



# Unsteady Boundary Layer Flow and Heat Transfer of a Casson Fluid past an Oscillating Vertical Plate with Newtonian Heating

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## Abstract

In this paper, the heat transfer effect on the unsteady boundary layer flow of a Casson fluid past an infinite oscillating vertical plate with Newtonian heating is investigated. The governing equations are transformed to a systems of linear partial differential equations using appropriate non-dimensional variables. The resulting equations are solved analytically by using the Laplace transform method and the expressions for velocity and temperature are obtained. They satisfy all imposed initial and boundary conditions and reduce to some well-known solutions for Newtonian fluids. Numerical results for velocity, temperature, skin friction and Nusselt number are shown in various graphs and discussed for embedded flow parameters. It is found that velocity decreases as Casson parameters increases and thermal boundary layer thickness increases with increasing Newtonian heating parameter.

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## Introduction

Non-Newtonian fluids are widely used in industries such as chemicals, cosmetics, pharmaceuticals, food and oil & gas [1]. Due to their numerous applications several scientists and engineers are working on them. Despite of the fact non-Newtonian fluids are not as easy as Newtonian fluids. It is due to the fact that in non-Newtonian fluids there does not exist a single constitutive relation that can be used to explain all of them. Therefore several constitutive equations or models are introduced to study their characteristics. The different non-Newtonian models include power law [2], second grade [3], Jeffrey [4], Maxwell [5], viscoplastic [6], Bingham plastic [7], Brinkman type [8], Oldroyd-B [9] and Walters-B [10] models. However, there is another model known as Casson model which is recently the most popular one. Casson [11] was the first who introduce this model for the prediction of the flow behavior of pigment oil suspensions of the printing ink type. Later on, several researchers studied Casson fluid for different flow situations and configurations. Amongst them, Mustafa et al. [12] studied the unsteady flow and heat transfer of a Casson fluid past a moving flat plate. Rao et al. [13] considered the thermal and hydrodynamic slip conditions on heat transfer flow of a Casson fluid past a semi-infinite vertical plate. Heat transfer flow of a Casson fluid past a permeable shrinking sheet with viscous dissipation was considered by Qasim and Noreen [14]. Recently, forced convection flow of a Casson fluid

past with surface heat flux over a symmetric porous wedge was investigated by Mukhopadhyay and Mandal [15]. Few other attempts for the Casson fluid can also be found in [16–21].

In all these studies mentioned above, the Newtonian heating condition was neglected at the boundary. The situation where the heat is transported to the convective fluid via a bounding surface having finite heat capacity is known as Newtonian heating (or conjugate convective flows). This configuration occurs in convection flows set up when the bounding surfaces absorb heat by solar radiation. Merkin [22] in his pioneering work studied the free convection boundary layer flow past a vertical plate with Newtonian heating. He found the asymptotic solution near the leading edge analytically and the full solution along the whole plate for free convection boundary layer over vertical surfaces numerically. On the other hand, the Newtonian heating situation occurs in many important engineering devices, such as heat exchanger and conjugate heat transfer around fins. Therefore, in view of such applications several authors have used the Newtonian heating condition in their convective heat transfer problems and have obtained the solutions either numerically [23–26] or analytical forms [27–33].

Most of the existing studies on unsteady boundary layer flow and heat transfer with Newtonian heating condition are limited to the Newtonian fluid or they are solved using any numerical or approximate technique. This motivates us to consider the Newtonian heating phenomenon in the present work for non-

Newtonian fluids. More exactly, our aim is to investigate unsteady boundary layer flow and heat transfer of a Casson fluid past an infinite oscillating vertical plate with Newtonian heating condition. The equations of the problem are first formulated and then transformed into their dimensionless forms where the Laplace transform method is applied to find the exact solutions for velocity and temperature.

**Mathematical Formulation**

Let us consider the heat transfer effect on unsteady boundary layer flow in a Casson fluid past an infinite oscillating vertical plate fixed at  $y=0$ , the flow being confined to  $y>0$ , where  $y$  is the coordinate axis normal to the plate. Initially, for time  $t=0$ , both plate and fluid are at stationary condition with the constant temperature  $T_\infty$ . At time  $t=0^+$ , the plate started an oscillatory motion in its plane ( $y=0$ ) according to

$$\mathbf{V} = UH(t) \cos(\omega t) \mathbf{i}; \quad t > 0, \tag{1}$$

where  $\mathbf{V} = u(y,t)\mathbf{i}$ ,  $U$  is the amplitude of the motion,  $H(t)$  is the unit step function,  $\mathbf{i}$  is the unit vector in the vertical flow direction and  $\omega$  is the frequency of plate oscillation. At the same time, the heat transfer from the plate to the fluid is proportional to the local surface temperature  $T$ . We assume that the rheological equation for an isotropic and incompressible Casson fluid, reported by Casson [11], is

$$\tau = \tau_0 + \mu \alpha^*$$

equivalently

$$\tau_{ij} = \begin{cases} 2 \left( \mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, & \pi > \pi_c \\ 2 \left( \mu_B + \frac{p_y}{\sqrt{2\pi_c}} \right) e_{ij}, & \pi < \pi_c \end{cases},$$

where  $\tau$  is the shear stress,  $\tau_0$  is the Casson yield stress,  $\mu$  is the dynamic viscosity,  $\alpha^*$  is the shear rate,  $\pi = e_{ij} e_{ij}$  and  $e_{ij}$  is the  $(i,j)^{th}$  component of the deformation rate,  $\pi$  is the product of the component of deformation rate with itself,  $\pi_c$  is a critical value of this product based on the non-Newtonian model,  $\mu_B$  is the plastic dynamic viscosity of the non-Newtonian fluid and  $p_y$  is the yield stress of fluid. Under these assumptions the unsteady boundary layer flow with heat transfer is governed by momentum and energy equations:

