COMPATIBILITY CONDITIONS AND NONABELIAN TENSOR PRODUCTS OF FINITE CYCLIC GROUPS OF p-POWER ORDER

MOHD SHAM BIN MOHAMAD

A thesis submitted in fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

MARCH 2012

ABSTRACT

The nonabelian tensor products of groups originated from a generalized Van Kampen Theorem and its construction has its origins in algebraic K-theory and in homotopy theory. In this research, cyclic groups of p-power order where p is a prime number are considered. The aim of this research is to prove that the nonabelian tensor products of some finite cyclic groups of p-power order are cyclic. This research starts with the characterization of automorphisms of cyclic groups of p-power order using number theoretical results where the order of the actions are considered. Then, the necessary and sufficient conditions for the actions to be compatible are determined for a pair of finite cyclic groups. Finally, by using a general expansion formula, the nonabelian tensor products of some cyclic groups of p-power order are proven to be cyclic. The results of this research show that the nonabelian tensor product of cyclic groups of p-power order where p is an odd prime with two-sided actions are cyclic. Furthermore, the nonabelian tensor product of cyclic groups of 2-power order with two-sided actions and both actions have order greater than two have been proven to be also cyclic.

ABSTRAK

Hasil darab tensor tak abelan bagi kumpulan bermula dari Teorem Van Kampen teritlak dan pembinaan hasil darab tensor tak abelan berasal dari teori-K aljabar dan teori homotopi. Dalam penyelidikan ini, kumpulan kitaran berperingkat kuasa bagi p dengan p adalah nombor perdana dipertimbangkan. Tujuan penyelidikan ini adalah untuk membuktikan bahawa hasil darab tensor tak abelan bagi sebahagian kumpulan kitaran berperingkat kuasa bagi p adalah kumpulan kitaran. Penyelidikan ini dimulakan dengan pengkelasan automorfisma kumpulan kitaran terhingga berperingkat kuasa bagi p dengan menggunakan teori nombor di mana peringkat bagi tindakan diambil kira. Setelah itu, syarat-syarat cukup dan perlu supaya tindakan tersebut adalah serasi antara satu sama lain bagi semua kumpulan kitaran terhingga ditentukan. Akhir sekali, dengan menggunakan rumus umum pengembangan tensor, hasil darab tensor tak abelan bagi sebahagian kumpulan kitaran berperingkat kuasa bagi p dibuktikan sebagai kumpulan kitaran. Keputusan penyelidikan ini menunjukkan hasil darab tensor tak abelan bagi kumpulan kitaran berperingkat kuasa bagi p dengan p adalah perdana ganjil dengan tindakan dua sisi adalah merupakan kumpulan kitaran. Tambahan pula, hasil darab tensor tak abelan bagi kumpulan kitaran berperingkat kuasa dua dengan tindakan dua sisi dan kedua-dua tindakan berperingkat lebih besar daripada dua telah juga dibuktikan sebagai kumpulan kitaran.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

1	–	identity element
a^{-1}	–	inverse of a
$\langle a \rangle$	–	cyclic subgroup generated by a
$[a, b]$	–	commutator of a and b
$[a, b, c]$	–	commutator of $[a, b]$ and c
$\mathrm{Aut}\,(G)$	–	automorphism group of G
C_n	–	the cyclic group of order n
D_n	–	the dihedral group of order $2n$
G	–	a finite group
$\lvert G \rvert$	–	the order of G
$G \otimes H$	–	the nonabelian tensor product of G and H
$G \cong H$	–	the groups G and H are isomorphic
${}^{g}h$	–	an action of g on h
$H \leq G$	–	H is a subgroup of G
mod	–	modulo
$\mathbb{N}$	–	natural numbers
Q_n	–	the quaternion group of order $2n$
$t \mid s$	–	t divides s
$Z\,(G)$	–	the center of G
$\mathbb{Z}$	–	set of integers
$\in$	–	element of

CHAPTER 1

INTRODUCTION

1.1 Introduction

This chapter is the introduction chapter that includes research background, problem statement, research objectives, scope of the study, significant of findings and thesis organization.

