Compatible Pair of Nontrivial Actions for Some Cyclic Groups of 2-Power Order

Sahimel Azwal Sulaiman[a], Mohd Sham Mohamad[b] and Yuhani Yusof[c]

[a,b,c]*Futures and Trends Research Groups, Faculty of Industrial Sciences & Technology,*
Universiti Malaysia Pahang, Lebuhraya Tun Razak, 26300 Gambang, Kuantan, Pahang Darul Makmur
[a]*titus1704@yahoo.com,* [b]*mohdsham@ump.edu.my,* [c]*yuhani@ump.edu.my*

Abstract. Compatible action played a very important verification before the nonabelian tensor product can be computed. This paper gives the some exact number of compatible pairs of actions for some cyclic groups of 2-power order. Some necessary and sufficient number theoretical conditions for a pair of cyclic groups of 2-power order with nontrivial actions act compatibly on each other are used to investigate some properties to find the exact number of compatible pairs of actions. Algorithms in Groups, Algorithms and Programming (GAP) software are constructed to create more examples on selected case. New results on compatible pair of nontrivial actions of order two and four are given.

Keywords: Compatible Actions, Cyclic Groups, Nonabelian Tensor Product

INTRODUCTION

Let G and H be groups which act on each other and each of which acts on itself by conjugation, then the actions are compatible if ${}^{({}^{g}h)}g' = {}^{g}({}^{h}({}^{g^{-1}}g'))$ and ${}^{({}^{h}g)}h' = {}^{h}({}^{g}({}^{h^{-1}}h'))$. This compatible action played a very important verification before the nonabelian tensor product can be computed. Brown and Loday [1] were the first to introduce the concept of the nonabelian tensor product of groups with the compatible actions. Compatible conditions are the prerequisite before nonabelian tensor product can be calculated.

This construction of the nonabelian tensor product has its origins in the algebraic K-theory and in homotopy theory. Several authors, starting with Brown, Johnson and Robertson [2], have studied group theoretical aspects of nonabelian tensor products extensively. A paper by Kappe [3] gives an overview of known results and literature.

Many authors are working with nonabelian tensor square (where $H = G$), and the compatibly conditions can be checked easily. It is interesting to check the compatible conditions when calculating tensor product since the groups are not the same or when the actions are nontrivial. Ellis and McDermott [4] had done this but specifically for quaternion group of order 32. Visscher [5] in 1998 had provided necessary and sufficient conditions for a pair of cyclic groups to act compatibly. He gave complete compatible conditions when one of the actions is trivial or both actions are trivial in cyclic groups of p-power order. In this research, all compatible actions involving at least one of the actions having order greater than two will be determined as a continuation from Visscher's.

Groups, Algorithms and Programming (GAP) software [6] is used to determine the conditions in which the actions satisfy the compatibility conditions in some finite cyclic groups.

SOME PREPARATORY RESULTS ON COMPATIBILITY CONDITIONS

The nonabelian tensor product $G \otimes H$ is defined only if g and h act on each other in such way that the actions satisfy certain compatibility condition. Start with the definition of an action a group g on a group h.

Definition 2.1 [5]
Let G and H be groups. An action of G on H is a mapping, $\Phi : G \to \mathrm{Aut}(H)$ such that
$$\Phi(gg')(h) = \Phi(g)(\Phi(g')(h))$$
for all $g, g' \in G$ and $h \in H$.

For a group g to act on a group h, by using the fact that the identity in g acts as the identity mapping on h, thus this implies that all elements in g act as automorphism of h on h. The definition of a compatible action is given as follows.

Definition 2.1 [5]
Let G and H be groups which act on each other. These mutual actions are said to be compatible with each other and with the actions of G and H on themselves by conjugation if

$$\left(^{g}h\right)g' = {}^{g}(^{h}(^{g^{-1}}g')) \text{ and } \left(^{h}g\right)h' = {}^{h}(^{g}(^{h^{-1}}h'))$$

for all $g, g' \in G$ and $h \in H$.

Since we are focusing on finite cyclic groups, the compatibility conditions which hold for abelian groups will be reduce to $\left(^{g}h\right)g' = {}^{h}g'$ and $\left(^{h}g\right)h' = {}^{g}h'$, for all $g, g' \in G$ and $h \in H$. For compatibility of actions of G and H on each other, it suffices for the compability conditions to hold on the generators of G and H. Visscher [7] has done this and he gave some necessary and sufficient number theoretical conditions for a pair of finite cyclic groups to act compatibly on each other as stated in Proposition 2.3.