$$\rho \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = \text{div } \mathbf{T} + \rho \mathbf{b}, \tag{2}$$

$$\rho C_p \frac{\partial T}{\partial t} = - \frac{\partial p}{\partial t} + k \nabla^2 T, \tag{3}$$

where  $\mathbf{T}$  is the Cauchy stress tensor,  $\rho$  is the fluid density,  $\rho \mathbf{b}$  is the body force,  $p$  is the pressure,  $C_p$  is the heat capacity at constant pressure and  $k$  is the thermal conductivity. Under the Boussinesq approximation along with the assumption that the pressure is uniform across the boundary layer, we get the following set of partial differential equations:

$$\frac{\partial u}{\partial t} = \nu \left( 1 + \frac{1}{\alpha} \right) \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty), \tag{4}$$

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2}, \tag{5}$$

with following initial and boundary conditions

$$u(y,0) = 0, \quad T(y,0) = T_\infty, \quad \text{for all } y \geq 0, \tag{6}$$

$$u(0,t) = H(t)U \cos(\omega t), \quad \frac{\partial T}{\partial y}(0,t) = -h_s T(0,t), \quad t > 0, \tag{7}$$

$$u(\infty,t) \rightarrow 0, \quad T(\infty,t) \rightarrow T_\infty, \quad t > 0, \tag{8}$$

in which  $u$  is the axial velocity,  $t$  is the time,  $\nu$  is the kinematic viscosity,  $\alpha$  is the Casson fluid parameter,  $g$  is the acceleration due to gravity,  $\beta$  is the volumetric coefficient of thermal expansion and  $h_s$  is the heat transfer coefficient. The geometry of the problem is presented in Figure 1.

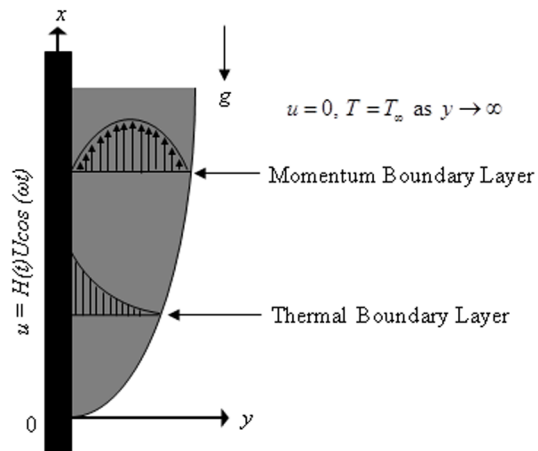
To reduce the above equations into their non-dimensional forms, we introduce the following non-dimensional quantities

$$y^* = \frac{U}{\nu} y, \quad t^* = \frac{U^2}{\nu} t, \quad u^* = \frac{u}{U}, \quad \theta = \frac{T - T_\infty}{T_\infty}, \quad \omega^* = \frac{\nu}{U^2} \omega. \tag{9}$$

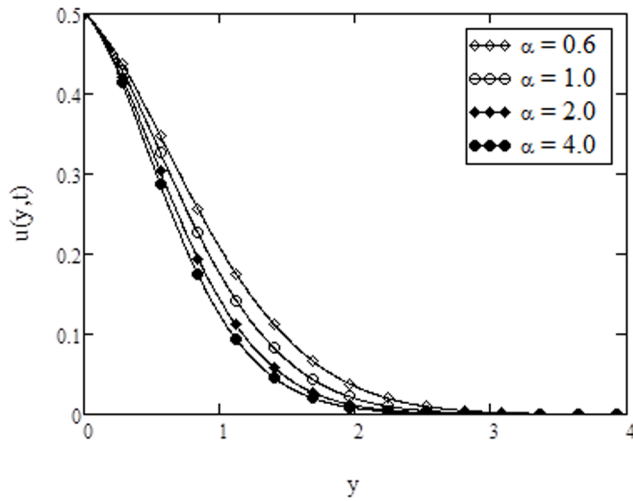
Substituting equation (9) into equations (4) and (5), we obtain the following non-dimensional partial differential equations (\* symbols are dropped for simplicity)

$$\frac{\partial u}{\partial t} = \left( 1 + \frac{1}{\alpha} \right) \frac{\partial^2 u}{\partial y^2} + Gr\theta, \tag{10}$$

$$\text{Pr} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2}. \tag{11}$$



**Figure 1. Physical model and coordinate system.**  
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**Figure 2. Velocity profiles for different values of  $\alpha$ , when  $Pr = 0.3, Gr = 3, \gamma = 0.5, t = 0.2, \omega t = \frac{\pi}{3}$ .**  
doi:10.1371/journal.pone.0108763.g002

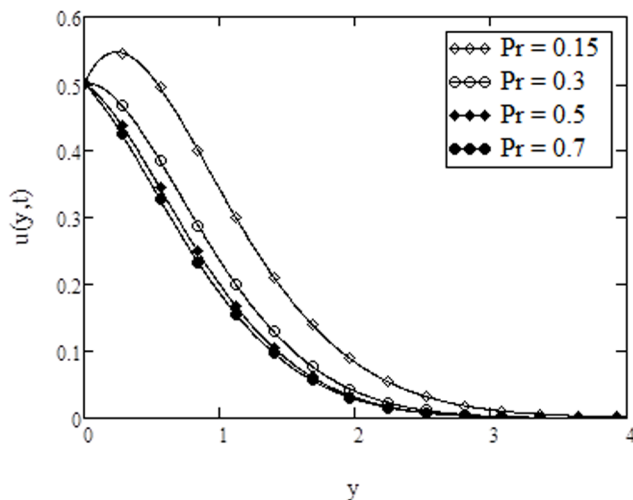
The corresponding initial and boundary conditions in non-dimensional form are