1.2 Research Background

The nonabelian tensor product for groups G and H, denoted by $G \otimes H$, originated in connection with a generalized Van Kampen Theorem and its construction has its origins in the algebraic K-theory and in homotopy theory. It was introduced by Brown and Loday in [1]. The nonabelian tensor product is defined for a pair of groups which act on each other provided the actions satisfy the compatibility conditions:

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \quad \text{and} \quad ^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$. If G and H are groups that act compatibly on each other, then the nonabelian tensor product, $G \otimes H$ is a group generated by the symbols $g \otimes h$ with relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)\,({}^{h}g \otimes {}^{h}h')$$

for all $g, g' \in G$ and $h, h' \in H$.

Starting with the paper of Brown *et al.* [2], many researchers had studied group theoretical aspects of nonabelian tensor products extensively. Brown *et al.* [2] focused on the group theoretic properties and the explicit computation of nonabelian tensor squares. They also provided a list of open problems concerning nonabelian tensor square and nonabelian tensor product.

Brown and Loday already established in [1] that the nonabelian tensor square, denoted by, $G \otimes G$ is finite for a finite group G. Then Ellis [3] extended the results for the nonabelian tensor products. In addition he showed that the nonabelian tensor product is of p-power order if G and H are of p-power order. McDermott [4] computed the nonabelian tensor product, $G \otimes H$ when G is a p-group and H is a q-group, where p and q are primes. However, he only focused on the bound of the order of $G \otimes H$ and gave some results on that case. Visscher [5] continued the study on the nonabelian tensor product of p-power order and focused on cyclic groups.

1.3 Problem Statement

In 1987, Brown *et al.* [2] gave an open problem in determining whether the tensor product of two cyclic groups is cyclic. Visscher [5] in 1998 had calculated and proved that the nonabelian tensor product of two cyclic groups is not necessarily cyclic and he found that the rank of the nonabelian tensor product of cyclic groups does not exceed two. He only covered the cases for one sided actions and completely determined the nonabelian tensor products of cyclic groups of 2-power order where both actions have order two. In this research, the cases that we looked into are the nonabelian tensor product of cyclic groups of p-power order where p is an odd prime with two-sided actions and the nonabelian tensor product of cyclic groups of 2-power order with two-sided actions and both actions have order greater than two.

1.4 Research Objectives

The objectives of this research are:

(i) to characterize all automorphisms of a cyclic group of p-power order.

(ii) to determine some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups of p-power order with nontrivial actions act compatibly on each other.

(iii) to develop some algorithms in Groups, Algorithms and Programming (GAP) software for finding the nonabelian tensor product of finite cyclic groups of p-power order.

(iv) to prove that the nonabelian tensor products of some finite cyclic groups of p-power order are cyclic.

1.5 Scope of the Study

In this thesis, the groups considered are limited to the finite cyclic groups of p-power order where p is prime with the actions are nontrivial.

1.6 Significance of Findings

The major contribution of this thesis consist of new theoretical results on determining the conditions in which the nonabelian tensor products of finite cyclic groups are cyclic. This thesis also provides some characterizations of automorphisms of cyclic groups of p-power order that are proved using number theoretical results. In addition, this characterization gives the general presentation for automorphisms of p-power order.

The next contribution is a new necessary and sufficient condition that a pair of cyclic groups of p-power order acts compatible on each other. This provides a new classification since the conditions depend on the order of the action. According to this necessary and sufficient condition, the number of compatible pair of two cyclic groups can be determined.

1.7 Thesis Organization

The first chapter serves as an introduction to the whole thesis. This chapter contains research background, problem statement, objectives of the research, scope of the study and significant of findings.