Proposition 2.3 [5]
Let $G = \langle x \rangle = C_m$ and $H = \langle y \rangle = C_n$ be finite cyclic groups. Then there exist mutual actions of G and H on each other such that $^{y}x = x^k$ and $^{x}y = y^l$ for $k,l \in \mathbb{Z}$ if and only if conditions (i) and (ii) below are satisfied. These action are compatible if and only condition (iii) is satisfied as well.

(i) $(k,m) = (l,n) = 1$

(ii) $k^n \equiv 1 \bmod m$ and $l^m \equiv 1 \bmod n$

(iii) $k^{l-1} \equiv 1 \bmod m$ and $l^{k-1} \equiv 1 \bmod n$.

In the next section, all compatible actions involving at least one of the actions having order greater than two will be determined.

COMPATIBLE CONDITIONS FOR NONTRIVIAL ACTION

In the previous section, we can see that Visscher had done the classification of compatibility condition focusing on cyclic groups and provide necessary and sufficient conditions for the general case without any specific condition.

Now, our aim is to determine the compatible conditions if the order of the action is more than two. All computations are doing by GAP. New GAP coding has been built to calculate the compatible actions.

Let $G = \langle x \rangle = C_m$ and $H = \langle y \rangle = C_m$ be finite cyclic groups. Let the actions of G and H on each other be such that $^{y}x = x^k$ and $^{x}y = y^l$ for $k,l \in \mathbb{Z}$. Now consider that both actions have order four. The compatible conditions that are satisfied by k and l are given in **Table 1** when $G = H$ and in **Table 2** when $G \neq H$.

TABLE 1. Values of k and l in which the action of order four satisfying the compatible conditions when $G = H$.

Groups	k	l	Groups	k	l
$G = H = C_{2^4}$	5	5	$G = H = C_{2^7}$	33	33
	13	13		97	97
$G = H = C_{2^5}$	9	9	$G = H = C_{2^8}$	65	65
	25	25		193	193
$G = H = C_{2^6}$	17	17	$G = H = C_{2^9}$	129	129
	49	49		385	385

TABLE 2. Values of k and l in which the action of order four satisfying the compatible conditions when $G \neq H$.

Groups	k	l
$G = C_{2^4}$	5	9
	5	25
and	13	9
$H = C_{2^5}$	13	25
$G = C_{2^4}$	5	17
	5	49
and	13	17
$H = C_{2^6}$	13	49
$G = C_{2^4}$	5	33
	5	97
and	13	33
$H = C_{2^7}$	13	97
$G = C_{2^4}$	5	65
	5	193
and	13	65
$H = C_{2^8}$	13	193
$G = C_{2^4}$	5	129
	5	385
and	13	129
$H = C_{2^9}$	13	385

Groups	k	l
$G = C_{2^5}$	9	5
	9	13
and	25	5
$H = C_{2^4}$	25	13
$G = C_{2^5}$	9	17
	9	49
and	25	17
$H = C_{2^6}$	25	49
$G = C_{2^5}$	9	33
	9	97
and	25	33
$H = C_{2^7}$	25	97
$G = C_{2^5}$	9	65
	9	193
and	25	65
$H = C_{2^8}$	25	193
$G = C_{2^5}$	9	129
	9	385
and	25	129
$H = C_{2^9}$	25	385

Groups	k	l
$G = C_{2^6}$	17	5
	17	13
and	49	5
$H = C_{2^4}$	49	13
$G = C_{2^6}$	17	9
	17	25
and	49	9
$H = C_{2^5}$	49	25
$G = C_{2^6}$	17	33
	17	97
and	49	33
$H = C_{2^7}$	49	97
$G = C_{2^6}$	17	65
	17	193
and	49	65
$H = C_{2^8}$	49	193
$G = C_{2^6}$	17	129
	17	385
and	49	129
$H = C_{2^9}$	49	385