$$u(y,0) = 0, \theta(y,0) = 0, \text{ for all } y \geq 0, \quad (12)$$

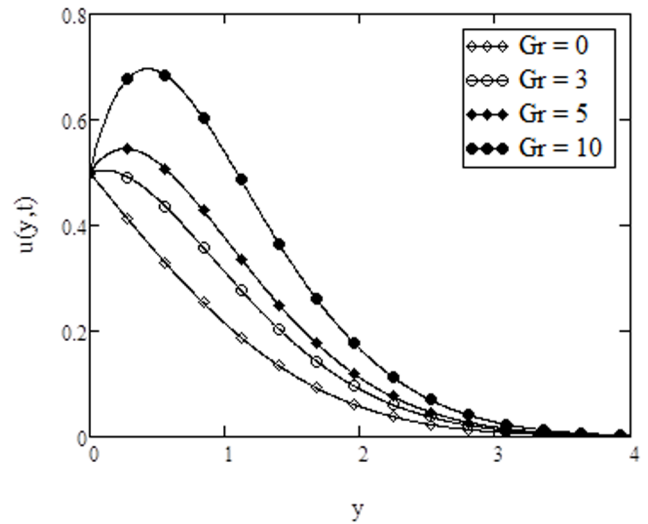
$$u(0,t) = H(t) \cos(\omega t), \frac{\partial \theta}{\partial y}(0,t) = -\gamma[1 + \theta(0,t)], t > 0, \quad (13)$$

$$u(\infty,t) \rightarrow 0, \theta(\infty,t) \rightarrow 0, t > 0, \quad (14)$$

where



**Figure 3. Velocity profiles for different values of  $Pr$ , when  $\alpha = 0.6, Gr = 5, \gamma = 0.5, t = 0.2, \omega t = \frac{\pi}{3}$ .**  
doi:10.1371/journal.pone.0108763.g003



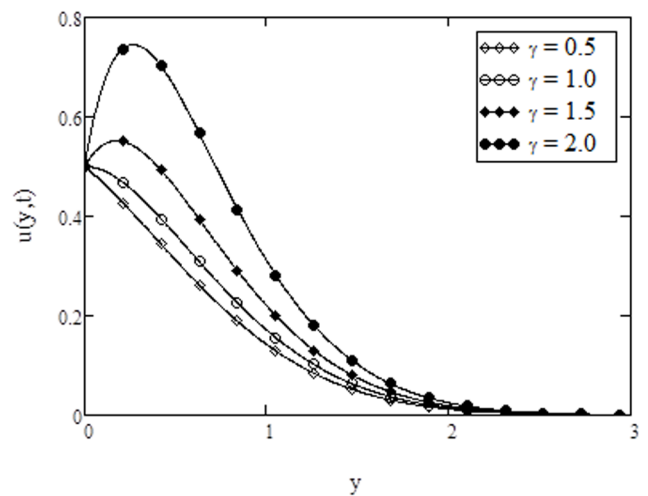
**Figure 4. Velocity profiles for different values of  $Gr$ , when  $\alpha = 0.6, Pr = 0.3, \gamma = 0.5, t = 0.3, \omega t = \frac{\pi}{3}$ .**  
doi:10.1371/journal.pone.0108763.g004

$$Gr = \frac{\nu g \beta T_{\infty}}{U^3}, Pr = \frac{\mu C_p}{k}, \gamma = \frac{h_s \nu}{U}$$

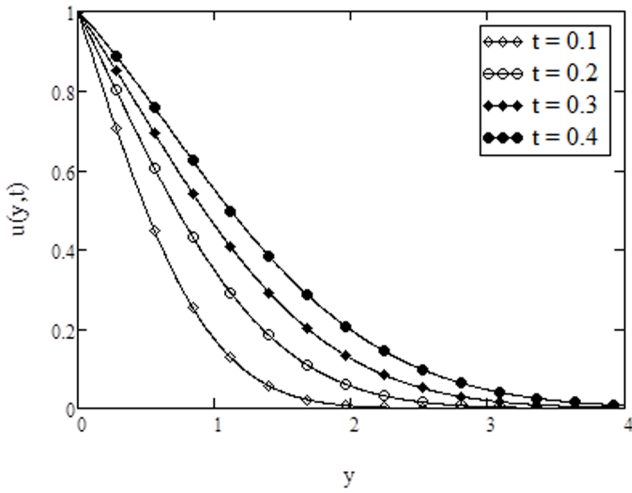
are the Grashof number, the Prandtl number and the conjugate parameter for Newtonian heating respectively. We note that equation (13) gives  $\theta = 0$  when  $\gamma = 0$ , corresponding to having  $h_s = 0$  and hence no heating from the plate exists [23,32].