Chapter 2 presents the literature review of this research. Various works by different researchers regarding the nonabelian tensor product are stated. It has been discovered for 27 years since 1984 where Brown and Loday [1] were the first to introduce the concept of tensor.

In Chapter 3, some definitions and preliminary results on automorphism groups, compatibility conditions, the nonabelian tensor products and Groups, Algorithms and Programming (GAP) algorithms are given. GAP algorithms to determine the compatible actions and to find the nonabelian tensor products are also given. All results in this chapter are used in the following chapters.

Chapter 4 shows the characterizations of all automorphisms of cyclic groups of p-power order. The propositions which characterize the automorphisms with certain order are given. This chapter is concluded with some results on characterizing all automorphisms of cyclic group of p-power order. The characterizations are divided into two parts, namely for p an odd prime and for $p = 2$.

In Chapter 5, compatible actions for cyclic groups of p-power order are presented. The main theorem in this chapter gives a necessary and sufficient condition for two finite cyclic groups acting on each other in a compatible way. This result is applied to finite cyclic groups of p-power order and classifies all compatible actions between them.

Chapter 6 contains some of the main results of this thesis concerning the nonabelian tensor product of cyclic groups of p-power order. This chapter shows that the nonabelian tensor product of these groups is cyclic with some exceptions. Various results on the group theoretic properties and explicit computation of the nonabelian tensor product of cyclic groups under certain families of actions, such as trivial actions, one sided action and both actions nontrivial, but each of order two are given. Next, to compute the nonabelian tensor product of cyclic groups of p-power order for two sided action, two separate cases are considered namely for the case p an odd prime and $p = 2$.

Chapter 7 presents the conclusion of this research and suggestions for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The nonabelian tensor product has been discussed since 1984. Brown and Loday were the first who introduced the concept of tensor in their paper [1]. Since the paper was written in Portuguese, thus the second paper by Brown *et al.* [2] in 1987 became the starting point for most investigations of the nonabelian tensor products. They also provided a list of open problems concerning nonabelian tensor product and tensor square. The detailed literature review is given in Section 2.2.

2.2 The Nonabelian Tensor Product of Groups

The nonabelian tensor product for groups G and H, denoted by $G \otimes H$, was originated in connection with a generalized Van Kampen Theorem and its construction has its origins in the algebraic K-theory and in homotopy theory. It was introduced by Brown and Loday in [1] and [6], extended the concepts by Whitehead in [8]. Starting with the paper of Brown *et al.* [2], many researchers had studied group theoretical aspects of nonabelian tensor products extensively. A paper by Kappe [9] gives an overview of known results and literature up to 1997.

Brown *et al.* in [2] focused on the group theoretic properties and explicit computation of nonabelian tensor squares. They also provided a list of open problems concerning nonabelian tensor squares and tensor products. Initially, this research topic was guided by eight open problems at the end of [2].

It is already established in [2] that the nonabelian tensor square, denoted by $G \otimes G$, is finite for a finite G. Brown *et al.* [2] also showed that the nonabelian tensor square of a nilpotent or solvable group is nilpotent or solvable. In addition, he showed that the nonabelian tensor product, $G \otimes H$ is of p-power order if G and H are of p-power order and given the results of the computation of the nonabelian tensor squares for groups of order up to 30 with the help of GAP.

In 1987, Ellis [3] extended the results for nonabelian tensor products but he did not provide any analytical proof. He also showed that the nonabelian tensor product, $G \otimes H$ is of p-power order if G and H are of p-power order. In 1991, Rocco [11] gives a bound for the order of $G \otimes G$ if G has order p^n. Later, Bacon *et al.* in [13] used GAP in computing the nonabelian tensor square of the 2-Engel group of rank 3. Besides that, Ellis and Leonard in [12] developed an algorithm that can handle the computation of the nonabelian tensor square of much larger groups. They calculated Burnside Groups $B(2,4)$ and $B(3,3)$, which have order 2^{12} and 3^7, respectively. They also developed a method that can be used to compute the nonabelian tensor product of a pair of groups, which are embeded as normal subgroups, or what is called as parent group of order up to 14. In 1998, McDermott in [4, 14] had developed an algorithm for computing the nonabelian tensor product of more general finite case. He gave the nonabelian tensor product for every pair of normal subgroups of the nonabelian quaternion group of order 32. In addition, he also gave the nonabelian tensor product of quarternion group and dihedral group both of order eight and split them into two cases, namely when the action acts compatibly on each other and when the actions do not act compatibly on each other. In addition, Ellis and McDermott in [14] improved Rocco's bound [11] and extended it to the case of nonabelian tensor product of prime power groups G and H.