Groups	k	l
$G = C_{2^7}$	33	5
	33	13
and	97	5
$H = C_{2^4}$	97	13
$G = C_{2^7}$	33	9
	33	25
and	97	9
$H = C_{2^5}$	97	25
$G = C_{2^7}$	33	17
	33	49
and	97	17
$H = C_{2^6}$	97	49
$G = C_{2^7}$	33	65
	33	193
and	97	65
$H = C_{2^8}$	97	193
$G = C_{2^7}$	33	129
	33	385
and	97	129
$H = C_{2^9}$	97	385

Groups	k	l
$G = C_{2^8}$	65	5
	65	13
and	193	5
$H = C_{2^4}$	193	13
$G = C_{2^8}$	65	9
	65	25
and	193	9
$H = C_{2^5}$	193	25
$G = C_{2^8}$	65	17
	65	49
and	193	17
$H = C_{2^6}$	193	49
$G = C_{2^8}$	65	33
	65	97
and	193	33
$H = C_{2^7}$	193	97
$G = C_{2^8}$	65	129
	65	385
and	193	129
$H = C_{2^9}$	193	385

Groups	k	l
$G = C_{2^9}$	129	5
	129	13
and	385	5
$H = C_{2^4}$	385	13
$G = C_{2^9}$	129	9
	129	25
and	385	9
$H = C_{2^5}$	385	25
$G = C_{2^9}$	129	17
	129	49
and	385	17
$H = C_{2^6}$	385	49
$G = C_{2^9}$	129	33
	129	97
and	385	33
$H = C_{2^7}$	385	97
$G = C_{2^9}$	129	65
	129	193
and	385	65
$H = C_{2^8}$	385	193

This lead the following results:

Theorem 3.1

Let $C_{2^\alpha} = \langle x \rangle$ and $C_{2^\beta} = \langle y \rangle$ be cyclic groups of 2-power order. Furthermore, let C_{2^α} and C_{2^β} act on each other so that $^y x = x^k$ and $^x y = y^l$ for $k, l \in \mathbb{N}$ with $(k,2) = (l,2) = 1$. Then C_{2^α} and C_{2^β} act compatibly on each other when $\alpha = \beta$ with $^{y^4} x = x$ and $^{x^4} y = y$, if and only if k and l are congruent to one of the following:

(i) $k = l \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$;

(ii) $k = l \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$.

Theorem 3.2

Let $C_{2^\alpha} = \langle x \rangle$ and $C_{2^\beta} = \langle y \rangle$ be cyclic groups of 2-power order. Furthermore, let C_{2^α} and C_{2^β} act on each other so that $^y x = x^k$ and $^x y = y^l$ for $k, l \in \mathbb{N}$ with $(k,2) = (l,2) = 1$. Then C_{2^α} and C_{2^β} act compatibly on each other when $\alpha \neq \beta$ with $^{y^4} x = x$ and $^{x^4} y = y$, if and only if k and l are congruent to one of the following:

(i) $k \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 + 2^{\beta-2} \bmod 2^\beta$;

(ii) $k \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-2} \bmod 2^\beta$;

(iii) $k \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 + 2^{\beta-2} \bmod 2^\beta$;

(iv) $k \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-2} \bmod 2^\beta$.

Next, let the actions of G and H be on each other such that $^y x = x^k$ and $^x y = y^l$ for $k, l \in \mathbb{Z}$. Now consider that one of the actions has order four and the other one has order two. The compatible conditions that are satisfied by k and l are given in **Table 3** when $G = H$ and in **Table 4** when $G \neq H$.

TABLE 3. Values of k for which the action has order four and l for which the action had order two satisfying the compatible conditions when the groups G and H are the same.

Groups	k	l	Groups	k	l	Groups	k	l
$G = H = C_{2^4}$	3	9	$G = H = C_{2^4}$	3	9	$G = H = C_{2^4}$	3	9
	5	9		5	9		5	9
	11	9		11	9		11	9
	13	9		13	9		13	9
$G = H = C_{2^5}$	7	17	$G = H = C_{2^5}$	7	17	$G = H = C_{2^5}$	7	17
	9	17		9	17		9	17
	23	17		23	17		23	17
	25	17		25	17		25	17

TABLE 4. Values of k for which the action has order four and l for which the action has order two satisfying the compatible conditions when the groups G and H are different.