**Method of Solution**

In order to obtain the exact solution of the present problem, we will use the Laplace transform technique. Applying the Laplace transforms with respect to time  $t$  to the equations (10)–(11), we get



**Figure 5. Velocity profiles for different values of  $\gamma$ , when  $\alpha = 0.6, Pr = 0.3, Gr = 3, t = 0.2, \omega t = \frac{\pi}{3}$ .**  
doi:10.1371/journal.pone.0108763.g005



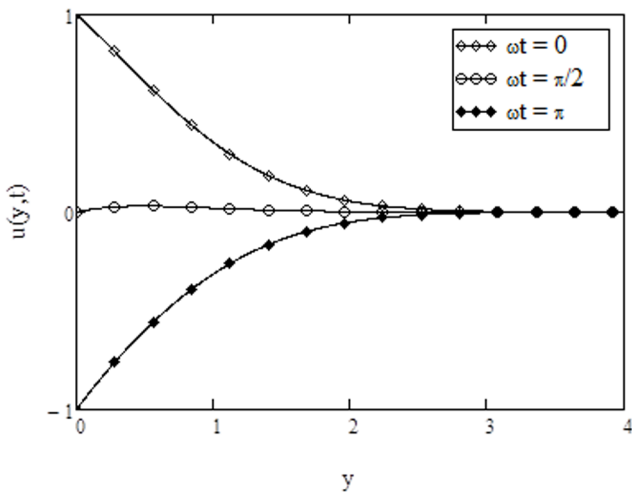
**Figure 6. Velocity profiles for different values of  $t$ ,** when  $\alpha=0.6, Pr=0.3, Gr=1, \gamma=0.5, \omega t=0$ .  
doi:10.1371/journal.pone.0108763.g006

$$q\bar{u}(y,q) - \bar{u}(y,0) = \left(1 + \frac{1}{\alpha}\right) \frac{d^2\bar{u}}{dy^2}(y,q) + Gr\bar{\theta}(y,q), \quad (15)$$

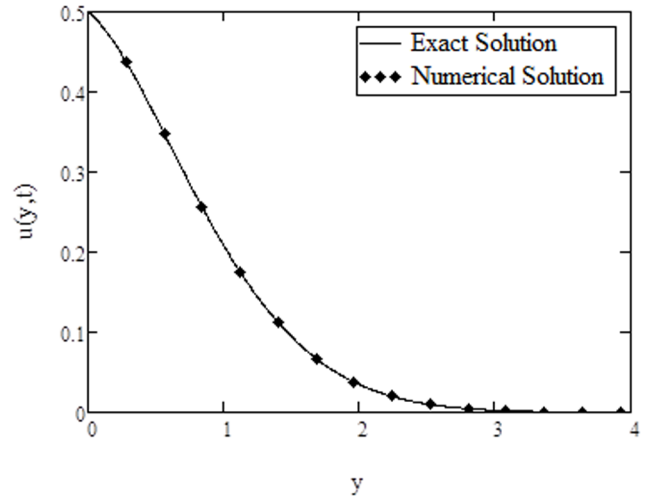
$$q\bar{\theta}(y,q) - \bar{\theta}(y,0) = \frac{1}{Pr} \frac{d^2\bar{\theta}}{dy^2}(y,q), \quad (16)$$

Here,  $\bar{u}(y,q) = \int_0^\infty e^{-qt} u(y,t) dt$  and  $\bar{\theta}(y,q) = \int_0^\infty e^{-qt} \theta(y,t) dt$  denote the Laplace transforms of  $u(y,t)$  and  $\theta(y,t)$ , respectively. Using the initial condition (12), we get

$$\left(1 + \frac{1}{\alpha}\right) \frac{d^2\bar{u}}{dy^2}(y,q) - q\bar{u}(y,q) + Gr\bar{\theta}(y,q) = 0, \quad (17)$$



**Figure 7. Velocity profiles for different values of  $\omega t$ ,** when  $\alpha=0.6, Pr=0.5, Gr=3, \gamma=0.5, t=0.2$ .  
doi:10.1371/journal.pone.0108763.g007



**Figure 8. Comparison of exact solution of velocity with numerical solution,** when  $\alpha=0.6, Pr=0.3, Gr=3, \gamma=0.5, t=0.2, \omega t = \frac{\pi}{3}$ .  
doi:10.1371/journal.pone.0108763.g008

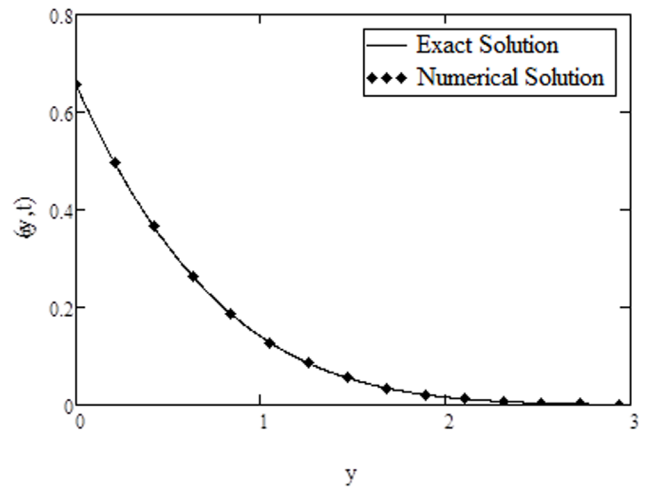
$$\frac{1}{Pr} \frac{d^2\bar{\theta}}{dy^2}(y,q) - q\bar{\theta}(y,q) = 0, \quad (18)$$

The corresponding transformed boundary conditions are

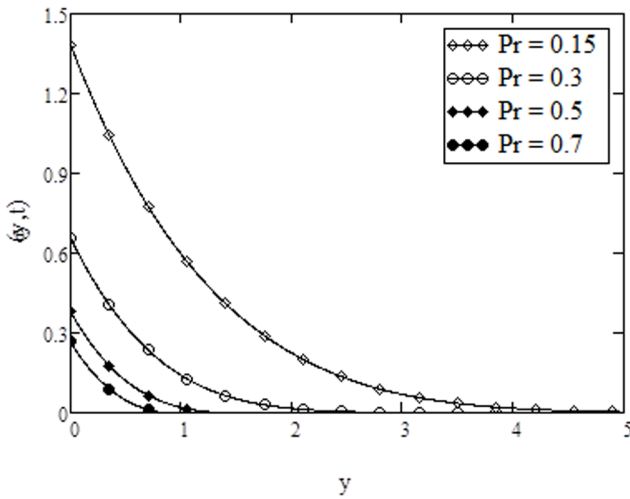
$$\bar{u}(0,q) = H(t) \frac{q}{q^2 + \omega^2}, \quad \frac{d\bar{\theta}}{dy}(0,q) = -\gamma \left[ \frac{1}{q} + \bar{\theta}(0,q) \right], \quad (19)$$

$$\bar{u}(\infty,q) \rightarrow 0, \quad \bar{\theta}(\infty,q) \rightarrow 0, \quad (20)$$