McDermott [4] computed the nonabelian tensor product, $G \otimes H$ when G is a p-group and H is a q-group, where p and q are primes. However, he only focused on the bound of the order of $G \otimes H$ and gave some results on that case. Visscher [5] continued the study on the nonabelian tensor product of p-power order and focused on cyclic groups. Visscher gave an explicit description of the action of a cyclic group of prime power order on another in the first part in his thesis [5] before he used the results to compute the nonabelian tensor product. The lemma that he used containing a pair of number theoretic results in [13] for $p \neq 2$ and in [15] for $p = 2$. Visscher computed some of the nonabelian tensor product of cyclic group of p-power order and gave a complete classification of all the nonabelian tensor product of cyclic groups of 2-power order with mutual nontrivial actions of order two. Visscher [5] also gave the bounds on the nilpotency class and solvability length of $G \otimes H$, provided such information is given in context with G and H. The bounds are given in terms of $D_H(G)$, the derive subgroup of G afforded by the action of H on G, and $D_G(H)$, the analogous subgroup of H.

In other point of view, Nakaoka [16] considered a group construction that is related to the nonabelian tensor product as a second proof to claim Ellis's result in [3]. Nakaoko in [16] constructed a group defined as follows:

Let G and H be groups acting compatibly on each other and H^{ϕ} an extra copy of H, isomorphic through $\phi : H \to H^{\phi}, h \mapsto h^{\phi}$ for all $h \in H$. Then the group $\eta(G, H)$ is defined as $\eta(G, H) =$

$$\left\langle G, H \,\middle|\, \left[g, h^{\phi}\right]^{g_1} = \left[g^{g_1}, (h^{g_1})^{\phi}\right], \left[g, h^{\phi}\right]^{h_1^{\phi}} = \left[g^{h_1}, (h^{h_1})^{\phi}\right], \forall g, g_1 \in G, \forall h, h_1 \in H \right\rangle.$$

It follows from Gilbert and Higgins [17] that there is an isomorphism from the subgroup $\left[G, H^{\phi}\right]$ of $\eta(G, H)$ onto the nonabelian tensor product, $G \otimes H$ such that $\left[g, h^{\phi}\right] \mapsto g \otimes h$, for $g \in G$ and $h \in H$. This isomorphism is useful to study the nonabelian tensor product inside of $\eta(G, H)$. Rocco in [11] used this method to settle a bound for the order of $G \otimes G$ when G is a finite p-group. Observe that $\left[G, H^{\phi}\right]$ is a normal subgroup of $\eta(G, H)$ and $\eta(G, H) = \left[G, H^{\phi}\right] GH^{\phi}$. Thus, if G and H are finite and solvable, then $\eta(G, H)$ is also finite and solvable.

Primoz [18] in 2008 proved that the exponent of the nonabelian tensor product of two locally finite groups can be bounded in terms of exponents of given groups. Several estimates for the exponents of nonabelian tensor squares are obtained. In particular, if the group G is nilpotent of class ≤ 3 and of finite exponent, then the exponent of its nonabelian tensor square divides the exponent of G. Then, in 2010, he introduced the notion of a powerful action of a p-group upon another p-group [19]. He also proved that the non-abelian tensor product of powerful p-groups acting powerfully and compatibly upon each other is again a powerful p-group. Also in 2010, Thomas [10] gave a homology free proof for the nonabelian tensor product of finite groups is finite which give an algebraic proof for Ellis' [3] result.