Groups	k	l
$G = C_{2^4}$ and $H = C_{2^5}$	3	17
	5	17
	11	17
	13	17
$G = C_{2^4}$ and $H = C_{2^6}$	3	33
	5	33
	11	33
	13	33
$G = C_{2^4}$ and $H = C_{2^7}$	3	65
	5	65
	11	65
	13	65
$G = C_{2^4}$ and $H = C_{2^8}$	3	129
	5	129
	11	129
	13	129
$G = C_{2^4}$ and $H = C_{2^9}$	3	257
	5	257
	11	257
	13	257

Groups	k	l
$G = C_{2^5}$ and $H = C_{2^4}$	7	9
	9	9
	23	9
	25	9
$G = C_{2^5}$ and $H = C_{2^6}$	7	33
	9	33
	23	33
	25	33
$G = C_{2^5}$ and $H = C_{2^7}$	7	65
	9	65
	23	65
	25	65
$G = C_{2^5}$ and $H = C_{2^8}$	7	129
	9	129
	23	129
	25	129
$G = C_{2^5}$ and $H = C_{2^9}$	7	257
	9	257
	23	257
	25	257

Groups	k	l
$G = C_{2^6}$ and $H = C_{2^4}$	15	9
	17	9
	47	9
	49	9
$G = C_{2^6}$ and $H = C_{2^5}$	15	17
	17	17
	47	17
	49	17
$G = C_{2^6}$ and $H = C_{2^7}$	15	65
	17	65
	47	65
	49	65
$G = C_{2^6}$ and $H = C_{2^8}$	15	129
	17	129
	47	129
	49	129
$G = C_{2^6}$ and $H = C_{2^9}$	15	257
	17	257
	47	257
	49	257

Groups	k	l
$G = C_{2^7}$ and $H = C_{2^4}$	31	9
	33	9
	95	9
	97	9
$G = C_{2^7}$ and $H = C_{2^5}$	31	17
	33	17
	95	17
	97	17
$G = C_{2^7}$ and $H = C_{2^6}$	31	33
	33	33
	95	33
	97	33
$G = C_{2^7}$ and $H = C_{2^8}$	31	129
	33	129
	95	129
	97	129
$G = C_{2^7}$ and $H = C_{2^9}$	31	257
	33	257
	95	257
	97	257

Groups	k	l
$G = C_{2^8}$ and $H = C_{2^4}$	63	9
	65	9
	191	9
	193	9
$G = C_{2^8}$ and $H = C_{2^5}$	63	17
	65	17
	191	17
	193	17
$G = C_{2^8}$ and $H = C_{2^6}$	63	33
	65	33
	191	33
	193	33
$G = C_{2^8}$ and $H = C_{2^7}$	63	65
	65	65
	191	65
	193	65
$G = C_{2^8}$ and $H = C_{2^9}$	63	257
	65	257
	191	257
	193	257

Groups	k	l
$G = C_{2^9}$ and $H = C_{2^4}$	127	9
	129	9
	383	9
	385	9
$G = C_{2^9}$ and $H = C_{2^5}$	127	17
	129	17
	383	17
	385	17
$G = C_{2^9}$ and $H = C_{2^6}$	127	33
	129	33
	383	33
	385	33
$G = C_{2^9}$ and $H = C_{2^7}$	127	65
	129	65
	383	65
	385	65
$G = C_{2^9}$ and $H = C_{2^8}$	127	129
	129	129
	383	129
	385	129

The result is given in the following theorem.

Theorem 3.3

Let $G = \langle x \rangle \cong C_{2^\alpha}$ and $H = \langle y \rangle \cong C_{2^\beta}$ be cyclic groups of 2-power order. Furthermore, let G and H act on each other so that $^y x = x^k$ and $^x y = y^l$ for $k, l \in \mathbb{N}$ with $(k, 2) = (l, 2) = 1$. Then G and H act compatibly on each other with the action of H on G is order 4 and the action of H on G of order 2, if and only if k and l are congruent to one of the following:

(i) $k \equiv 1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$;

(ii) $k \equiv 1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$;

(iii) $k \equiv -1 + 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$;

(iv) $k \equiv -1 - 2^{\alpha-2} \bmod 2^\alpha$ and $l \equiv 1 - 2^{\beta-1} \bmod 2^\beta$.

CONCLUSION

In this paper, three theorems have been stated accordingly to the order of the actions that satisfy compatible actions, which covered when the both actions have order 4, and when one of the actions has order 4 while the other has order 2

ACKNOWLEDGMENTS

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