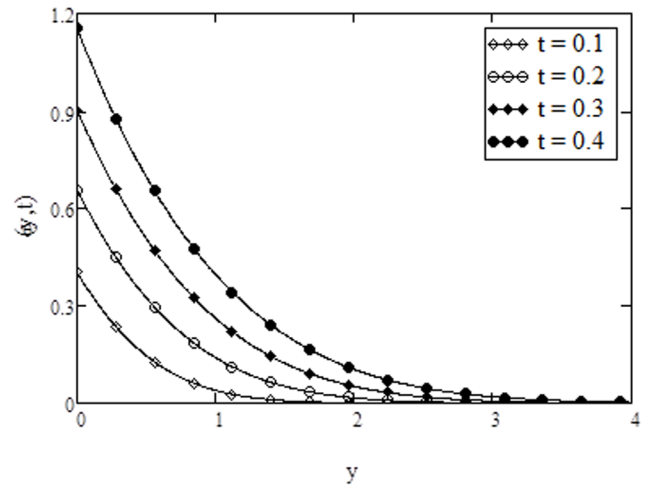
The solutions of equations (17) and (18) subject to the boundary conditions (19) and (20) are



**Figure 9. Comparison of exact solution of temperature with numerical solution,** when  $Pr=0.3, \gamma=0.5, t=0.2$ .  
doi:10.1371/journal.pone.0108763.g009



**Figure 10. Temperature profiles for different values of Pr, when  $\gamma=0.5, t=0.2$ .**  
doi:10.1371/journal.pone.0108763.g010



**Figure 12. Temperature profiles for different values of t, when Pr = 0.3,  $\gamma=0.5$ .**  
doi:10.1371/journal.pone.0108763.g012

$$\bar{u}(y, q) = \frac{H(t)}{2(q+i\omega)} e^{-y\sqrt{qa_1}} + \frac{H(t)}{2(q-i\omega)} e^{-y\sqrt{qa_1}} + \frac{a_1 a_2 a_3}{q^2(\sqrt{q}-a_2)} e^{-y\sqrt{qa_1}} - \frac{a_1 a_2 a_3}{q^2(\sqrt{q}-a_2)} e^{-y\sqrt{qPr}}, \quad (21)$$

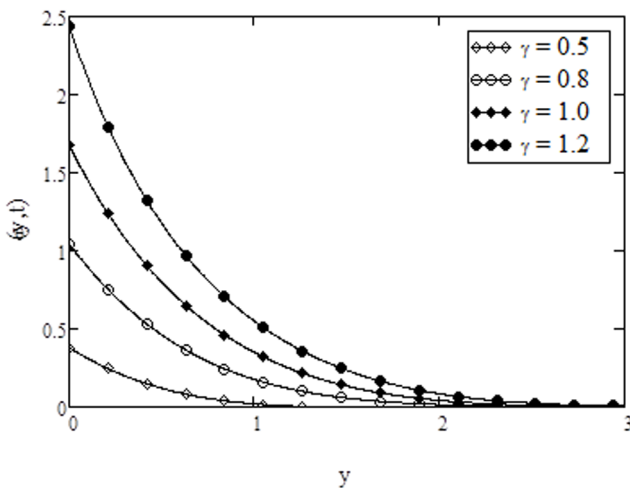
$$\bar{\theta}(y, q) = \frac{a_2}{q(\sqrt{q}-a_2)} e^{-y\sqrt{qPr}}. \quad (22)$$

By taking the inverse Laplace transforms of above equations, we obtain

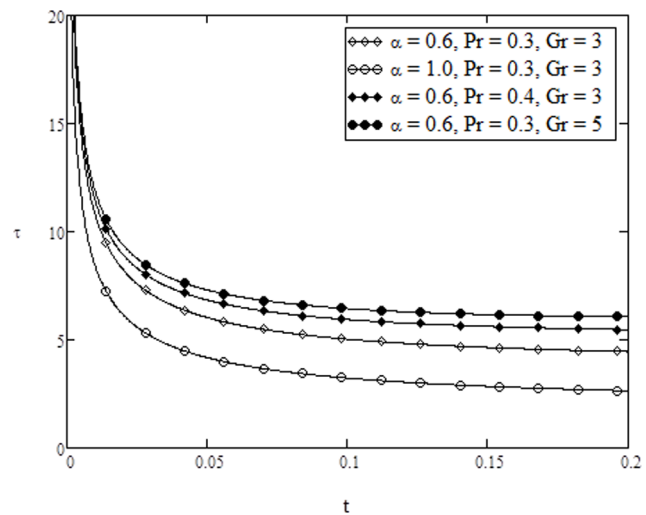
$$\theta(y, t) = F_2(y\sqrt{Pr}, t, a_2), \quad (23)$$

$$u(y, t) = \frac{H(t)}{2} [F_1(y\sqrt{a_1}, t, -i\omega) + F_1(y\sqrt{a_1}, t, i\omega)] + \frac{a_1 a_3}{a_2^2} [F_2(y\sqrt{a_1}, t, a_2) - F_2(y\sqrt{Pr}, t, a_2)] - \frac{a_1 a_3}{a_2} [F_3(y\sqrt{a_1}, t) - F_3(y\sqrt{Pr}, t)] - a_1 a_3 [F_4(y\sqrt{a_1}, t) - F_4(y\sqrt{Pr}, t)]. \quad (24)$$