Recently in 2011, Russo [20] had constructed the nonabelian tensor product in the classes of groups such as the class of all finite groups, nilpotent groups, soluble groups, polycyclic groups, locally finite groups, Chernikov or soluble minimax groups [21] to form the nonabelian tensor product in the same class.

In this research, we are interested in finding the conditions for the actions that act compatibly on each other. Previously, there are only three papers which focus on these conditions, namely [4], [5] and [14]. The action for the nonabelian tensor square is defined as conjugation. Using conjugation in Q_{32} as the basis for all actions, McDermott in [4] and [14] gave different actions, which arise between each pair of subgroups by exhibiting the images of the generators of Q_{32} under the actions of the generators of the same group. For the nonabelian tensor product, he gave all possible pairs of actions between D_4 and Q_8 by finding all possible images of their generators in the respective automorphism group. There are 28 possibles actions of Q_8 on D_4 and 76 possibles actions of D_4 on Q_8, giving all together 2128 distinct pairs of actions between D_4 on Q_8. He found that only 292 are compatible pairs. Besides that, Visshier [5] had started in determining the nonabelian tensor product of cyclic groups. He had determined the characterization on the compatibility condition and provided some necessary and sufficient conditions for a pair of cyclic groups of p-power order where p an odd prime, as well as when $p = 2$ to act compatibly on each other. However, the order of the actions as one of the conditions is not included in the characterizations.

As a continuation of Visscher's work, this research is focused on the determination of the nonabelian tensor product of finite cyclic groups of p-power order with two-sided actions of order greater then two and the actions are compatible.

This research starts with the characterization of automorphisms of cyclic groups of p-power order using number theoretical results where the order of the actions are considered. Then, the necessary and sufficient conditions for the actions to be compatible are determined for a pair of finite cyclic groups. Finally, by using a general expansion formula, the nonabelian tensor products of some cyclic groups of p-power order are proven to be cyclic.

2.3 Conclusion

In this chapter, literatures on the nonabelian tensor products of groups are presented. From the literatures, non of the references have shown that the nonabelian tensor product of finite cyclic groups are cyclic except Visscher [5] but he only focused on nonabelian tensor product of one-sided action and when both actions have order two. Therefore, the computation of the nonabelian tensor products for other groups will enrich some new samples in this area.

The next chapter gives some related preliminaries results for this research.

CHAPTER 3

PRELIMINARY RESULTS

3.1 Introduction

In this chapter, some definitions and preliminary results on automorphism groups, compatibility conditions, the nonabelian tensor products and Groups, Algorithms and Programming (GAP) algorithms are given. GAP algorithms to determine the compatible actions and to find the nonabelian tensor products are also given. All results in this chapter are used in the following chapters.

3.2 Automorphism Groups

Let G be a finite cyclic group generated by $g \in G$. Any automorphism of G is given by a mapping $\sigma : g \to g^t$, where t is an integer with $\gcd(t, |g|) = 1$. In further applications, some information on the arithmetic nature of t depending on the order of σ in case G is a cyclic group of p-power order is needed in finding the conditions for compatible actions stated in the next chapter.

Some familiar results related to φ-function which can be found in [22] are stated as follows:

Definition 3.1 [22] **Euler's φ-function**
For $m \geq 1$, the Euler's φ-function, denoted by $\varphi(m)$, is the number of positive integer not exceeding m that are relatively prime to m.

Theorem 3.1 [22] If an integer $m > 1$ has the prime factorization $m = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$, then

$$\begin{aligned} \varphi(m) &= \left(p_1^{k_1} - p_1^{k_1-1}\right)\left(p_2^{k_2} - p_2^{k_2-1}\right)\cdots\left(p_r^{k_r} - p_r^{k_r-1}\right) \\ &= m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right). \end{aligned}$$

Theorem 3.2 [22] **Euler's Theorem**

If $m > 1$ and $\gcd(a, m) = 1$, then $a^{\varphi(m)} \equiv 1 \bmod m$, where $\varphi(m)$ is the Euler's φ-function.

