiti Teknologi PETRONAS - Universiti Malaysia Pahang - Universitas Islam Riau – Telkom University) (Cost Center: 015ME0-333) and Ministry of Higher Education under the Fundamental Research Grant Scheme (FRGS/1/2014/SG04/UTP/03/1).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

[1] Abdullah AR. The four point explicit decoupled group (EDG) method: a fast Poisson solver. Int J Comput Math 1991;38:61–70.
[2] Adibi H, Assari P. Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind. Math Probl Eng 2010;2010:138408.
[3] Chen Z, Xu Y, Yang H. Fast collocation methods for solving ill-posed integral equations of the first kind. Inverse Probl 2008;24:065007.
[4] Hasan MK, Othman M, Abbas Z, Sulaiman J, Ahmad F. Parallel solution of high speed low order FDTD on 2D free space wave propagation. In: Gervasi O, Gavrilova M, editors. Lecture Notes in Computer Science. Volume 4706. Berlin/Heidelberg, Germany: Springer Science+Business Media; 2007. p. 13–24.
[5] Kang S-H, Koltracht I, Rawitscher G. Nyström-clenshaw-curtis quadrature for integral equations with discontinuous kernel. Math Comput 2002;72:729–56.
[6] Koh WS, Sulaiman J, Mail R. Quarter-sweep projected modified gauss-seidel algorithm applied to linear complementarity problem. Am. J. Appl. Sci. 2010;7:790–4.
[7] Kress R. Numerical Analysis. New York: Springer-Verlag; 1998.
[8] Maleknejad K, Mollapourasl R, Nouri K. Convergence of numerical solution of the Fredholm integral equations of the first kind with the degenerate kernel. Appl Math Comput 2006;181:1000–7.
[9] Muthuvalu MS, Sulaiman J. Half-Sweep Geometric Mean method for solution of linear Fredholm equations. Matematika 2008;24:75–84.
[10] Muthuvalu MS, Aruchunan E, Ali MKM, Chew JVL, Sunarto A, Lebelo R, Sulaiman J. Complexity reduction approach for solving second kind of Fredholm integral equations. Symmetry (Basel) 2022;14:1017.
[11] Muthuvalu MS, Sulaiman J. Half-Sweep Arithmetic Mean method with composite trapezoidal scheme for solving linear Fredholm integral equations. Appl Math Comput 2011;217:5442–8.
[12] Muthuvalu MS, Sulaiman J. Numerical solutions of first kind linear Fredholm integral equations by using Quarter-Sweep SOR with piecewise linear collocation method. Aust J Basic Appl. Sci 2011;5:18–24.
[13] Muthuvalu MS, Sulaiman J. The quarter-sweep geometric mean method for solving second kind linear fredholm integral equations. Bull Malays Math Sci Soc 2013;36:1009–26.
[14] Othman M, Abdullah AR. An efficient four point modified explicit group poisson solver. Int J Comput Math 2000;76:203–17.
[15] Rabbani M, Maleknejad K, Aghazadeh N, Mollapourasl R. Computational projection methods for solving Fredholm integral equation. Appl Math Comput 2007;191:140–3.
[16] Rashed MT. An expansion method to treat integral equations. Appl Math Comput 2003;135:65–72.
[17] Sulaiman J, Abdullah MH, Saudi A, Hasan MK, Othman M. A numerical simulation on water quality model using Quarter-Sweep Geometric Mean (QSGM) iterative method. In: Proceedings of the 2nd International Conference on Science and Technology: Applications in Industry and Education; 2008. p. 1853–9.
[18] Sulaiman J, Hasan MK, Othman M, et al. Red-black half-sweep iterative method using triangle finite element approximation for 2D Poisson equations. In: Shi Y, et al., editors. Lecture Notes in Computer Science. Volume 4487. Berlin/Heidelberg, Germany: Springer Science+Business Media; 2007. p. 326–33.
[19] Sulaiman J, Hasan MK, Othman M. Red-black EDGSOR iterative method using triangle element approximation for 2D Poisson equations. In: Gervasi O, Gavrilova M, editors. Lecture Notes in Computer Science. Volume 4707. Berlin/Heidelberg, Germany: Springer Science+Business Media; 2007. p. 298–308.
[20] Sulaiman J, Hasan MK, Othman M, Karim SAA. MEGSOR iterative method for the triangle element solution of 2D Poisson equations. Procedia Comput Sci 2010;1:377–85.
[21] Sulaiman J, Othman M, Yaacob N, Hasan MK. Half-sweep geometric mean (HSGM) method using fourth-order finite difference scheme for two-point boundary problems. In: Proceedings of the First International Conference on Mathematics and Statistics; 2006. p. 25–33.