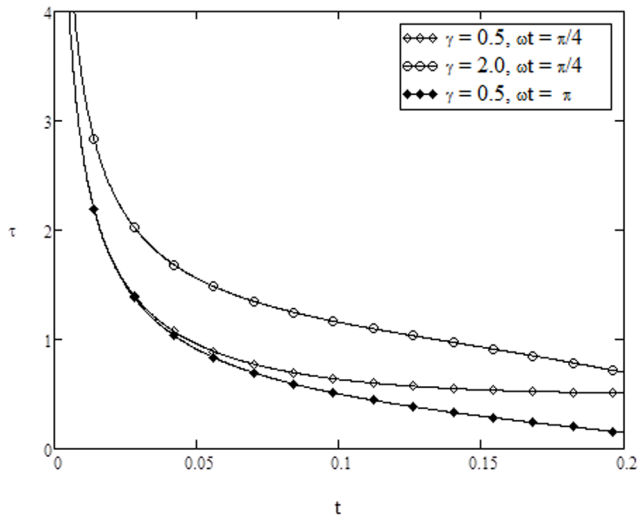
The solution for velocity given in equation (24) is not valid, when Pr = 1 and  $\alpha \rightarrow \infty$ . In this case, the solution obtained is given by



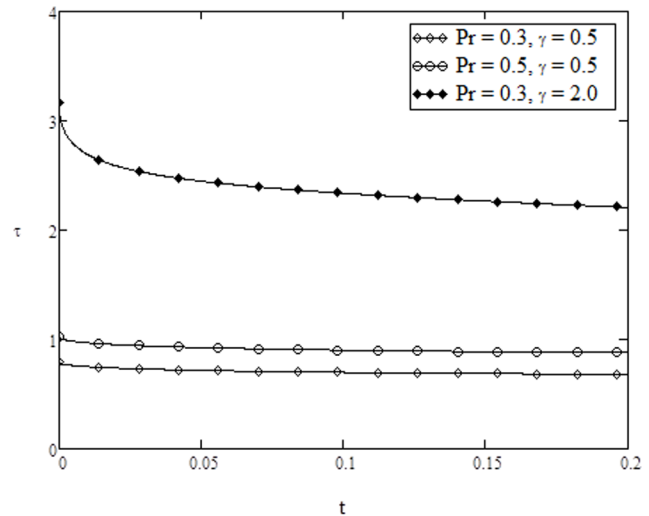
**Figure 11. Temperature profiles for different values of  $\gamma$ , when Pr = 0.5,  $t=0.2$ .**  
doi:10.1371/journal.pone.0108763.g011



**Figure 13. Skin-friction variation for different values of  $\alpha$ , Pr and Gr, when Pr = 0.3,  $\gamma=0.5$ .**  
doi:10.1371/journal.pone.0108763.g013



**Figure 14. Skin-friction variation for different values of  $\gamma$  and  $\omega t$ , when  $\alpha=0.6$ ,  $Pr=0.3$ ,  $\gamma=0.5$ .**  
doi:10.1371/journal.pone.0108763.g014



**Figure 15. Nusselt number variation for different values of  $Pr$  and  $\gamma$ .**  
doi:10.1371/journal.pone.0108763.g015

$$u(y, t) = \frac{H(t)}{2} [F_1(y, t, -i\omega) + F_1(y, t, i\omega)] + \frac{yGr}{2\gamma} [F_2(y, t, \gamma) - \gamma F_3(y, t)], \tag{25}$$

where

$$a_1 = \frac{\alpha}{1+\alpha}, a_2 = \frac{\gamma}{\sqrt{Pr}}, a_3 = \frac{Gr}{Pr - a_1},$$

$$F_1(\xi, t, \psi) = \frac{1}{2} e^{\psi t} \left[ e^{-\xi\sqrt{\psi}} \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} - \sqrt{\psi t} \right) + e^{\xi\sqrt{\psi}} \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} + \sqrt{\psi t} \right) \right],$$

$$F_2(\xi, t, \psi) = e^{(\psi^2 t - \xi\psi)} \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} - \psi\sqrt{t} \right) - \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} \right),$$

$$F_3(\xi, t) = 2\sqrt{\frac{t}{\pi}} e^{-\frac{\xi^2 t}{4t}} - \xi \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} \right),$$

$$F_4(\xi, t) = \left( \frac{\xi^2}{2} + y \right) \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} \right) - \xi \sqrt{\frac{t}{\pi}} e^{-\frac{\xi^2 t}{4t}},$$

$$F_5(\xi, t) = \operatorname{erf} c \left( \frac{\xi}{2\sqrt{t}} \right),$$

$F_i, i=1$  to  $5$ , are dummy functions of the dummy variables  $\xi$  and  $\psi$ .