In view of Euler's Theorem, note that $a^{\varphi(m)} \equiv 1 \bmod m$ whenever $\gcd(a, m) = 1$. However, there are often powers smaller than $\varphi(m)$ such that a to that power is congruent to 1 modulo m [22]. This prompts the following definition.

Definition 3.2 [22] **Order of an Integer Modulo m**

Let $m > 1$ and $\gcd(a, m) = 1$. The order of a modulo m is the smallest positive integer k such that $a^k \equiv 1 \bmod m$.

From now on, the order of a modulo m will be written as $k = \mathrm{ord}_m(a)$. Note that if $\gcd(a, m) = 1$, then $\mathrm{ord}_m(a)$ can be defined.

Theorem 3.3 [22] Let a be an integer with $\gcd(a, m) = 1$ and $\mathrm{ord}_m(a) = k$. Then $a^h \equiv 1 \bmod m$ if and only if $k|h$; in particular $k|\varphi(m)$.

Another basic fact regarding the order of an integer is given as follows.

Theorem 3.4 [22] If $\mathrm{ord}_m(a) = k$, then $a^i \equiv a^j \bmod m$ if and only if $i \equiv j \bmod k$.

This leads to the following corollary since there are k different numbers of i.

Corollary 3.1 [22] If $\mathrm{ord}_m(a) = k$, then the integers $a, a^2, ..., a^k$ are pairwise incongruent mod m.

It is possible to express the order of any power of a in terms of the order of a as stated in the next theorem.

Theorem 3.5 [22] If $\mathrm{ord}_m(a) = k$ and $h > 0$, then

$$\mathrm{ord}_m\left(a^h\right) = \frac{k}{\gcd(h,k)}.$$

Now, consider $\mathrm{ord}_m(a) = m$. Then there are m pairwise incongruent mod m numbers of integers which are $a, a^2, ..., a^m$. Let A be a set with $A = a, a^2, ..., a^m$. Then A can be proved to be group in particularly a finite cyclic group of order m.

Next, let $\langle g \rangle$ be a finite cyclic group of order n, then Theorem 3.5 becomes $\left|g^k\right| = \dfrac{|g|}{\gcd(k, |g|)}$ for any $k \in \mathbb{N}$.

This fact can be formally written in the following corollary.

Corollary 3.2 [22] Let G be group and $g \in G$ with $|g| < \infty$. Then $\left|g^k\right| = \dfrac{|g|}{\gcd(k, |g|)}$ for any $k \in N$.

Hence, from this point on, some number theoretical results are applied.

Next, Dummit and Foote in [23] gave that the automorphism group of cyclic group of order p-power is a direct product of two cyclic groups as stated in the following.

Theorem 3.6 [23] Let p be an odd prime and $\alpha \in \mathbb{Z}^+$. If G is a cyclic group of order p^α, then $\mathrm{Aut}(G) \cong C_{p-1} \times C_{p^{\alpha-1}} \cong C_{(p-1)p^{\alpha-1}}$ and $|\mathrm{Aut}\,(\mathrm{G})| = \varphi(p^\alpha) = (p-1)\,p^{\alpha-1}$.

In addition, they also gave the group isomorphic to the automorpshim groups of 2-power order stated as follows.

Theorem 3.7 [23] Let G be a cyclic group of order 2^n, $n \geq 3$. Then $\mathrm{Aut}(G) \cong C_2 \times C_{2^{n-2}}$ and $|\mathrm{Aut}\,(G)| = \varphi\,(2^n) = 2^{n-1}$.

Next we state all known results on compatible actions that will be used in Chapter 5.