The dimensionless expression for skin friction evaluated from equation (24) is given by

$$\tau' = - \left( 1 + \frac{1}{\alpha} \right) \frac{\partial u}{\partial y} \Big|_{y=0},$$

$$\tau^* = \frac{\tau'}{\rho U^2} = - \left( 1 + \frac{1}{\alpha} \right) \frac{\partial u^*}{\partial y^*} \Big|_{y^*=0},$$

$$\begin{aligned} &= \frac{H(t)}{2a_1} \left[ \sqrt{-i\omega} a_1 e^{-i\omega t} [1 - \operatorname{erfc}(\sqrt{-i\omega t})] + \sqrt{i\omega} a_1 e^{i\omega t} [1 - \operatorname{erfc}(\sqrt{i\omega t})] \right] \\ &+ a_3 \left( \frac{\sqrt{a_1} - \sqrt{Pr}}{a_2} \right) e^{a_2^2 t} [1 - \operatorname{erfc}(a_2 \sqrt{t})] - \frac{a_3 \sqrt{Pr}}{a_2} [e^{a_2^2 t} - 1] \\ &+ \frac{a_1 a_3}{a_2} [e^{a_2^2 t} - 1] - 2a_3 \sqrt{Pr} \sqrt{\frac{t}{\pi}} + a_3 \sqrt{a_1} \sqrt{\frac{t}{\pi}} - a_1 a_3 \sqrt{\frac{t}{\pi}} + \frac{1}{a_1} \sqrt{\frac{a_1}{\pi}}. \end{aligned} \tag{26}$$

where  $\tau'$  is the dimensional skin friction. The dimensionless expression of Nusselt number is given by

$$\begin{aligned} Nu &= - \frac{v}{U_0(T - T_\infty)} \frac{\partial T}{\partial y} \Big|_{y=0} = \frac{1}{\theta(0, t)} + 1, \\ &= a_2 \sqrt{Pr} \left( 1 + \frac{1}{e^{a_2^2 t} [1 + \operatorname{erf}(a_2 \sqrt{t})] - 1} \right). \end{aligned} \tag{27}$$

### Limiting Cases

The solutions obtained here are more general. In this section, we consider some of their limiting cases.

#### Solution in case of Newtonian fluid

If  $\alpha \rightarrow \infty$ , the solution for velocity given in equation (24) reduces to the corresponding solution for Newtonian fluid given by

Table 1. Numerical results for velocity.

$y$	$t$	$\alpha$	$Pr$	$Gr$	$\gamma$	$\omega t$	$u(y, t)$
0	0.2	0.6	0.3	3	0.5	$\pi/3$	0.500
1	0.2	0.6	0.3	3	0.5	$\pi/3$	0.208
2	0.2	0.6	0.3	3	0.5	$\pi/3$	0.035
1	0.4	0.6	0.3	3	0.5	$\pi/3$	0.417
2	0.4	0.6	0.3	3	0.5	$\pi/3$	0.162
1	0.2	1.0	0.3	3	0.5	$\pi/3$	0.176
2	0.2	1.0	0.3	3	0.5	$\pi/3$	0.020
1	0.2	0.6	0.5	3	0.5	$\pi/3$	0.187
2	0.2	0.6	0.5	3	0.5	$\pi/3$	0.029
1	0.2	0.6	0.3	5	0.5	$\pi/3$	0.236
2	0.2	0.6	0.3	5	0.5	$\pi/3$	0.040
1	0.2	0.6	0.3	3	1.0	$\pi/3$	0.283
2	0.2	0.6	0.3	3	1.0	$\pi/3$	0.047
1	0.2	0.6	0.3	3	0.5	$\pi/2$	0.042
2	0.2	0.6	0.3	3	0.5	$\pi/2$	0.008

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$$\begin{aligned}
 u(y, t) = & \frac{H(t)}{2} [F_1(y, t, -i\omega) + F_1(y, t, i\omega)] \\
 & + \frac{a_3}{a_2^2} [F_2(y, t, a_2) - F_2(y\sqrt{Pr}, t, a_2)] \\
 & - \frac{a_3}{a_2} [F_3(y, t) - F_3(y\sqrt{Pr}, t)] \\
 & - a_3 [F_4(y, t) - F_4(y\sqrt{Pr}, t)].
 \end{aligned} \quad (28)$$

It is important to note that the above solution (28) for Newtonian fluid over an impulsively moved plate when  $\gamma=1$  is similar to that obtained by [27].

### Solution in the absence of free convection

In the absence of free convection, which is numerically corresponds to  $Gr=0$ , the equation (24) reduces to

$$u(y, t) = \frac{H(t)}{2} [F_1(y\sqrt{a_1}, t, -i\omega) + F_1(y\sqrt{a_1}, t, i\omega)]. \quad (29)$$

### Solution of Stokes first problem

By making  $\omega \rightarrow 0$  into equation (24), we get the classical solution

$$\begin{aligned}
 u(y, t) = & F_5(y\sqrt{a_1}, t) + \frac{a_1 a_3}{a_2^2} [F_2(y\sqrt{a_1}, t, a_2) - F_2(y\sqrt{Pr}, t, a_2)] \\
 & - \frac{a_1 a_3}{a_2} [F_3(y\sqrt{a_1}, t) - F_3(y\sqrt{Pr}, t)] \\
 & - a_1 a_3 [F_4(y\sqrt{a_1}, t) - F_4(y\sqrt{Pr}, t)].
 \end{aligned} \quad (30)$$

corresponding to the Stokes first problem for Casson fluid over an impulsively motion of the plate.

### Graphical Results and Discussion

Exact solutions for the problem of unsteady boundary layer heat transfer flow of an incompressible Casson fluid past an infinite oscillating vertical plate with Newtonian heating condition are obtained. For the physical behavior of embedded parameters such as Casson parameter  $\alpha$ , Prandtl number  $Pr$ , Grashof number  $Gr$ , conjugate parameter for Newtonian heating  $\gamma$ , time  $t$  and phase angle  $\omega t$ , these solutions are plotted in graphs (Figures 2–15) and discussed in details.