3.3 Compatible Actions

Firstly, the definition of an action of a group G on a group H is given in the following:

Definition 3.3 [5] **Action**
Let G and H be groups. An action of G on H is a mapping $\Phi : G \to \mathrm{End}(H)$ such that

$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$

for all $g, g' \in G$ and $h \in H$.

For the case of cyclic groups, an action Φ of a group G on a group H will also be required to have the property that $\Phi(1_G) = id_H$, the identity mapping on H. Such an action is typically called a monoid action. Thus, from this point on, an action will be a homomorphism Φ from G to $\mathrm{Aut}\,(H)$. In addition, the action will be written as

$${}^{g}h \stackrel{def}{=} \Phi\,(g)\,(h)\,.$$

In the context of this thesis, only actions of the following type are considered:

$$\Phi : G \to \mathrm{Aut}\,(H)\,,$$

where Φ is a homomorphism of G into $\mathrm{Aut}\,(H)$ and is denoted by

$$^{g}h = \Phi\,(g)\,(h)\,.$$

The conjugation action of a group on itself is written as ${}^{g}g' = gg'g^{-1}$ for $g, g' \in G$. Observe that conjugation is trivial for abelian groups.

A compatible action between two groups is defined in the following:

Definition 3.4 [6] **Compatible Action**
Let G and H be groups which act on each other. These mutual actions are said to be compatible with each other and with the actions of G and H onto themselves by conjugation if

$$^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g')) \tag{3.1}$$

and

$$^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h')) \tag{3.2}$$

for all $g, g' \in G$ and $h, h' \in H$.

In the next proposition, it can be shown that the dual of the compatibility condition always holds.

Proposition 3.1 [5] Let G and H be groups which act on each other and which act on themselves by conjugation. Then the following conditions always hold:

$$^{({}^{h}g)}g' = {}^{h}({}^{g}({}^{h^{-1}}g')) \quad \text{and} \quad {}^{({}^{g}h)}h' = {}^{g}({}^{h}({}^{g^{-1}}h'))$$

for all $g, g' \in G$ and $h, h' \in H$.

Proof Let $g, g' \in G$ and $h, h' \in H$. Then

$$\begin{aligned}
{}^{({}^{h}g)}g' &= ({}^{h}g)g'({}^{h}g)^{-1} \\
&= {}^{h}g \cdot g' \cdot {}^{h}g^{-1} \\
&= {}^{h}(g \cdot {}^{h^{-1}}g' \cdot g^{-1}) \\
&= {}^{h}({}^{g}({}^{h^{-1}}g')).
\end{aligned}$$

The second condition follows in a similar way. □

In case of abelian groups, the compatibility conditions can be simplified. This result is given in the next proposition.

Proposition 3.2 [5] Let G and H be groups which act on each other and on themselves by conjugation. If G and H are abelian, then the mutual actions are compatible if and only if

$$^{(^g h)}g' = {}^h g' \quad \text{and} \quad ^{(^h g)}h' = {}^g h'$$

for all $g, g' \in G$ and $h, h' \in H$.

Proof Let G and H be abelian groups. Observe that conjugation is trivial for an abelian group. Thus the first compatibility condition becomes

$$\begin{aligned} ^{(^g h)}g' &= {}^g(^h(^{g^{-1}}g')) \\ &= g(^h(g^{-1}g'g))g^{-1} \\ &= {}^h g'. \end{aligned}$$

The second condition can also be proved in a similar way. □

The next result gives a characterization of compatible actions if one action is trivial. The proof can be found in [5].

Proposition 3.3 [5] Let G and H be groups. Furthermore, let G act trivially on H, that is ${}^g h = h$ for all $g \in G$ and $h \in H$. Then an action $\Phi : H \to \mathrm{Aut}\,(G)$ is compatible with the trivial action if and only if $\Phi\,(H) \subseteq C_{\mathrm{Aut}(G)}\,(\mathrm{Inn}\,(G))$.

In the event where G is abelian, the following corollary shows that the trivial action is always compatible with any other action.