The velocity profiles for different values of Casson parameter  $\alpha$  are shown in Figure 2. From this figure, it is observed that velocity decreases with increasing values of  $\alpha$ . Further, it is noticed that Casson parameter does not have any influence as the fluid moves away from the bounding surface. The velocity profiles are shown in Figure 3 for different values of Prandtl number  $Pr$ . It is observed that velocity decreases with increasing Prandtl number. This situation is in consistence with the physical observation because fluids with large Prandtl number have high viscosity and small thermal conductivity, which makes the fluid thick and hence causes a decrease in velocity of fluid. In addition, the curves show that velocity of fluid is maximum near the plate and approaches to zero as  $y \rightarrow \infty$  (for away from the plate). It is also found from Figures 2 and 3, that the behavior of  $\alpha$  and  $Pr$  on the velocity profiles are quite identical with that found in figure 7 and 9, of Rao et al. [13]. The effects of Grashof number  $Gr$  on the velocity



**Table 2.** Numerical results for temperature.

$y$	$t$	$Pr$	$\gamma$	$\theta(y,t)$
0	0.2	0.3	0.5	0.656
1	0.2	0.3	0.5	0.140
2	0.2	0.3	0.5	0.016
0	0.4	0.3	0.5	1.155
1	0.4	0.3	0.5	0.396
0	0.2	0.5	0.5	0.379
1	0.2	0.5	0.5	0.020
0	0.2	0.3	1.0	2.827
1	0.2	0.3	1.0	0.893

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profiles are shown in Figure 4. The trend shows that velocity increases with increasing values of  $Gr$ . It is true physically also because the role of Grashof number in heat transfer flow is to increase the strength of the flow. Here  $Gr=0$  corresponds to the absence of free convection, while  $Gr>0$  represents to the cooling problem. Moreover, the cooling problem is of great importance and mostly encountered in engineering applications, such as in the cooling of electronic components and nuclear reactors. For different values of conjugate parameter for Newtonian heating  $\gamma$ , the velocity profiles are plotted in Figure 5. An increase in conjugate parameter for Newtonian heating may reduce the fluid density and increases the momentum boundary layer thickness, as a result, the velocity increases within the boundary layer. Further, the behavior of Grashof number and conjugate parameter on the velocity profiles are quite identical with that found in figures 7 and 8 of Jain [33]. Figure 6 demonstrates the effect of time  $t$  on the velocity profiles. It is found that velocity increases with an increase in  $t$ . The velocity profiles for different values of phase angle  $\omega t$  are depicted in Figure 7. It is found that the velocity shows an oscillatory behavior. The oscillations near the plate are of great significance; however, these oscillations reduce for large values of the independent variable  $y$  and approach to zero as  $y$  approaches to infinity. The numerical results for velocity and temperature are computed in Table 1 and Table 2 respectively. Furthermore, Figures 8 and 9 are prepared to show the comparison of the present analytical results for velocity and temperature given by equations (24) and (23) with the numerical results in Table 1 and Table 2. It is found that the analytical results are quite identical with the numerical results.

The variation of temperature for different values of Prandtl number  $Pr$  are plotted in Figure 10. It is found that temperature of

the fluid decreases with increasing values of  $Pr$ . This is in agreement with the physical fact that with increasing Prandtl number, the viscosity of the fluid increases, the fluid become more thick which reduces the heat transfer. From Figure 11, it is observed that an increase in the conjugate parameter for Newtonian heating increases the thermal boundary layer thickness and as a result the surface temperature of the plate increases. It is also observed that there is a sharp rise in temperature with the increase of conjugate parameter. Note that the variations in temperature due to conjugate parameter are identical to the published work of [31,33]. It is observed from Figure 12 that the fluid temperature increases with an increase in time  $t$ .

On the other hand, variation of skin friction and Nusselt number verses time are plotted in Figures 13–15 for various parameters of interest. It is found from Figure 13 that skin friction increases with increasing value of  $Pr$  whereas it decreases with increasing value of  $\alpha$  and  $Gr$ , when  $\gamma$  and  $\omega t$  are fixed. From Figure 14, it is noticed that the skin friction increases with increasing values of conjugate parameter  $\gamma$ , while reverse effect is observed for phase angle  $\omega t$ . Finally, the Nusselt number increases as  $Pr$  and  $\gamma$  are increased as shown in Figure 15. Finally, for the comparison of the present results with those existing in the literature we have plotted Table 3. It is found that for  $\alpha \rightarrow \infty$ , our results are quite identical with those obtained in [32], when  $R=0$ (in the absence of thermal radiation).

## Conclusions

In this paper, exact solutions of unsteady boundary layer flow and heat transfer of a Casson fluid past an oscillating vertical plate with Newtonian heating are obtained using the Laplace transform technique. The results obtained have shown that the effect of

**Table 3.** Comparison of skin friction calculated in the present work at  $\alpha \rightarrow \infty$  and in [32], when  $R=0$ .

$t$	$Pr$	$Gr$	$\gamma$	$\omega t$	Present Results	Results of [32]
0.01	0.35	5	1	0	5.5818	5.5818
0.02	0.35	5	1	0	3.8620	3.8620
0.01	0.50	5	1	0	5.5956	5.5956
0.01	0.35	10	1	0	5.5218	5.5218
0.01	0.35	5	2	0	5.5027	5.5027
0.01	0.35	5	1	$\pi/2$	4.4819	4.4819

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number increases the velocity but reduces the skin friction. However, the velocity is decreased when the Casson parameter is increased. Moreover, in the particular case of Newtonian fluid, the analytical results obtained in the present work were compared with those available in the literature, obtaining an excellent agreement with those given in [32]. A significant finding of this study is that flow separation can be controlled by increasing the value of Casson fluid parameter as well as by increasing Prandtl number.

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## Author Contributions

AH IK. Conceived and designed the experiments: AH IK RMT. Performed the experiments: IK AH MZ. Analyzed the data: AH IK MZ. Contributed reagents/materials/analysis tools: AH IK MZ RMT. Wrote the paper: AH IK.